Chapter 6

A generalized Rutherford model calculation for sheared flow effects on NTMs

6.1 Introduction

As tokamak experiments move into the high temperature (high $\beta$) long pulsed regime, one of the major threats to their confinement properties arises from the neoclassical tearing modes. These modes dominate over the classical tearing modes since the driving term for the latter is considerably diminished due to the decrease in resistivity at high temperatures. The NTM driving term on the other hand is proportional to $\beta$ and it starts becoming important at higher temperatures. Unlike the classical tearing instability the NTM instability is sub-critical i.e. it is a nonlinear instability requiring a critical threshold amplitude of perturbation for it to develop. Analytic descriptions of the NTM have to therefore take into account these characteristic features of the instability. The most successful analytic theory for NTMs to date is the generalized Rutherford model (GRM) which is an adaptation of the quasilinear description developed for classical tearing modes [80] and into which the neoclassical driving sources are incorporated. The GRM model has received a great deal of attention in recent times [16, 71, 83, 106, 118-123] and continues to undergo further extensions and refinements. For a physical understanding of some of the numerical results obtained in the previous chapter regarding the interaction of NTMs with equilibrium flows it is
appropriate then to attempt a model calculation within the framework of the GRM. We carry out such a calculation in this chapter. We are guided in our efforts by some past studies which have tried to assess the influence of plasma rotation on NTMs [47, 49, 51, 124] but have restricted themselves to flows that are primarily generated by perpendicular drifts. In our study we include both parallel and perpendicular drifts that can contribute to a general toroidal flow and obtain results which are in good qualitative agreement with our numerical results [125-128].

The chapter is organized as follows. Section (6.2) briefly outlines the derivation of the island evolution equation for a single helicity tearing mode in the presence of equilibrium sheared flows and neoclassical current contributions. Some of the details of the calculation are provided in the Appendix 6.5. In section (6.3) we describe the stability results. The chapter concludes with a brief summary and discussion in section (6.4).

### 6.2 Rutherford Model Equations in Presence of Sheared Flows

We consider a single helicity magnetic perturbation moving across an equilibrium magnetic field in the vicinity of a resonant surface where the helicity of the perturbation matches the pitch of the equilibrium field (i.e. \( q(r_s) = m/n, q \) is the safety factor), the magnetic field can be expressed in terms of an effective flux function \( \psi \):

\[
\psi = -\frac{B_0 x^2}{L_s} + \tilde{\psi} \cos \xi \tag{6.1}
\]

Here \( B_0 \) is the average equilibrium toroidal magnetic field, \( x = r - r_s \) is the distance from the rational surface, \( L_s = qR/s \) is the shear length and \( s = r_s q' / q \). Note that for \( m \geq 2 \), when the constant \( \tilde{\psi} \) approximation holds, the magnetic island half-width is given by,

\[
W = \left( \frac{4L_s \tilde{\psi}}{B_0} \right)^{1/2} \tag{6.2}
\]

The nonlinear evolution equation of the magnetic island is derived from the matching conditions.
CHAPTER 6: A GENERALIZED RUTHERFORD MODEL CALCULATION

obtained by integrating Ampere's equation across the nonlinear region.

\[ \int_{-\pi}^{\pi} d\xi \cos \xi \int_{-\infty}^{\infty} dx J_{||} = \frac{c}{4\pi} \Delta_{c} \pi \psi \]  
\[ \int_{-\pi}^{\pi} d\xi \sin \xi \int_{-\infty}^{\infty} dx J_{\perp} = \frac{c}{4\pi} \Delta_{c} \pi \psi \]

(6.3)  
(6.4)

where the matching parameters \( \Delta_{c} \) are determined from the outer (linear) region and are assumed to be given by ideal MHD equations. The longitudinal current needs to be obtained from the parallel Ohm's law,

\[ J_{||} = \sigma_{neo} \left( -\nabla_{\phi} \Phi + \frac{1}{c} \frac{\partial}{\partial t} \psi(t) \right) - \frac{\mu_{e} c}{\nu_{ei} B_{0}} \frac{dp}{dx} \]

(6.5)

where \( \mu_{e} \) is the viscosity coefficient, \( \nu_{ei} \) is the electron-ion collision frequency, \( B_{0} \) is the poloidal magnetic field, \( p \) is the plasma pressure and \( \sigma_{neo} \) is the neoclassical conductivity. The last term on the RHS is the perturbed bootstrap current which is responsible for driving the neoclassical tearing modes. For the low frequency tearing mode, quasi-neutrality condition holds so that we have,

\[ \nabla_{\perp} J_{\perp} = 0 \]

(6.6)

where the perpendicular component of the total current is proportional to the plasma inertia through the ion polarization drift. In the standard Rutherford model inertial terms are neglected and \( J_{\perp} \) is the only current contribution that is retained. In that case eqn.(6.6) leads to \( B \cdot \nabla J \approx 0 \) which requires that the longitudinal current is a surface function of the magnetic flux. This assumption introduces a considerable simplification in the calculation of \( J_{||} \) from the Ohm's law and there is no contribution from the \( \Phi \) term. When inertial terms are retained we need to solve eqn.(6.6) which with the substitution for \( J_{\perp} \) takes the form,

\[ \nabla_{\parallel} J_{\parallel} = -\frac{c^{2}}{4\pi \psi^{2}} \frac{d\psi}{dt} \nabla_{\perp}^{2} \left( \phi + \frac{p_{i}}{en} \right) = 0 \]

(6.7)

where the operator \( \frac{d\psi}{dt} \) can be written as,

\[ \frac{d\psi}{dt} = \frac{\partial}{\partial t} + \left( \nu_{K} + \nu_{00} \right) \cdot \nabla = -\psi_{s} \left( \frac{\omega - \omega_{pi}}{B_{0}} \right) + \frac{k_{0} \psi_{s} \phi_{0}}{B_{0}} \frac{\partial}{\partial \xi} \]

(6.8)
Here \( \{\alpha, \beta\} \) denotes a Poisson bracket and \( \omega_{\alpha \beta} \) is the ion diamagnetic frequency, \( \omega_{\alpha \beta} = k_B T_\beta p_\alpha / (e B p_\beta) \) [17]. \( v_E \) is the perpendicular drift component and \( v_{B0} \) is the parallel component of the equilibrium sheared flow.

To solve (6.7) we need to get an expression for \( \Phi \), the electrostatic potential. Note that to the lowest order in Ohm's law, \( E_\parallel = \nabla_\parallel \Phi + \frac{1}{c} \frac{\partial}{\partial t} \psi(t) \approx 0 \), which implies a near cancellation of the electrostatic field by the inductive electric field. By transforming the coordinates from \((x, y) \rightarrow (\psi, \xi)\) we can write \( \nabla_\parallel \) as,

\[
\nabla_\parallel = \frac{k_\psi}{B_0} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial \xi} \right)
\]

where \( k_\psi = m/r_g \) is the poloidal wave vector and derivative with respect to \( \xi \) is to be evaluated at a constant \( \psi \).

Using the above expression for \( \nabla_\parallel \) and integrating \( E_\parallel \approx 0 \), in \( \xi \) we get.

\[
\Phi = \frac{B_0 \omega_\parallel}{c k_\psi} + f(\psi)
\]

where \( f(\psi) \) the integration constant is an arbitrary function of \( \psi \), and needs to be determined from the boundary conditions. In carrying out the integration one also uses the identity,

\[
\frac{\dot{\psi} \sin \xi}{\dot{\psi}_x} = \left( \frac{\partial x}{\partial \xi} \right)_c
\]

The first term on the right side of eqn.(6.10) is related to the electric field due to the moving magnetic field. In a reference frame where the island is stationary, the electrostatic potential becomes a function of \( \psi \). So, the plasma flow due to the \( E \times B \) drift arising from this electric field will always be tangent to the magnetic field and never pierce the magnetic surface. If the flow has a shear we can account for it by writing \( \Phi \) as,

\[
\Phi = \Phi_0' x^2 + \frac{\Phi_0''}{2} x^2 + \dot{\phi}
\]
CHAPTER 6: A GENERALIZED RUTHERFORD MODEL CALCULATION

where \( \omega_E = k_0 \Phi'_0 / B_0 \) is the drift frequency due to the radial equilibrium electric field created by the magnetic perturbation moving across \( B_0 \) and \( \tilde{\Phi} \) is the perturbed \( \Phi \). Equating both the expressions for \( \Phi \) (i.e. using expressions (6.10) and (6.12)) we can write,

\[
\tilde{\Phi} = \frac{B_0}{ck_0} (\omega - \omega_E) x - \frac{B_0 \omega_E}{ck_0} x^2 + f(\psi)
\]

(6.13)

An appropriate choice of \( f(\psi) \) such that \( \tilde{\Phi} \) vanishes for large \( x \) (i.e. far away from the island) is,

\[
f(\psi) = -\frac{B_0}{ck_0} (\omega - \omega_E) \lambda(\psi) - \frac{B_0 \omega_E}{ck_0} \frac{x^2}{2} \lambda^2(\psi)
\]

(6.14)

Substituting for \( f(\psi) \) we get,

\[
\tilde{\Phi} = \frac{B_0}{ck_0} (\omega - \omega_E)(x - \lambda) - \frac{B_0 \omega_E}{ck_0} \frac{x^2}{2} (x - \lambda^2)
\]

(6.15)

Here the function \( \lambda(\psi) \) is chosen to be zero inside the magnetic separatrix and \( \lambda(\psi) \rightarrow x \) in the region \( x \gg W \) [46, 47]. We have taken the following form of \( \lambda(\psi) \) in the outer region,

\[
\lambda(\psi) = \frac{W}{\sqrt{2}} \left[ \left( \frac{\psi}{\psi'} \right)^{1/2} - 1 \right]
\]

(6.16)

where \( W \) is the magnetic island half width as defined in (6.2).

Following standard procedure we now integrate eqn. (6.7) using eqns. (6.12) and (6.15) to obtain the parallel component of current as,

\[
J_\parallel = A(\psi)(\cos \xi - < \cos \xi >) + \frac{\sigma_t}{c} \frac{\partial \tilde{\Phi}}{\partial t} < \cos \xi > - \frac{\mu_e}{\nu_e B_0} \left( \frac{\partial \psi}{\partial \psi} \right) \frac{\partial \psi}{\partial \psi}
\]

(6.17)

where,

\[
A(\psi) = \frac{cB_0^4 W^2}{8\pi k_0^2} \frac{\omega_B^2}{2} \frac{\partial^2 \lambda}{\partial \psi^2} - \frac{\omega - \omega_E - \omega_p}{\omega_E} \frac{\partial \lambda}{\partial \psi} + \frac{k_0 \psi_0}{B_0} \left( \frac{\omega_B^2}{2} \frac{\partial^2 \lambda}{\partial \psi^2} - \frac{\omega - \omega_E}{\partial \psi^2} \right)
\]

(6.18)
and the flux surface average operator \( \langle \cdots \rangle \) is defined as,

\[
\langle \cdots \rangle = \oint \frac{1}{\partial \xi} d\xi
\]

We now use eqn. (6.17) in the matching condition (6.3), to arrive at the following island evolution equation,

\[
G_1 \frac{\partial W}{\partial t} = D_R^{mco} \left[ \frac{\Delta' T^*}{4} + G_2 \sqrt{\frac{\beta p L_p}{W}} \frac{W^2}{W^2 + W_l^2} G_3 \sqrt{W^2 + 0.65W_l^2} + \frac{L_1^2}{L_0^2} \right] (6.20)
\]

Similarly from matching condition (6.4) we get,

\[
G_W \frac{\partial}{\partial t} \left[ W(\omega - \omega_E) + \frac{\omega_E}{2} W^2 \right] = -6G_1 \frac{\mu_e}{W} (\omega - \omega_E) + \frac{1}{2} \frac{\mu_e}{W^2} \frac{\partial}{\partial \xi} \left( \frac{V}{2} \right)^2 W^2 \eta^2 + \xi^2
\]

where, \( D_R^{mco} = \frac{c^2}{4\pi\sigma_{neo}} \) is the magnetic diffusion coefficient calculated using the neo resistivity, \( \beta_p = 8\pi p_e/B_0^2 \), \( L_p = -(d\eta/d\xi)^{-1} \), \( L_1 = \frac{(d\eta/d\xi)^{-1}}{L_p} \), \( L_2 = \frac{L_1}{L_0} \), and \(-2(q^2 - 1) \eta^4/(W^4)\) is the resistive interchange parameter, \( s = \eta^2/\eta^2 \) is the neo resistivity, and \( \bar{\eta} \) is the average parallel flow velocity. The coefficients \( G_1 \) to \( G_6 \) are defined in the Appendix 6.5 and their numerical values are: \( G_1 = 0.41, G_2 = 0.58, G_3 = 6.35, G_4 = 2.3, G_5 = 0.74, G_6 = 0.77 \). \( G_W = 0.82 \) and \( G_V = 2.31 \). For evaluating the neoclassical contribution we have adopted the standard procedure outlined in [47] where \( \mu_e \approx \sqrt{e} \eta \) for the long mean-free-path regime has been used and \( \omega_* = \omega_{*p} + k_0 \omega_T [47] \), with \( \omega_{*T} = k_0 C_{T1}/e B_0 \). The factor \( W^2/(W^2 + W_l^2) \) in the neoclassical term is the usual effect associated with finite radial thermal diffusion and sets a critical island width \( W_x = \sqrt{\frac{4\pi L_0}{m} \frac{\xi_1}{\xi}} \frac{1}{4} \), below which radial transport becomes significant and the pressure is no longer flattened across the island. In the above expression for the island width evolution equation (6.20), the term proportional to \( \omega_E^2 \) arise purely from the flow shear contributions and the term proportional
to \( \bar{v}_B \omega_E \) is due to the combination of parallel flow and the flow shear.

The equation (6.21) is the evolution equation for the mode frequency obtained from the second matching condition (6.4) using the higher order \( J_4 \) equation [47]. The second term of the l.h.s of the equation (6.21) is the flow shear term.

### 6.3 Results

As shown in the equation (6.20), the flow shear will introduce few new terms in the classical neoclassical equations. The term involving \( G_3 \) is usual NTM driving term arising from the perturbed poloidal parallel current and that of involving \( G_4 \) is the Glasser-Green-Johnson (GGJ) term representing the destabilizing term due to the toroidal curvature [116]. Both the above mentioned terms are being modified due to the ratio of perpendicular to parallel heat transport i.e. through \( W_4 \). The next term involving \( G_5 \) is the destabilizing term which is destabilizing in nature except for the window \( 0 < \omega_E / \omega_p < \omega_p / \omega_E < 1 \). The \( \omega_p / \omega_E \) term represents the influence of the rotation frequency on the growth of the tearing modes.

The term involving \( G_5 \) shows the effect of flow shear on tearing modes. As the term \( G_5 \) depends to the \( \omega_E^2 \) so this doesn't depend on the sign of the shear but always destabilizes the tearing mode. The one i.e. the term with \( G_6 \) is arising due to parallel shear flow. This term is proportional to the parallel shear \( \omega_E^2 \) as well as the average parallel flow velocity \( \bar{v}_p \). So the nature of the effect depends on the type of flow shear. In a typical tokamak case, it is destabilizing because \( \omega_E^2 \) is negative.

So the last three terms of the equation (6.20) basically show the effect of the equilibrium shear flow in the neoclassical tearing modes. The nature of the last two terms are slightly different from the \( G_1 \) term.

The \( G_4 \) term goes as \( \frac{1}{\eta_T} \). So for NTMs which begin with a finite threshold, the effect of this term fades sooner because it decreases more rapidly than the driving NTM term. The numerator of the term also remains small because the island rotation frequency \( \omega \) is close to the flow frequency \( \omega_E \) i.e. \( (\omega - \omega_E) \).
remains small. But the shear terms go as $\frac{1}{W}$, which are similar to the way the NTM driving term goes. So their effect remains similar throughout the evolution of NTM and can influence the full NTM dynamics from threshold to saturation. Secondly, the $G_4$ term mainly depends on the flow frequency, however the new terms also depend on the flow gradient. It should be noted that the parallel flow term is finite when both $\delta i_{10}$ and $\omega_f$ are finite, e.g., in case of sheared toroidal flow.

### 6.4 Summary and Discussion

In this chapter, we have investigated the effect of flow shear on neoclassical tearing modes. We take the generalized Rutherford equations i.e., the usual quasilinear Rutherford model along with the next to the next term because it is important in presence of the equilibrium flow. We consider a more general type of flow where both parallel component and perpendicular drift component are present. We have shown that the flow frequency, the flow gradient can also influence the growth of the neoclassical tearing modes. We expect two terms related to the flow shear. These terms have similar dependence of $W$ as in NTM driven term.

The $G_4$ term is always destabilizing in nature irrespective of the sign of the flow gradient. However, the parallel flow term depends on the sign of the flow gradient. In case of a tokamak with a parallel flow, in general, this term also gives destabilizing effect. With positive flow shear the term gives stabilizing effect. In the absence of shear a constant parallel flow has no effect on the modes—essentially it just scales the parallel shift of the mode frequency. The effect of $G_5$ term arises due to a combination of the flow gradient with the bulk flow. These results qualitatively agree with our numerical findings in the last chapter (chapter 5).

In this calculation we have assumed $\Delta'$ to be constant and given by the ideal outer region dynamics. For the outer region dynamics the effect of flow is generally ignored. However, that may not be true and the equilibrium sheared flow may significantly change the value of $\Delta'$ as has been shown earlier in simple geometries [30]. In the next chapter we explore this issue and determine the influence of the global flow profile on NTM stability through changes effected on the parameter $\Delta'$. 

**CHAPTER 6: A GENERALIZED RUTHERFORD MODEL CALCULATION**
6.5 Appendix

Calculations of the numerical coefficients in the island evaluation equations

The G coefficients in the equation (6.20) have been evaluated numerically. Here \( G_1 \) is given by the integral,

\[
G_1 = \frac{1}{2\sqrt{2}\pi} \int_1^\infty d\alpha \langle \cos \xi \rangle^2 \int_0^\infty \frac{d\alpha}{(\cos \xi + \alpha)^{1/2}}
\]

and \( \alpha = -(\psi' / \dot{\psi}) \). Now by putting \( k = 2k^2 - 1 \) we have put,

\[
G_1 = \frac{1}{\pi} \int_0^1 dk \langle \cos \xi \rangle_{\text{in}}^2 K(k^2) + \frac{1}{\pi} \int_1^\infty dk \langle \cos \xi \rangle_{\text{out}}^2 K(1/k^2)
\]

where,

\[
\langle \cos \xi \rangle_{\text{in}} = 2 \frac{E(k^2)}{K(k^2)} - 1, \quad k < 1
\]

\[
\langle \cos \xi \rangle_{\text{out}} = 1 + 2k^2 \left[ \frac{E(k^2)}{K(k^2)} - 1 \right], \quad k > 1
\]

and E,K are the elliptic integrals of first and second kind respectively. Here, the inner region is correspond to \( k < 1 \) where the magnetic field lines are closed and the outer region is correspond to \( k > 1 \) with an open magnetic field lines.

Similarly other coefficients are given by,

\[
G_4 = -\frac{8\sqrt{2}}{\pi} \int_{-1}^1 d\alpha \left( \langle \cos^2 \xi \rangle - \langle \cos \xi \rangle^2 \right) \frac{\partial^2 g}{\partial \alpha \partial \theta^2} \int_{-\pi}^0 d\xi \left( \frac{1}{\cos \xi + \alpha} \right)^{1/2}
\]

\[
= -\frac{8}{\pi} \int_0^\infty \frac{dk}{k^2} \left( \langle \cos^2 \xi \rangle_{\text{out}} - \langle \cos \xi \rangle_{\text{out}}^2 \right) \frac{\partial g}{\partial k} \left( \frac{\partial g}{\partial k} \right) K(1/k^2)
\]

\[
G_5 = \frac{\sqrt{2}}{\pi} \int_{-1}^1 d\alpha \left( \langle \cos^2 \xi \rangle - \langle \cos \xi \rangle^2 \right) \frac{\partial^2 g}{\partial \alpha \partial \theta^2} \int_{-\pi}^0 d\xi \left( \frac{1}{\cos \xi + \alpha} \right)^{1/2}
\]

\[
= \frac{1}{\pi} \int_0^\infty \frac{dk}{k^2} \left( \langle \cos^2 \xi \rangle_{\text{out}} - \langle \cos \xi \rangle_{\text{out}}^2 \right) \frac{\partial g}{\partial k} \left( \frac{\partial g}{\partial k} \right) K(1/k^2)
\]
\[ G_6 = \frac{\sqrt{3}}{\pi} \int_{-1}^{\infty} d\alpha (\langle \cos^2 \xi \rangle - \langle \cos \xi \rangle^2) \frac{\partial^2 g_2}{\partial \alpha^2} \int_{-\pi}^{\pi} \frac{d\xi}{(\cos \xi + \alpha)^{1/2}} \]

\[ = \frac{1}{\pi} \int_{1}^{\infty} \frac{dk}{k} ((\langle \cos^2 \xi \rangle_{\text{out}} - \langle \cos \xi \rangle^2_{\text{out}}) \frac{\partial}{\partial k} \left( \frac{1}{4k} \frac{\partial g_2}{\partial k} \right) K \left( \frac{1}{k^2} \right) \]

\[ (6.29) \]

Where, \( \lambda(\psi) = \frac{1}{\sqrt{2}} g(\alpha) \) and \( g(\alpha) = (\sqrt{\alpha} - 1) \) for outer region. All the integrals are contributed only in the outer regions because \( g(\alpha) \) is zero in the inner region.

Here,

\[ \langle \cos^2 \xi \rangle_{\text{out}} = -\frac{1}{3} \frac{E(\frac{1}{L})}{K(\frac{1}{L})} k^2(2k^2 - 1) + \frac{8}{3} k^2(k^2 - 1) + 1 \]

\[ (6.30) \]

The calculation of \( G_3 \) is shown in ref. [116] and calculation of \( G_{1W} \) \& \( G_{1V} \) are shown in ref. [47]. For a neoclassical term, we have calculated \( G_2 \) as,

\[ G_2 = -1.46 \sqrt{2\pi} \int_{-1}^{\infty} d\alpha \langle \cos \xi \rangle \sqrt{\frac{\Theta(\alpha - 1)}{E(\frac{1}{L})}} \frac{\partial \Theta(\alpha - 1)}{\partial \xi (\cos \xi + \alpha)^{1/2}} \]

\[ = -1.46 \pi \int_{1}^{\infty} \frac{dk}{E(\frac{1}{L})} \langle \cos \xi \rangle_{\text{out}} \]

\[ (6.31) \]

Here the step function \( \Theta \) is taking care of the fact that the pressure gets flattened inside the island.