Chapter 7

Sheared flow effects on the outer layer dynamics of NTMs

7.1 Introduction

The temporal evolution of magnetic islands due to unstable classical or neo-classical tearing modes is strongly influenced by the plasma and field dynamics within a narrow boundary layer centered on the mode rational surface. The mode rational surfaces in a tokamak are those where the pitch of the perturbation magnetic field matches the pitch of the equilibrium field. Such a resonance condition favors the growth of the tearing mode and the growth rate is controlled by such factors as the resistivity, viscosity, and inertial effects close to the resonant surface. The overall global structure of the mode is however determined by the ideal solutions in the rest of the plasma (the outer layer) and resembles the ideal ideal MHD modes such as kink modes. Since in ideal MHD magnetic fields remain frozen in the flow, the presence of any equilibrium flow can be expected to have a significant influence on the evolution of the modes. Since resistivity is very small even within the inner layer the existence of flows can profoundly change the dynamics in this region and thereby affect the growth rate of the tearing mode. Many past studies have delineated this effect in the inner layer dynamics for the linear classical tearing mode [26-28]. In the previous chapter we have also highlighted such an effect in the nonlinear Rutherford model. Similar effects can also be expected to occur for the outer layer dynamics and a few past studies carried out for
simple slab geometries or simple flow profiles in cylindrical geometry have pointed out the significance of this effect for linear classical tearing modes [30, 35, 129]. Since $\Delta'$ - the stability index determined from the outer layer solutions - appears in the Rutherford model and plays an important role in determining the threshold island widths for neoclassical tearing modes, it is important to assess the influence of flows on this quantity. In this chapter we carry out such an investigation by deriving a generalized version of the outer layer equations in a cylindrical geometry that incorporates both inertial contributions of flow and finite $\delta$ terms and show through numerical solutions that the value of $\Delta'$ can be significantly influenced by the combination of the velocity and magnetic field profiles. Our findings should prove useful in extending the applicability of the Rutherford model and also in the interpretation of numerical investigations carried out on more complex codes like NIMROD. Based on our understanding of these flow induced effects one can hope to develop experimental strategies that can exploit flows for mitigation or better control of island growths in long pulsed tokamak experiments.

The plan of the chapter is as follows. Section (7.2) details the derivation of the generalized Newcomb equation in the presence of an equilibrium flow for a cylindrical geometry. The method of numerical solution of this equation to determine $\Delta'$ is presented in section (7.3). Detailed results showing the parametric dependence of $\Delta'$ on various profile coefficients are given in section (7.4). Section (7.5) provides a concluding summary.

### 7.2 Newcomb equation in the presence of flow

We consider a uniform density compressible plasma in a cylindrical geometry $(r, \theta, z)$ that has a uniform equilibrium flow along the $z$ axis and a sheared poloidal flow along the $\theta$ direction. To describe the outer layer dynamics we consider the ideal MHD model equations given by.
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\[ \rho \left[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = -\nabla P + \mathbf{J} \times \mathbf{B} \]  \hspace{1cm} (7.1)

\[ \mathbf{E} + \mathbf{V} \times \mathbf{B} = 0 \]  \hspace{1cm} (7.2)

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \; ; \; \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]  \hspace{1cm} (7.3)

\[ \nabla \cdot \mathbf{B} = 0 \; ; \; \nabla \cdot \mathbf{V} = 0 \]  \hspace{1cm} (7.4)

We assume the equilibrium quantities to be of the form.

\[ \mathbf{B}_0 = B_{00}(r)\hat{e}_\theta + B_{0z}(r)\hat{e}_z \; ; \; \mathbf{V}_0 = V_{00}(r)\hat{e}_\theta + V_{0z}\hat{e}_z \]

The ideal equilibrium, obtained from the momentum eqn.(7.1), is then given by the relation.

\[ \frac{3}{2} \frac{dP_0}{dr} = \frac{V_{00}^2}{r} - \frac{B_{0\theta}^2}{r} - B_{00} \frac{dB_{0\theta}}{dr} - B_{0z} \frac{dB_{0z}}{dz} \]  \hspace{1cm} (7.5)

where we have normalized the magnetic field by \( B_{0z} \), the velocity field by \( V_{0} \), the pressure by \( P_{00} = \beta B_{00z}^2/2\mu_0 \) with \( P_{00} \) being the peak value of the pressure, \( B_{0z} \) the peak value of the axial magnetic field and \( \beta = 2\mu_0 P_{00}/B_{0z}^2 \).

We next linearize eqns.(7.1) and (7.2) about these equilibrium quantities and neglect the contributions from the time derivative term. This is appropriate for the ideal external region where resistivity can be neglected and the mode growth term, which scales inversely as some power of the resistivity, is also very small. The perturbed quantities then obey the following set of equations.

\[ (\mathbf{V}_0 \cdot \nabla)\mathbf{V}_1 + (\mathbf{V}_1 \cdot \nabla)\mathbf{V}_0 = -\nabla p_1 + (\mathbf{B}_0 \cdot \nabla)\mathbf{B}_1 + (\mathbf{B}_1 \cdot \nabla)\mathbf{B}_0 \]  \hspace{1cm} (7.6)

\[ (\mathbf{B}_1 \cdot \nabla)\mathbf{V}_0 - (\mathbf{V}_0 \cdot \nabla)\mathbf{B}_1 + (\mathbf{B}_0 \cdot \nabla)\mathbf{V}_1 - (\mathbf{V}_1 \cdot \nabla)\mathbf{B}_0 = 0 \]  \hspace{1cm} (7.7)
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where \( p_1^* = \frac{1}{r} \beta_1 \cdot B_0 \). We assume the perturbed quantities to have a spatial dependence of the form, \( f_1(r, \theta, z) = f_1(r) e^{i \rho (m \theta + k_z z)} \). We further define,

\[
F = k \cdot B_0 \quad ; \quad G = k \cdot V_0
\]

where \( k = (0, m/r, k_z) \). The radial components of eqns (7.6) and (7.7) give,

\[
-FB_{1r} + \frac{2}{r} \left( V_{1\theta} V_{1\theta} - B_{1\theta} B_{1\theta} \right) + GV_{1r} - \frac{1}{r} \frac{dp_1^*}{dr} = 0 \quad (7.8)
\]

\[
V_{1r} = \frac{G}{F} B_{1r} \quad (7.9)
\]

Further, the \( \theta \) and \( z \) components of eqns (7.6) and (7.7) can be used to obtain the following equations:

\[
\frac{ip_1^*}{r} = \frac{B_{1r}}{r} \left( H \frac{\partial F}{\partial r} + \frac{2m B_{1\theta} H}{r^2} \right) - \frac{HF}{r} \frac{\partial B_{1r}}{\partial r} - \frac{V_{1r}}{r} \left( H \frac{\partial G}{\partial r} \right) + \frac{2mV_{1\theta} H}{r^2} - \frac{H(G \frac{\partial V_{1r}}{\partial r})}{r} \right) \quad (7.10)
\]

\[
\frac{1}{r} \left( V_{1\theta} V_{1\theta} - B_{1\theta} B_{1\theta} \right) = \frac{2m(ip_1^*)}{r^2} \left( V_{1\theta} G - B_{1\theta} F \right) + \left[ \frac{2(FB_{1r} + GV_{1r})}{(F^2 - G^2)} \left( \frac{B_{1\theta}^2}{r^2} - \frac{V_{1\theta}^2}{r^2} \right) \right]
\]

\[
- \frac{4V_{1\theta} B_{1\theta}}{r^2} \left( GB_{1r} + FV_{1r} \right) - \left( \frac{\partial V_{1\theta}^2}{\partial r} - \frac{\partial B_{1\theta}^2}{\partial r} \right) \left( \frac{FB_{1\theta}^2}{r^2} - \frac{G^2}{r^2} \right) \left( B_{1\theta} \frac{\partial V_{1\theta}}{\partial r} \right) - \frac{\partial B_{1\theta}}{\partial r} \left( GB_{1r} - FV_{1r} \right) \right) \quad (7.11)
\]

where \( H = r^3/(k_z r^2 + m^2) \). Using eqn (7.10) and eqn (7.11) to substitute for \( p_1^* \) and \( \left( V_{1\theta} V_{1\theta} - B_{1\theta} B_{1\theta} \right) \) in eqn (7.8) and after some rearrangement of terms one gets.
\[
F \frac{d}{dr} \left( H \frac{d\psi}{dr} \right) - \frac{2mH G}{r^2} \left( V_{00} - \frac{G}{F} B_{00} \right) \frac{d\psi}{dr} - \frac{\psi^2}{d^2r} \left( H \frac{dF}{dr} \right) - F \psi \left[ r + \frac{2m}{r^2 F^2} \right] \frac{H}{dr}
\]

\[
+ \frac{2mB_{00} H}{r^2} \left( \frac{HF}{r} \right) \left( B_{00} - \frac{G}{F} V_{00} \right) - \frac{2}{rF^2} \left( B_{00}^2 + V_{00}^2 \right) + \frac{4}{rF^2} \left( \frac{G}{F} \right) V_{00} B_{00} + \frac{1}{F^2} \left( \frac{dV_{00}}{dr} \right)
\]

\[
- \frac{d^2 B_{00}}{dr^2} + \frac{r d}{F dr} \left( \frac{2mB_{00} H}{r^2} \right) - \frac{r d}{F dr} \left( \frac{HF}{r^2} \right) - \frac{H}{F} \frac{dF}{dr} - \frac{G}{F} \left( B_{00} \frac{dV_{00}}{dr} - V_{00} \frac{dB_{00}}{dr} \right)
\]

\[
= \frac{G}{dr} \left( \frac{dW}{dr} \right) - \frac{2mH}{r^2} \left( V_{00} - \frac{G}{F} B_{00} \right) \frac{dW}{dr} - W \frac{dF}{dr} \left( \frac{H}{dF} \right) - \frac{G}{F} \left( B_{00} \frac{dV_{00}}{dr} - V_{00} \frac{dB_{00}}{dr} \right)
\]

\[
+ \frac{2mV_{00} H}{r^2} \left( \frac{HG}{r} \right) \left( B_{00} - \frac{G}{F} V_{00} \right) + \frac{2}{rF^2} \left( B_{00}^2 + V_{00}^2 \right) - \frac{4}{rF^2} \left( \frac{G}{F} \right) V_{00} B_{00} + \frac{1}{F^2} \left( \frac{dV_{00}}{dr} \right)
\]

\[
- \frac{d^2 B_{00}}{dr^2} + \frac{r d}{G dr} \left( \frac{2mV_{00} H}{r^2} \right) - \frac{r d}{G dr} \left( \frac{HG}{r^2} \right) - \frac{H}{G} \frac{dF}{dr} - \frac{G}{F} \left( B_{00} \frac{dV_{00}}{dr} - V_{00} \frac{dB_{00}}{dr} \right)
\]

(7.12)

where \( \alpha = 1 - (G^2/F^2) \) and we have simplified the notation somewhat by using \( G \) for normalized \( B_{1r} \) and \( W \) for normalized \( V_{1r} \). Further, using eqn.(7.5) and eqn.(7.9) in the above equation (7.12) and after affecting some simplifications one can get the following equation in the single variable \( \epsilon \):

\[
H \frac{d^2 \psi}{dr^2} + \left( \frac{dH}{dr} + h_f \right) \frac{d\psi}{dr} = \frac{g}{F^2} + \frac{g_f}{F^2} + \frac{1}{F} \frac{dF}{dr} \left( \frac{H}{dF} \right) \psi \quad \quad \quad (7.13)
\]

where,

\[
g = \frac{(\alpha m^2 - 1) r F^2}{\alpha (k^2 r^2 + m^2)} + \frac{k_2^2 r^2}{\alpha (k_2^2 r^2 + m^2)} \left( \alpha r F^2 + F \frac{2(k_2 - m B_{00})}{k_2^2 r^2 + m^2} - \frac{dH_{00}}{dr} \right)
\]

\[
h_f = \frac{2H G}{\alpha F} \left( \frac{G}{F} \frac{dF}{dr} - \frac{1}{dF} \frac{dG}{dr} \right)
\]

\[
g_f = \frac{2H G}{\alpha F} \left( \frac{G}{F} \frac{dF}{dr} - \frac{1}{dF} \frac{dG}{dr} \right) + \frac{4}{r^2} \left( \frac{G}{F} \frac{dF}{dr} - \frac{1}{dF} \frac{dG}{dr} \right) + \frac{4}{r^2} \left( \frac{G}{F} \frac{dF}{dr} - \frac{1}{dF} \frac{dG}{dr} \right) + \frac{2V_{00}}{r^2} \frac{dV_{00}}{dr}
\]

\[
+ \frac{G}{\alpha r^2} \left( \frac{2m H V_{00}}{r^2} + \frac{4}{r^2} \left( \frac{G}{F} \frac{dF}{dr} - \frac{1}{dF} \frac{dG}{dr} \right) \right) \left( \frac{V_{00} - G}{F} B_{00} \right)
\]

\[
- \frac{2m H G}{r^2} \left( \frac{dG}{dr} - \frac{2m V_{00}}{r} \right) \left( \frac{B_{00} - G}{F} V_{00} \right)
\]
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For $G = 0$ and $\beta = 0$ eqn.(7.13) reduces to the standard outer layer equation that has been analyzed in the paper by Furth, Rutherford and Selberg [130]. More recently, Nishimura et al.[131] have extended the results of [130] to include finite $\beta$ effects and have shown that finite $\beta$ can have a stabilizing effect on $\Delta'$. The effect of equilibrium sheared flows on $\Delta'$ has been examined in the past by Chen and Morrison [30] but only in a simple slab geometry. For $G$ finite and in the limit of a slab geometry ($r \to \infty$, $d/dr \to d/dx$) our eqn.(7.13) reduces to the set of equations that have been discussed in Chen and Morrison [30]. Note that in the slab limit the finite $\beta$ contribution disappears. Thus eqn.(7.13) represents a more generalized description of the outer layer dynamics that takes into account finite $\beta$ contributions, cylindrical curvature effects as well as sheared flow effects.

7.3 Numerical evaluation of $\Delta'$

To investigate the influence of flows on the outer layer dynamics we have solved eqn.(7.13) numerically to determine $\Delta'$ for a model set of profiles of $B_{bg}$ and velocity $V_{bg}$. An advanced shooting method, first developed by Nishimura et al [131] for the finite $\beta$ problem, has been adopted for this purpose. The algorithm involves numerical integration of the equation away from the singular layer towards the boundaries.

The following analytic expressions representing asymptotic solutions for $\psi$ near the resonant surface:

![Figure 7.1: Profiles of $B_{bg}$ and $V_{bg}$](image)
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Figure 7.2: Typical $p$ profile

Figure 7.3: Typical $q$ profile

Figure 7.4: Typical $F$ profile
Figure 7.5: Typical G profiles for different $\nu_f$

Figure 7.6: Eigenfunction $\psi$ for $(n=2, m=1)$ mode
have been used to launch the numerical solutions.

\[ \psi = A_{i} |s|^{h+1} - B_{i} |s|^{-h} \quad \text{for } x < x_{s} \]  
\[ \psi = A_{r} |s|^{h+1} + B_{r} |s|^{-h} \quad \text{for } x > x_{s} \]

where \( x = r/r_{s} \), \( r_{s} \) is the location of the singular layer and \( s = x - x_{s} \),

\[ h = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4D_{s}} \]

The Mercier coefficient \( D_{s} \) is given as.

\[
D_{s} = -\frac{g_{s}^{2}}{d_{s}^{2} \alpha x_{s}} \left[ \frac{\beta dP_{0}}{dx} + \frac{2x}{Hk_{2}^{2}r_{s}^{2}} V_{09} \frac{dV_{09}}{dx} + \frac{2}{Hk_{2}^{2}r_{s}^{2}} \left( \frac{m^{2}}{(k_{2}^{2}x^{2} + m^{2})} - \frac{2}{\alpha} \right) V_{09} \right]
+ \left( \frac{4 G B_{09}}{\alpha F x} - \frac{2m}{\alpha x k_{2}^{2} r_{s}^{2}} \frac{dG}{dx} \right) \left( V_{09} - \frac{G}{F} V_{09} \right)
\times \left( B_{09} - \frac{G}{F} V_{09} \right) + \frac{4}{\alpha H k_{2}^{2} r_{s}^{2} F} V_{09} B_{09} \right|_{x=r_{s}}.
\]

We iterate the constants \( A \) and \( B \) until the solution satisfies the appropriate boundary conditions. \cite{131}

The value of \( \Delta' \) is then obtained as.

\[ \Delta' = \frac{A_{r}}{B_{r}} - \frac{A_{i}}{B_{i}} \]

In the absence of flow our numerical values of \( \Delta' \) agree with those of Nishimura et al. \cite{131} for a choice of their model profile.

### 7.4 Results

We now present our numerical results of \( \Delta' \) values obtained for the \((m=2, n=1)\) tearing mode. We have used the following equilibrium profile for the normalized poloidal magnetic field,

\[ B_{09}(x) = \frac{r_{s}}{R_{09}} \frac{x}{(1 + x^{2})^{1/2}} \]

\[ q(x) = q_{0} (1 + x^{2})^{1/2} \]
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\[ v=1.0, \nu=1.0 \]

\[ \Delta \]

Figure 7.7: Change of \( \Delta' \) with \( \beta \) for different \( V_{0z} \)

\[ v=1.0, \beta=0.2 \]

\[ \Delta \]

Figure 7.8: Change of \( \Delta' \) with \( V_{0z} \) for different \( V_{0z} \) profiles
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where R the major radius is taken to be a constant quantity and \( \nu \) is an index that controls the flatness of the magnetic profile. To account for finite \( \beta \) effects we have chosen the normalized pressure profile to be,

\[
P_{\text{th}}(x) = 1 - \left( \frac{x}{R_{\text{th}}} \right)^2
\]

The equilibrium pressure balance is ensured by providing a small variation in \( B_{\text{th}} \). The plasma boundary is chosen to be at \( x_0 = 2 \) and by construct \( x_\alpha = 1 \). Note that since \( q(x_\alpha) = m/n = 2 \), for the (2, 1) tearing mode, the index \( \nu \) and the quantity \( q_0 \) are related as \( q_0 = 2^{1-(m/n)} \). For the velocity profile we have chosen the following model form,

\[
V_{\text{th}}(x) = \frac{r_v V_{\infty}}{R_{\text{th}}^\nu} \left( 1 + \frac{x}{R_{\text{th}}} \right)^{3/2} \quad q_e(x) = q_e(0)(1 + x^{2\nu_f})^{1/2}
\]

where \( q_v = r_v V_{\infty}/(RV_{\infty}) \), and \( \nu_f \) is an index that controls the flatness of the flow velocity profile. For convenience we choose \( q_e(0) = m/2^{(1/\nu_f)} \) which makes the magnetic and velocity profiles to have a similar behavior near the singular layer. Figure (7.1) shows two sets of profiles of \( B_{\text{th}} \) and \( V_{\text{th}} \) plotted for values of \( (\nu = 1.0, 3.0) \) and \( (\nu_f = 1.0, 3.0) \) respectively. Typical profiles of \( p, q, F \) and \( G \) are shown in figure (7.2), figure (7.3), figure (7.4) and figure (7.5) respectively. Figure (7.6) depicts a typical eigenfunction for the \( (m = 2, n = 1) \) mode. In figure (7.7) we have plotted the variation of \( \Delta' \) with \( \beta \) for various values of the flow velocity \( V_{\text{th}} \). The flatness profile indices \( \nu \) and \( \nu_f \) are held constant at the value of unity.

The solid curve (no flow case) corresponds to the previous result of Nishimura et al. [131] and shows the stabilizing effect of finite \( \beta \) on \( \Delta' \). Due to a factor of 2 difference in the definition of \( \beta \) between our normalization and that adopted in [131] the x axis scale is expanded by a factor of 2 in our case. When finite flow velocity is turned on (at the same values of \( \nu \) and \( \nu_f \) we notice two differences from the no flow result. At low \( \beta \) finite flow has a slightly destabilizing effect but the threshold \( \beta \) at which the curve begins to sharply drop to negative values is decreased as seen from the two other curves in the figure. Thus one can access higher \( \beta \) values more easily in the presence of flows. This trend however is strongly influenced by the shape of the velocity profile. This is shown in figure (7.8) where the variation of \( \Delta' \) with \( V_{\text{th}} \) is shown at a fixed value of \( \beta = 0.2, \nu = 1 \) and for different values of \( \nu_f \). As \( \nu_f \) increases we
Figure 7.9: Change of $\Delta'$ with $\nu$ for different $V_0\theta$ profiles

Figure 7.10: Change of $\Delta'$ with $\beta$ for different $V_0\theta$ profiles
see that there is a change in the behavior of $\Delta'$ beyond a threshold value of $\nu_f$ and flow begins to have a destabilizing effect. This sensitivity to the profile parameter is also seen for the magnetic field. In figure (7.9) we show how $\Delta'$ changes with the magnetic field flatness parameter $\nu$ for a zero $\beta$ plasma. As can be seen there is a dramatic rise in the value of $\Delta'$ as $\nu$ increases i.e. as the magnetic field profile gets more peaked. At a given value of $\nu$ if $\nu_f$ is raised then $\Delta'$ decreases somewhat indicating that raising the peakedness of the flow profile has a stabilizing influence. The stabilizing influence is more pronounced at higher values of $\nu$. Figure (7.10) shows the effect of finite $\beta$ on $\Delta'$ with velocity profiles of different $\nu_f$ for a given and $\nu$ and for a fixed value of $\nu = 1$. The figure shows that $\beta$ has a stabilizing influence on $\Delta'$ at a given value of $\nu$ = 1 in agreement with the results of [131]. Increasing $\nu_f$ at this value of $\nu$ and for a given value of $\beta$ provides a further stabilizing influence but the incremental effect is small. This is further supported by figure (7.11) where the curve of $\Delta'$ vs $\nu_f$ for a given $\beta$ is seen to be quite flat. The above picture changes however when the value of $\nu$ is increased to higher values. As figure (7.12) demonstrates, for a value of $\nu$ exceeding a critical value increasing $\beta$ can have a destabilizing effect (e.g. the curves for $\nu = 2, 3$ with $\nu_f = 2$). This is very similar to the turnover behavior that is observed in figure (7.8) for $\Delta'$ vs $V_{0z}$ curve where when $\nu_f$ is changed keeping $\beta$ and $\nu$ constant.
7.5 Summary and Discussion

These numerical results suggest that the combination of the magnetic and velocity profile variations along with finite $\beta$ effects can profoundly influence the magnitude of $\Delta'$ and consequently the stability of the tearing mode. This global dependence of $\Delta'$ needs to be appropriately accounted for when estimating stability thresholds or saturation widths of magnetic islands in the nonlinear Slatertype. Our model outer layer equation (7.13) provides a means for estimating $\Delta'$ in the presence of tearing flow, particularly for large aspect ratio machines [132, 133]. When toroidal effects become important it is necessary to generalize the equation to account for the additional geometric effects. Our present calculations were done with simple model profiles and in a limited parametric space to highlight the sensitivity of $\Delta'$ to equilibrium profile parameters. A more direct utility of our equation would be to estimate $\Delta'$ using realistic equilibrium profiles obtained from MHD equilibrium codes.

Figure 7.12: Change of $\Delta'$ with $\beta$ for different $B_{0\theta}$ profiles