Chapter 4

Generalized reduced MHD model and the NEAR code

4.1 Introduction

As remarked in the preceding chapter, in order to obtain a better understanding and detailed information about the effect of sheared equilibrium flows on neoclassical tearing modes it is desirable to carry out a numerical time evolution study of the modes in an appropriate model system. We have indeed carried out such an investigation and the results will be presented and discussed in the next chapter. In this chapter, we provide a description of the model system of equations that we have solved and some highlights of the numerical code used for the study. The resistive MHD equations of a plasma provide a complete framework for describing the dynamical behaviour of neoclassical (or classical) tearing modes. However, from a practical point of view the full set of equations are somewhat unwieldy since they also support many other modes with diverse space and time scales. In particular, solving the MHD equations in their full form is not desirable from the computational point of view. It is more convenient in fact to work with a reduced set of MHD equations [68, 103, 104] that have been obtained by eliminating the fast magnetosonic waves which otherwise severely constrain the computational speed without contributing much to the MHD instabilities related to the tearing mode. Among the various available reduced MHD equations, we have chosen to work with the one developed recently by Kruger et al. [68] which they have named as
'generalized reduced MHD' (GRMHD) equations. Unlike some previous models, the GRMHD equations are not derived on the basis of using the inverse aspect ratio as an expansion parameter. Instead they have adopted $k_y/k_\perp$ as a small expansion parameter (where $k_y$ and $k_\perp$ are the parallel and perpendicular wavelengths respectively). Thus their model equations are valid at any aspect ratio. Further their analytic reduction method treats equilibrium, parallel and perpendicular time scales separately and also permits elimination of fast time scales associated with perpendicular wave motion. The model equations thus evolve scalar potential quantities on a time scale associated with the parallel wave vector (shear-Alfvén wave time scale), which is the time scale of interest for resistive MHD instabilities like tearing modes. Their derivation also respects MHD equilibrium constraints, maintains energy conservation and divergence-free magnetic fields to all orders. In addition the model equations also permit incorporation of sub-Alfvénic equilibrium shear flows and neoclassical closures in a straightforward manner and are therefore well suited for studying the nonlinear evolution of neoclassical tearing modes in the presence of flows. In view of all these advantages we have used the GRMHD equations for our time evolution studies of tearing modes. The numerical code modified and used for this purpose is 'NEAR': a neoclassical version of an earlier code called FAR.

The plan of this chapter is as follows. Section (4.2.1) describes the basic features of the generalized reduced MHD equations and the manner of incorporation of flows is discussed in section (4.2.2). In section (4.2.3) we present the stress tensor terms obtained from a specific neoclassical scheme, and which provide the basic bootstrap current source in Ohm's law. Section (4.3), provides some details about the operational aspects of the numerical stability code 'NEAR' and discusses its present capabilities. Finally in section (4.4) we summarize what we have discussed in the present chapter.
4.2 Model Equations

4.2.1 GRMHD Equations

In this section we discuss in some detail the generalized reduced MHD equations originally derived by Kruger et al. [68]. The basic MHD equations that have been used in the derivation are the Ohm's law, the equation of motion and the pressure equation. Furthermore the anisotropic neoclassical stress tensors and the heat flows have been included in the equations. The basic equations can be written as:

\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{B} + \mathbf{E} (\nabla \cdot \mathbf{V}) - (\mathbf{B} \cdot \nabla) \mathbf{V} = -\nabla \times \eta \mathbf{V} \times \mathbf{B} - \nabla \times \frac{1}{n e} \nabla \cdot \mathbf{\Pi}_e
\]

\[
\rho \left[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = -\nabla (p + \frac{\gamma}{2} \rho^2) + (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \cdot \mathbf{\Pi} \tag{4.1}
\]

\[
\frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla) p + \Gamma p (\nabla \cdot \mathbf{V}) = (\Gamma - 1) \left[ \eta p - \mathbf{\Pi} \cdot \nabla \mathbf{V} + \frac{1}{n e} \mathbf{\Pi}_e : \nabla \mathbf{V} \right] - \mathbf{\Pi}_e : \nabla \mathbf{V} - \mathbf{q}
\]

where \( \mathbf{B}, \mathbf{V}, p, \rho, \mathbf{\Pi}, \mathbf{\Pi}_e \) and \( t \) are magnetic field, plasma velocity, plasma pressure, mass density, total stress tensor, electron stress tensor and time respectively. In the original model the continuity equation involving density evolution is also considered. However we omit it here since in our numerical simulations we have treated the density as being constant.

The reduced equations are obtained from the above MHD equations (4.1) by a systematic expansion procedure using the following ordering scheme:

\[
\frac{\lambda_\perp}{\lambda_\parallel} \sim \epsilon \quad \text{and} \quad \frac{\lambda_\perp}{a} \sim \epsilon \tag{4.2}
\]

where \( \epsilon \ll 1 \), \( \lambda_\perp, \lambda_\parallel \) are the perpendicular and parallel wavelengths, \( a \) is the minor radius and directions are with respect to the equilibrium magnetic field.

By incorporating this ordering, the variables can formally be written as,
\[
\begin{align*}
\vec{B} &= \vec{B}_0(\vec{x}, t) + \epsilon \vec{B}_1 \left( \frac{\vec{x}_\perp}{\epsilon}, \frac{\vec{x}_\parallel}{\epsilon}, \frac{t_\perp}{\epsilon}, \frac{t_\parallel}{\epsilon} \right), \\
\vec{V} &= \epsilon \vec{V}_1 \left( \frac{\vec{x}_\perp}{\epsilon}, \frac{\vec{x}_\parallel}{\epsilon}, \frac{t_\perp}{\epsilon}, \frac{t_\parallel}{\epsilon} \right), \\
p &= p_0(\vec{x}, t) + \epsilon p_1 \left( \frac{\vec{x}_\perp}{\epsilon}, \frac{\vec{x}_\parallel}{\epsilon}, \frac{t_\perp}{\epsilon}, \frac{t_\parallel}{\epsilon} \right), \\
\Pi &= \epsilon \Pi_0(\vec{x}, t) + \epsilon \Pi_1 \left( \frac{\vec{x}_\perp}{\epsilon}, \frac{\vec{x}_\parallel}{\epsilon}, \frac{t_\perp}{\epsilon}, \frac{t_\parallel}{\epsilon} \right), \\
\Pi_\epsilon &= \epsilon^2 \Pi_{0\epsilon}(\vec{x}, t) + \epsilon^2 \Pi_{1\epsilon} \left( \frac{\vec{x}_\perp}{\epsilon}, \frac{\vec{x}_\parallel}{\epsilon}, \frac{t_\perp}{\epsilon}, \frac{t_\parallel}{\epsilon} \right)
\end{align*}
\]

Here the subscript 0 is the equilibrium quantity and the subscript 1 is the perturbed quantity. Unlike the perturbed quantities, the equilibrium quantities are not distinguished in parallel and perpendicular directions.

Similarly the spatial and time scales of perturbed quantities are ordered as,

\[
\nabla f_1 = \left( \frac{1}{\epsilon} \nabla_{\perp} + \nabla_{\parallel} \right) f_1(\vec{x}_\perp, \vec{x}_\parallel)
\]

and

\[
\frac{\partial f_1}{\partial t} = \left( \frac{1}{\epsilon} \frac{\partial}{\partial t_{\perp}} + \frac{\partial}{\partial t_{\parallel}} \right) f_1
\]

The constraint of the equations comes from the requirement to eliminate the fast perpendicular time scale by making \( \frac{\partial f_1}{\partial t_{\perp}} = 0 \) for perturbed quantities. This leads to,

\[
\nabla_{\perp} \cdot \vec{V}_1 = O(\epsilon), \quad p_1 + \vec{B}_0 \cdot \vec{B}_1 = O(\epsilon)
\]

Using the above mentioned ordering and constraint of equations, the MHD equations (4.1) in the limit of \( \beta \sim \delta^{1/2} (\delta \ll 1) \), can be reduced to the following simplified set of evolution equations.
\[
\frac{d\Psi}{dt} - (\mathbf{b}_0 + \mathbf{b}_1) \cdot \nabla \phi_1 - \mathbf{b}_1 \cdot \nabla \phi_0 = \eta \mathbf{j}_\parallel - \frac{1}{ne} \mathbf{b}_0 \cdot \nabla \cdot \mathbf{\Pi}
\]

\[
\frac{dU}{dt} + (\nabla \cdot \nabla)U = (\mathbf{B}_0 \cdot \nabla) \mathbf{j}_0 + (\mathbf{B}_1 \cdot \nabla) \frac{\mathbf{j}_\parallel}{B_0} + \nabla \cdot \frac{\mathbf{B}_0 \times \nabla p_1}{B_0^2}
\]

\[
+ \nabla \cdot \frac{\mathbf{B}_0}{B_0^2} \times \nabla \cdot \mathbf{\Pi}
\]

(4.7)

\[
\frac{dp_1}{dt} + (\nabla \cdot \nabla)p_0 + \nabla \cdot \nabla \cdot \mathbf{V}_1 = \left( \frac{\eta}{ne} \right) \left[ \mathbf{J}_\parallel \cdot \nabla \mathbf{V}^2 - \mathbf{\Pi} \cdot \nabla \mathbf{V} + \mathbf{\Pi} \cdot \nabla \mathbf{J} + \nabla \cdot \mathbf{J} \cdot \mathbf{\Pi} \right]
\]

where,

\[
U = \frac{\nabla \cdot \nabla \phi}{B_0^2}
\]

(4.8)

and

\[
\nabla \cdot \mathbf{q} = -\chi \mathbf{\nabla}^2 p_1 - (\chi \mathbf{\nabla} \cdot \mathbf{\nabla}) [\mathbf{b}_1 \cdot \nabla (\mathbf{b}_0 \cdot \nabla p_0) + \mathbf{b}_0 \cdot \nabla (\mathbf{b}_0 \cdot \nabla p_1 + \mathbf{b}_1 \cdot \nabla p_1)]
\]

\[
+ \mathbf{b}_0 \cdot \nabla \mathbf{\nabla} (\mathbf{b}_0 \cdot \nabla p_1) + \mathbf{b}_1 \cdot \nabla \mathbf{\nabla} (\mathbf{b}_1 \cdot \nabla p_0) + \mathbf{b}_0 \cdot \nabla \mathbf{\nabla} (\mathbf{b}_0 \cdot \nabla p_1)
\]

(4.9)

and

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \nabla \cdot \mathbf{V} ; \mathbf{\Phi}_\parallel = \Phi_0 + \Phi_1 ; \mathbf{p}_\parallel = p_0 + p_1 ; \mathbf{b}_\parallel = \mathbf{b}_0 + \mathbf{b}_1 = \frac{\mathbf{b}_0}{B_0} + \frac{\mathbf{b}_1}{B_0} + \mathbf{\nabla} \cdot \nabla \mathbf{\nabla} \psi
\]

\[
\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1 ; J_\parallel = J_0 + J_1 = \mathbf{b}_0 \cdot \mathbf{\nabla} \times \mathbf{B}_0 + \nabla^2 \psi
\]

\[
\mathbf{V}_1 = \frac{\mathbf{B}_0 \times \nabla \Phi_1}{B_0^2} = \mathbf{\nabla} \cdot \mathbf{\nabla} \mathbf{\nabla} \psi
\]

(4.10)

Eqn.(7.2) is Ohm's law, with \( \Psi \) representing the magnetic flux, \( \phi \) the electrostatic potential, \( \mathbf{J} \) the current density, \( \eta \) the resistivity, and \( n \) the number density. The last term on the right hand side is
the contribution from the bootstrap current and is the driving term for the neoclassical tearing mode.

Eqn. (4.7) is the vorticity equation and eqn (4.8) is the pressure evolution equation where heat flow terms have been retained with \( q \) representing the heat flux and \( \chi_\perp \), \( \chi_\parallel \) representing the heat transport coefficients in the perpendicular and parallel directions respectively. Finally eqn (4.8) is the evolution equation for the parallel velocity component.

In this ordering the energy integral remains conserved and the equations retain the sound waves. The magnetic field is defined such that all its components are divergence free. However as we have treated density \( \rho \) as a constant, energy is not exactly conserved. But it is observed numerically that the deviation is small and acceptable [105].

4.2.2 Equilibrium sheared flow

It has been shown by Kruger et al [68] that it is possible to self-consistently add the sub-Alfvenic equilibrium flow in lower order of the generalized reduced MHD equations, discussed in the previous section (4.2.1).

Here the equilibrium velocity \( \mathbf{V}_0 \) is chosen in such a way that we can write,

\[
\frac{\mathbf{B}_0}{\sqrt{\rho \mu_0}} \cdot \nabla \sim \mathbf{V}_0 \cdot \nabla_\perp
\]  
(4.11)

So,

\[
\frac{V_0}{V_A} \sim \frac{k_\parallel}{k_\perp} \sim \epsilon
\]  
(4.12)

In such a case we can add the equilibrium velocity in the lower order equation and accordingly change the velocity ordering in (4.3) as,
\[ \vec{V} = \epsilon \vec{V}_0 + \epsilon \vec{V}_1 \]  

(4.13)

Then using the equilibrium relation \( \vec{V} \times (\vec{V}_0 \times \vec{B}_0) = 0 \) it has been shown in \([15, 68]\) that we can write \( \vec{V}_0 \) in the form,

\[ \vec{V}_0 = \frac{\vec{B}_0 \times \vec{\nabla} \phi_0}{B_0^2} + \vec{V}_{0\parallel} \vec{j}_0 \]

\[ = \Omega(\phi_0) R^2 \vec{\nabla} \zeta \quad (4.14) \]

Here the toroidal flow frequency \( \Omega(\phi_0) \) can be related with other flow parameters as,

\[ \Omega = \frac{\partial \phi_0}{\partial \psi_0}, \quad V_{0\parallel} = \Omega / B_0 \quad (4.15) \]

If we compare the form of equilibrium flow velocity \( \vec{V}_0 \) in (4.14) with that of the perturbed flow velocity \( \vec{V}_1 \) in (4.10), it is obvious that both forms are similar in nature. This allows us directly to add the equilibrium flow in (4.10) by just replacing \( \phi \) by \( \phi_0 + \phi \) and adding \( \vec{V}_{0\parallel} \vec{j}_0 \) term. This is not possible in conventional reduced MHD where \( \vec{V}_1 = \vec{V} \zeta \times \vec{\nabla} \phi \) which only allows equilibrium poloidal flow.

So the flow velocity \( \vec{V} \) can be written as,

\[ \vec{V} = \Omega R^2 \vec{\nabla} \zeta + \frac{\vec{B}_0 \times \vec{\nabla} \phi_0}{B_0^2} + \vec{V}_0 \vec{j}_0 \quad (4.16) \]

This eqn. (4.16) is used along with the main set of generalized reduced MHD equations (4.7) in presence of equilibrium sheared toroidal flows.

### 4.2.3 Neoclassical closure terms

The neoclassical tearing mode is driven unstable by a perturbation of the bootstrap current which arises from the viscous damping of the poloidal electron flow and which is proportional to the cross-field pres-
sure gradient. To study the evolution of neoclassical tearing modes it is therefore necessary to retain the stress tensor terms in the reduced MHD equations in order to provide the drive term. However the most conventional reduced MHD formulation doesn’t differentiate between the parallel and toroidal directions or perpendicular and poloidal directions. But this differentiation is very important in neoclassical MHD physics [106]. So it is difficult to introduce proper neoclassical physics in conventional reduced MHD formulation [92, 106]. In the generalized reduced MHD formulation discussed in section (4.2.1), the neoclassical physics could be easily introduced because this formulation does differentiate between the above mentioned directions. In section (4.2.1) we have not discussed any particular form for viscous stress tensor. In this section we will discuss about the specific forms of the closure which give the right asymptotic behavior.

For the neoclassical viscous stress tensor closure, we have used the following ansatz [108] in the GRMHD eqns (4.7),

$$\nabla \cdot \Pi_s = \rho_s \mu_s \langle B^2 \rangle \frac{\nabla_s \cdot \nabla \Theta}{(B_0 \cdot \nabla \Theta)^2} \nabla \Theta,$$

(4.17)

where $\mu_s$ is the viscous damping coefficient for species $s$ and it depends on the collisionality of the plasma [106-108].

As for ion viscous stress tensor we can write,

$$\nabla \cdot \Pi_i = \rho_i \mu_i \langle B^2 \rangle \frac{\nabla \Theta}{(B_0 \cdot \nabla \Theta)},$$

(4.18)

where,
\[ \bar{u}_\theta = \frac{\vec{V} \cdot \nabla \Theta}{\hat{B}_0 \cdot \nabla \Theta}, \] (4.19)

It has shown in refs [105] that this form of ion viscous stress tensor gives rise to poloidal flow damping from parallel momentum balance equation. This term along with viscous term in the vorticity equation give rise to neoclassical enhancement of the polarization current [105, 106].

To get a simplified form of the electron stress tensor we use the relation for the electron velocity as,

\[ \vec{V}_{e1} = \vec{V} = \frac{\vec{J}}{n e} = -\frac{1}{n e} \left( \hat{J}_0 \hat{B}_0 + \frac{\hat{B}_0 \times \nabla \psi}{\hat{B}_0^2} \right). \] (4.20)

So the electron stress tensor can be written as,

\[ \vec{\nabla} \cdot \Pi_e = -\frac{\rho_e e_e}{n e} \langle B^2 \rangle \frac{\vec{J}_0}{(\hat{B}_0 \cdot \nabla \Theta)}, \] (4.21)

where,

\[ \vec{J}_0 = \frac{\vec{J} \cdot \nabla \Theta}{\hat{B}_0 \cdot \nabla \Theta}. \] (4.22)

This form of electron viscous stress tensor in parallel Ohm's law yields the neoclassical enhancement of the plasma resistivity and the perturbed bootstrap current [105]. So this is the basic term which is responsible for neoclassical tearing modes.

It was shown in ref. [105] that \( \bar{u}_\theta \) and \( \vec{J}_0 \) can be expressed as,

\[ \bar{u}_\theta = \frac{\vec{V}_e B_0}{\hat{B}_0^2} + \frac{I}{\hat{B}_0^2} \frac{\partial \phi}{\partial \psi_0}, \] (4.23)

\[ \vec{J}_0 = \frac{\nabla^2 \phi B_0}{\hat{B}_0^2} + \frac{I}{\hat{B}_0^2} \frac{\partial \phi}{\partial \psi_0}. \] (4.24)
where \( j = i, e \) and \( \mu_j \) is the viscous damping frequency of each species \( j \).

This closure ansatz is appropriate for the long mean free path (low collisional) limit and convenient for demonstrating the energy conservation [106]. Actually the contribution of viscous heating term in the pressure equation gets canceled by the negative contribution of the stress tensor terms in other equations. In that way these forms of viscous stress tensors maintain energy conservation.

4.3 NEAR code

In this section we will briefly describe the numerical code which we have used for simulation. The generalized reduced MHD equations mentioned in earlier section (4.2.1) have been programmed into an initial value code, called NEAR, which is a neoclassical version of an older code called FAR [106].

The NEAR code is a toroidal initial-value code. It uses \( \{ \rho, \Theta, \zeta \} \) coordinate system where \( \rho \) is a flux coordinate, \( \Theta \) is a poloidal angle and \( \zeta \) is a toroidal angle. The code uses a central finite difference scheme in the \( \rho \) direction and Fourier decomposition in the \( \Theta \) and \( \zeta \) directions. In this representation we can write the operator \( \vec{B}_0 \cdot \vec{\nabla} \sim (m - nq) \) i.e. it can be calculated very accurately. For higher magnetic Reynolds number \( S = \tau_R/\tau_A \), the resistive layer increasingly becomes very localized where \( m - nq \sim 0 \) and the gradients becomes large within this layer. It demands more accurate calculation of \( \vec{B}_0 \cdot \vec{\nabla} \) operator as we increase \( S \). This is one of the critical problems for MHD simulation. However, Fourier representation which is used here allows us to do simulation for relatively higher \( S \) particularly in linear cases.

For a up-down symmetric tokamak equilibria without flow, the perturbed quantities \( \psi_0, P_1, \chi_1 \) are even functions and can be written as,
\[ f_i = \sum_i f_i(p_j)\cos(m_j\theta + n_i\zeta) \quad (4.25) \]

Similarly the odd perturbed functions \( \phi_1, U_1, V_{11} \) can be written as,

\[ f_i = \sum_i f_i(p_j)\sin(m_j\theta + n_i\zeta) \quad (4.26) \]

However in the presence of flow, the symmetry of the equations w.r.to \( \zeta \) is broken and we need both sine and cosine components to represent the perturbed variables. So the size of linear matrix which goes as square of the number of harmonics will increase by 4 times and the time of convolutions (which dominates for nonlinear runs) which also goes as the square of the number of harmonics will increase by 4 times. This implies that addition of flow really can increase the computational time significantly.

Now to describe the time advancement scheme, the MHD equations can be written symbolically as,

\[ \vec{L} \cdot \frac{\partial \vec{x}}{\partial t} = \vec{R} \cdot \vec{x} + \vec{N}(\vec{x}) \quad (4.27) \]

with \( \vec{x} = [\Psi_1, \phi_1, p_1, U_1, \chi_1, V_{11}] \)

where \( \vec{L} \) and \( \vec{R} \) are linear operators and \( \vec{N} \) is the nonlinear term. Among the six equations, the two equations involving \( \phi_1 \) and \( \chi_1 \) are dummy equations. They are used for relating \( U \) to \( \phi_1 \) and \( V_{11} \) to \( p_1 \) and \( \psi_1 \) respectively [105]. Here the linear operators are a series of block tridiagonal matrices. They are tridiagonal due to central-differencing in \( \rho \) and gets blocked due to separation of toroidal harmonics for an axisymmetric equilibria. The time-differencing scheme which has been used here is,

\[ \left( \vec{L} - \frac{\Delta t}{2} \vec{R} \right) \cdot \vec{x}^{n+\Delta t} = \left( \vec{L} + \frac{\Delta t}{2} \vec{R} \right) \cdot \vec{x}^n + \Delta t \vec{N}(\vec{x}) \quad (4.28) \]

To find out values between two time steps an interpolation has been carried out. So the linear problem can be solved implicitly and that allows us to take large time steps without introducing numerical errors.
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This is done using a block tridiagonal solver [110]. But nonlinear terms are introduced explicitly in the time advanced equation (4.28). So as nonlinear terms become larger the time steps get limited by the CFL criteria [111]. In this simulation we have used the conducting wall boundary conditions which is discussed in [109]. The simulation is started with some initial profile of one or more variables. After some few initial transients the profiles take the shapes of the eigenfunctions. The more closely we can provide the initial profiles to the final mode shapes, the less time it takes to decay the initial transients.

To begin with we should provide the equilibrium values of the variables. The equilibrium profiles are generated by the equilibrium code called TOQ [112]. It solves the Grad-Shafranov equation and generates the equilibrium profiles. An interface code originally developed by Kruger [105] is used which maps the equilibrium profiles generated by TOQ to flux coordinates \( \rho \) of the stability code NEAR.

The Grad-Shafranov equation gets modified in the presence of flow. It has been shown in refs. [13, 105] that density \( \rho \) in presence of an equilibrium sheared toroidal flow can be written as,

\[
\rho = \rho_{nf}(\psi_0) \exp \left( \frac{\Gamma}{2} M_s(\psi_0)(\hat{R}^2 - \hat{R}_{\text{axis}}^2) \right)
\]

where \( \hat{R} = \hat{R}/R_0 \) and \( \hat{R}_{\text{axis}} = R_{\text{axis}}/R_0 \) and \( M_s(\psi_0) \) is the mach number related to toroidal flow frequency \( \Omega \) as,

\[
M_s(\psi_0) = \frac{\Omega^2 R_0^2}{\Gamma} = \frac{V_c^2(\psi_0)}{V_s^2(\psi_0)}
\]

where \( V_c \) is the toroidal flow velocity and \( V_s \) is the ion sound velocity. Using the relation \( \rho_0 = \rho_{nf}(\psi_0) \) we have the pressure equation as,

\[
p_0 = p_{nf}(\psi_0) \exp \left( \frac{\Gamma}{2} M_s(\psi_0)(\hat{R}^2 - \hat{R}_{\text{axis}}^2) \right)
\]
These relations show that pressure and density are shifted outwards from the magnetic flux surface as shown in figure 4.1. This outward shift takes place due to the centrifugal force exerted by the toroidal flow [105]. The relation (4.31) shows that pressure in presence of toroidal flow is not a flux function but it can be expressed in terms of two flux functions \( \rho_{nf}(\psi_0) \) and \( M_s(\psi_0) \). Using the pressure relation (4.31) in the equilibrium force balance we get the modified Grad-Shafranov equation as [105].

\[
-\Delta^*\psi_0 = I' + R^2 \left[ p'_{nf} + \Gamma \rho_{nf} M_s M'_s (\hat{R}^2 - \hat{R}^2_{azis}) \right] \exp\left( \frac{1}{2} M^2_s(\psi_0)(\hat{R}^2 - \hat{R}^2_{azis}) \right) \quad (4.32)
\]

This pressure relation (4.31) is also used to calculate \( \sigma = J_{psi}/B_{phi} \) in the right hand side of the vorticity equation involving \( U \) of eqns.(4.7) where the \( \sigma \) can be written as

\[
\sigma = -\frac{I'P_0}{B^2_0} - I' \quad (4.33)
\]
Finally let us discuss the normalizations which are used to de-dimensionalize the variables. The de-dimensionalized equations are given in ref. [105]. The normalizations parameters for equilibrium quantities are given by,

\[ R_* = R_0 \]
\[ B_* = B_0 \]
\[ p_* = p_{mf}(0) \]
\[ X_* = \epsilon R_* \]
\[ Z_* = \epsilon R_* \]
\[ t_* = B_* R_* \]
\[ \psi_* = -\epsilon^2 R_*^2 B_* \]

where \( X = R - R_{\text{axis}} \), \( R_0 \) = geometric center of plasma, \( B_0 \) = vacuum magnetic field at \( H_0 \) and \( \epsilon = \) inverse aspect ratio. Then, the normalizations of the independent variables are given by,

\[ t_* = \tau_\Lambda = \frac{\sqrt{4\pi \rho_m R_*}}{B_*} \]
\[ \rho_* = \epsilon^2 R_* \]  \hspace{1cm} (4.35)

where \( \rho_m \) is the mass density. The normalizations of the dependent variables are given by.
\[ \Psi_* = -\epsilon^2 R, B_* \]
\[ \psi_* = -\epsilon^2 R^2 B_* \]
\[ \chi_* = -\epsilon^2 R^2 B_* \]
\[ \phi_* = -\frac{\epsilon^2 R^2 B_*}{\zeta A} \]
\[ V_{||*} = \frac{\epsilon^2 R^2 B_*}{\zeta A} \]
\[ p_* = p_{ref}(0) \]

Here the -ve signs are used with the potentials because of FAR convention. Now the other parameters such as resistive time \( \tau_R \), magnetic Reynolds number \( S \) and equilibrium \( \beta \) are given by:

\[ \tau_R = \frac{4\pi^2 R^4}{\eta^2} \]
\[ S = \frac{\tau_R}{\tau_A} = \frac{\sqrt{\frac{4\pi}{3}} R^2}{\sqrt{\rho_m \eta^2}} \]
\[ \beta_0 = \frac{p_*}{B^2 / (8\pi)} \]

Then the perturbed quantities in FAR are expressed as:

\[ \tilde{Q} = \tilde{Q}_{m,0} e^{\imath \varpi t} (\imath m \Theta + \imath n \zeta) \]

So it is uses an uncommon convention of '+in'. But to keep conventional definition of \( \eta \) it is neutralized by defining,

\[ \frac{\partial}{\partial \zeta} > - \frac{\partial}{\partial \zeta} \]
4.4 Summary and Discussion

In this chapter we have discussed about the model equations which have been used in our numerical studies. Here we have used the generalized reduced MHD equations which are based on smallness of $k_i/k_\perp$ and valid for any aspect ratios. We describe how the sub-Alfvenic equilibrium toroidal sheared flow are added easily in these GRMHD equations. We also describe the neoclassical closures which are used for viscous stress tensor terms and explain how it can give the right physics in the framework of the generalized reduced MHD equations. Finally we discuss about the numerical codes which we have used for solving these model equations. Here we have discussed about the stability code NEAR and very briefly about the equilibrium code TOQ. It has been shown that how we can add the effect of centrifugal force of the equilibrium toroidal flow through the exponential factor of the pressure expression.

In the next chapter we will discuss how we have benchmarked the codes by comparing the results with the known facts. We will also present and discuss the new results we have obtained regarding the time evolution of classical and neoclassical tearing modes in the presence of equilibrium sheared toroidal flows.