Chapter 3

Nonlinear stationary states of NTMs in the presence of sheared flows

3.1 Introduction

Neoclassical tearing modes have attracted a great deal of attention in recent years due to the constraint they impose on the attainment of the ideal MHD beta limit in high temperature long pulse tokamak discharges [6, 82, 91]. They arise in the collision-less regime where the growth rate of the classical tearing mode is significantly enhanced by the perturbation in the bootstrap current due to the local flattening of the pressure profile inside the magnetic island. The basic dynamical behavior of this mode can be understood from an extension of the Rutherford theory [89] to include neoclassical effects. Several such analytic studies and some numerical modeling have considerably advanced our understanding of the evolution of these modes [68, 92, 93]. However a number of issues related to the origin of excitation of the mode, its excitation threshold, its nonlinear behavior and its interaction with error fields and equilibrium shear flows have not been satisfactorily resolved and need to be better understood [11]. The influence of shear flows is a particularly important issue as it can occur in a tokamak discharge under a variety of conditions, e.g. due to unbalanced parallel injection of neutral beams leading to large scale toroidal rotation in the plasma or poloidal flows arising from turbulence and/or radial electric fields. They are known to have a strong influence on the onset and nonlinear evolution of resistive tearing modes. A
number of past studies have examined the effect of flows on tearing modes, particularly in the linear regime and for simplified geometries [30]. There have also been a few nonlinear studies [38, 39] but the problem is quite complex, particularly in realistic toroidal geometries and is an important area of present and future study for major numerical initiatives such as NIMROD [94].

The major theme of the present thesis is to address this issue of the interaction of sheared flows with neoclassical tearing modes through both analytical and numerical means. Before we embark on a detailed numerical investigation of the problem it is useful to develop a general analytical picture of the possible final states of the system. The center manifold reduction method [39] is a useful and powerful tool to carry out such an analysis and in this chapter we employ it to investigate the effect of sheared flows on the final nonlinear states of the neoclassical tearing modes. The neoclassical tearing modes are most easily modeled by a suitably generalized set of resistive MHD equations. Using the center manifold technique we reduce such a set of resistive MHD equations to a set of coupled nonlinear amplitude equations and examine the time asymptotic states of these equations. These states represent the possible final nonlinear states of the neoclassical tearing mode and we delineate the domain of existence of these states and their stability properties through a detailed bifurcation analysis.

The plan of this chapter is as follows. Section (3.2) describes the basic set nonlinear MHD equations which are used to model the dynamics of the NTM. In the next section (3.3), the center manifold reduction method is used to convert the MHD equations to amplitude equations for two modes which are close to marginal stability. The details of the calculation of the coefficients of the amplitude equations are shown in Appendix 3.6. Section (3.4) is devoted to a bifurcation analysis of the amplitude equations in order to determine the time asymptotic states of the modes. The results are displayed in the form of bifurcation diagrams. The last section (3.5) provides a brief summary and some discussion of the main results of this chapter.
3.2 Model Equations

We start with an extended model of the resistive MHD equations which includes a bootstrap current contribution in the Ohm’s law and which is evolved self-consistently through a pressure equation. This set of equations is related to the generalized reduced magnetohydrodynamic equations of Kruger et al [68] and in the absence of flows has been used by Yu and Gunter [92] to numerically study the nonlinear evolution of the neoclassical tearing mode. The basic equations consist of the equation of motion, Ohm’s law and the pressure evolution equation and are given as.

\[
\begin{align*}
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) &= -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \nabla^2 \mathbf{v} \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} = -\nabla \times \left( \mathbf{j} - \mathbf{v} \times \mathbf{B} \right) \\
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) p &= \mathbf{v} \cdot \mathbf{\dot{v}} = \nabla \cdot \left( \chi_n \nabla p + \chi_\perp \nabla_\perp p \right) 
\end{align*}
\]

(3.1)

The magnetic field \( \mathbf{B} \) can be expressed in terms of a flux function \( \psi \) as, \( \mathbf{B} = \mathbf{\hat{z}} \times \nabla \psi \). Similarly, the velocity \( \mathbf{v} \) can be expressed in terms of a scalar potential \( \phi \) as, \( \mathbf{v} = \mathbf{\hat{z}} \times \nabla \phi \). If we write \( \mathbf{v}, \phi, p \) as the sum of equilibrium (with subscript 0) and perturbed parts (with subscript 1) then the equations (3.1) can be written as.

\[
\begin{align*}
\frac{\partial}{\partial t} \nabla^2 \phi_1 + (\mathbf{\hat{r}}_0 \cdot \nabla) \nabla^2 \phi_0 + (\mathbf{\hat{r}}_1 \cdot \nabla) \nabla^2 (\phi_0 + \phi_1) &= \mathbf{\hat{z}} \cdot (\nabla \phi_0 \times \nabla j_{z1}) + \mathbf{\hat{z}} \cdot \nabla \psi_1 \times \nabla (j_{z0} + j_{z1}) + \rho \nabla^4 \phi_1 \\
\frac{\partial \psi_1}{\partial t} + (\mathbf{\hat{r}}_0 \cdot \nabla) \psi_0 + (\mathbf{\hat{r}}_1 \cdot \nabla) (\psi_0 + \psi_1) &= -\gamma (j_{z1} - j_{z0}) \\
\frac{\partial p_1}{\partial t} + (\mathbf{\hat{r}}_0 \cdot \nabla) p_0 + (\mathbf{\hat{r}}_1 \cdot \nabla) (p_0 + p_1) &= \nabla \cdot \chi_n \nabla p_1 + \nabla \cdot \chi_\perp \nabla_\perp p_1
\end{align*}
\]

(3.2)

(3.3)

(3.4)

where \( j_{z1} = -\nabla^2 \psi_1 \), \( j_{z0} = -g \frac{\partial^2 \psi_0}{\partial \mathbf{r}^2} \) is the perturbed bootstrap current and other notations are standard. \( g \) is a smooth function of the minor radius which vanishes at the center and the plasma edge.

We assume a simple slab geometry where \( x \) corresponds to the radial direction and all perturbations are assumed to be periodic in the \( y \) and \( z \) directions (corresponding to the poloidal and toroidal directions).
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All equilibrium quantities are assumed to be a function only of \( z \). Equations (3.2-3.4) can be written in a more compact form as,

\[
\frac{\partial}{\partial t} \vec{R} = L \vec{R} + \mathcal{N}(\psi_1, \psi_1, p_1),
\]

(3.5)

where \( \vec{R} \) is the column vector \( (\psi_1, \psi_1, p_1)^T \), \( \vec{R} = (\nabla_z^2 \psi_1, \psi_1, p_1)^T \), \( L \) is a linear operator matrix,

\[
\begin{pmatrix}
\mu \nabla_z^4 - \phi_0''(x) \frac{\partial}{\partial y} \nabla_z^2 + \phi_0''(x) \frac{\partial}{\partial y} \\
\phi_0'(x) \frac{\partial}{\partial y} \\
p_0'(x) \frac{\partial}{\partial y}
\end{pmatrix}
\begin{pmatrix}
\phi_0''(x) \frac{\partial}{\partial y} \\
\phi_0'(x) \frac{\partial}{\partial y} \\
p_0'(x) \frac{\partial}{\partial y}
\end{pmatrix}
\]

and \( \mathcal{N} \) the nonlinear vector, \(( -\{\phi_1, \nabla_z^2 \psi_1\} - \{\psi_1, \nabla_z^2 \psi_1\} )^T \).

In the above, the superscript \( T \) stands for the transposed quantity, the primes denote differentiation with respect to \( z \) (\( z = 0 \) corresponds to the mode rational surface) and \( \{ a, b \} \) represents a Poisson bracket.

### 3.3 Center Manifold Reduction

The method of center manifold reduction is commonly used for nonlinear dynamics studies to find local time-asymptotic states of a dynamical system \([95, 96]\). We will describe this method very briefly here and more details can be found in Ref. \([39, 95, 96]\). Figure 3.1 describes the center manifold method graphically for a general system of dynamical equations given by \([39]\),

\[
\dot{X} = AX + N(X, Y),
\]

(3.6)

\[
\dot{Y} = BY + M(X, Y)
\]

(3.7)

where \( A, B \) are linear operators that represent the stable and marginally stable states and \( M, N \) are nonlinear operators such that \( M(0, 0) = N(0, 0) = 0 \). Thus \((X, Y) = (0, 0)\) is an equilibrium state. For the equations linearized around the equilibrium point there exists an invariant space given by \((X, Y) = (X, 0)\). When the nonlinear terms are added, the center manifold theorem states that there still exists an invariant subspace \((X, Y) = [X, h(x)]\) which is tangent to the center eigenspace at the equilibrium point.
Figure 3.1: Graphical illustration of center manifold

\[(X,Y) = (0,0), \text{ where } h(0) = 0, \frac{\partial h}{\partial X}(0) = 0. \text{ This invariant subspace is called the center manifold and it has}
\text{the dimension of } X. \text{ The center manifold is locally attractive and for finding the local time asymptotic}
\text{states the idea is to reduce the system to a lower dimension, namely that of the center manifold. The}
\text{center manifold can be expressed as,}
\[(X,Y) = [X, h(X)]
\text{ (3.8)}
\text{The dynamics on the center manifold can thus be expressed as,}
\[X = AX + N[X, h(X)]
\text{ (3.5)}
\text{In most cases } h(x) \text{ can be approximated by using a Taylor series expansion near } (X,Y) = (0,0).

Now we consider an equilibrium situation where the parameters of the magnetic field and flow are
\text{such that an } m = 2, n = 1 \text{ and its first harmonic are simultaneously marginally stable at the } q = 2
\text{surface (m and n are the poloidal and toroidal mode numbers). Such a situation is possible for a variety}
\text{of model equilibria as has been discussed in the literature [39, 97-99].}

We will examine the nonlinear interaction of these modes by first reducing Eq.(3.5), using the center
manifold technique, to a set of amplitude equations of the form,
\[\dot{Z}_r = f_r(Z_1, Z_1, Z_2, Z_2, Z_0)
\text{ (3.10)}\]
where $Z_{1,2}$ are the complex amplitudes of the two perturbed modes, bar denotes a complex conjugate quantity, overdot denotes time derivative and $Z_0$ denotes the distance of the system parameters from their critical values at marginal equilibrium. As the modes are not exactly on the imaginary axis so to apply the central manifold method we need to shift the parameters by $Z_0$. If the modes are close to marginal state, $Z_0$ should be very small. Note that $\dot{Z}_0 = 0$. The physical quantities are expanded as,

$$R(x, y, z, t) = \sum_{r=1,2} Z_r(t) R_r(x) e^{i \nu r \kappa} + c.c. + \sum_{r,x=1,2, r \leq x} Z_r(t) Z_x(t) R_{r,x}(x) + c.c. + \sum_{r,x,y=1,2, r \leq x \leq y} Z_r(t) Z_x(t) Z_y(t) R_{r,x,y}(x) + c.c. + \ldots$$  \hspace{1cm} (3.11)

where the $R_r$ are the linear eigenmodes of the eq.(3.5) and the functions $R_{r,s}, R_{r,s,q}$ and further higher order ones are chosen orthogonal to $R_r$, $\beta \kappa = (k_y y + k_z z)$, where $\kappa$ is the helical coordinate and $J$ is the helical mode number corresponding to an $m = 2, n = 1$ mode. Close to the marginal state the functions $f_r$ can be Taylor expanded in a power series of the amplitudes,

$$f_r = \sum_{s=0,1,2} A_r^s Z_s + c.c. + \sum_{s,p=0,1,2, r \leq p} A_r^p Z_s Z_p + c.c. + \ldots$$  \hspace{1cm} (3.12)

Substituting (3.11) and (3.12) in Eq.(3.5) and matching terms order by order in the amplitudes $Z$, up to say third order terms, one can obtain expressions for the various coefficients $A_r^s, A_r^p$ etc. In general, there are a large number of coefficients even with a truncation to third order and their evaluation is a laborious task. However the constraint imposed by the symmetry of the system can make many of these coefficients vanish. Our model equations are invariant to translation in $y$ (actually to the helical coordinate in a toroidal system) so that as discussed in [39, 100] Eq (3.10) can be reduced to the following generic form,

$$\dot{Z}_1 = (\lambda_1 + i \omega_1) Z_1 + a_1 Z_1 Z_2 + b_1 Z_1 |Z_1|^2 + c_1 Z_1 |Z_2|^2$$  \hspace{1cm} (3.13)

$$\dot{Z}_2 = (\lambda_2 + i \omega_2) Z_2 + a_2 Z_2^2 + b_2 Z_2 |Z_1|^2 + c_2 Z_2 |Z_2|^2$$  \hspace{1cm} (3.14)

The method of deriving the expressions for the coefficients is quite standard (see [39] for instance) and the details of calculations are given in the Appendix 3.6. These coefficients are complex and their
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where $Z_{1,2}$ are the complex amplitudes of the two perturbed modes, bar denotes a complex conjugate, quantity, overdot denotes time derivative and $Z_b$ denotes the distance of the system parameters from their critical values at marginal equilibrium. As the modes are not exactly on the imaginary axis so to apply the central manifold method we need to shift the parameters by $Z_b$. If the modes are close to marginal state, $Z_b$ should be very small. Note that $Z_0 = 0$. The physical quantities are expanded as,

$$ R(x,y,z,t) = \sum_{r=1,2} Z_r(t) R_r(x) e^{i\omega t} + c.c. + \sum_{r,s=1,2, r \neq s} Z_r(t) Z_s(t) R_{rs}(x) + c.c. + \sum_{r,s=1,2, r \neq s, \xi \leq \eta} Z_r(t) Z_s(t) Z_{\xi\eta}(x) + c.c. + ... \quad (3.11) $$

where the $R_r$ are the linear eigenmodes of the eq (3.5) and the functions $R_{rs}$. $R_{\xi\eta}$ and further higher order ones are chosen orthogonal to $R_r$. $\zeta = (k_y y + k_z z)$, where $\zeta$ is the helical coordinate and $J$ is the helical mode number corresponding to an $m = 2, n = 1$ mode. Close to the marginal state the functions $f_r$ can be Taylor expanded in a power series of the amplitudes,

$$ f_r = \sum_{p=0,1,2} A_r^p Z_r + c.c. + \sum_{r,s=0,1,2, r \neq s, p} A_r^p Z_s + c.c. + ... \quad (3.12) $$

Substituting (3.11) and (3.12) in Eq.(3.5) and matching terms order by order in the amplitudes $Z$, up to say third order terms, one can obtain expressions for the various coefficients $A_r^p$, $A_r^p$ etc. In general, there are a large number of coefficients even with a truncation to third order and their evaluation is a laborious task. However the constraint imposed by the symmetry of the system can make many of these coefficients vanish. Our model equations are invariant to translation in $y$ (actually in the helical coordinate in a toroidal system) so that as discussed in [39, 100] Eq.(3.10) can be reduced to the following generic form.

$$ \dot{Z}_1 = (\lambda_1 + i\omega_1) Z_1 + a_1 \dot{Z}_1 Z_2 + b_1 Z_1 |Z_1|^2 + c_1 Z_1 Z_2 |Z_1|^2 \quad (3.13) $$

$$ \dot{Z}_2 = (\lambda_2 + i\omega_2) Z_2 + a_2 \dot{Z}_2 Z_1 + b_2 Z_2 |Z_2|^2 + c_2 Z_1 Z_2 |Z_2|^2 \quad (3.14) $$

The method of deriving the expressions for the coefficients is quite standard (see [39] for instance) and the details of calculations are given in the Appendix 3.6. These coefficients are complex and their
imaginary contributions arise solely due to the presence of flow. The mathematical origin of this can be traced to a symmetry breaking in the system of equations—this case the breaking of reflection symmetry by the flow terms. The real parts of \( \lambda_j \) provide a measure of the distance of the system parameters from the marginal state, while the imaginary contributions arise from the Doppler shift contribution to the natural mode frequencies due to the flow. The frequencies are further modified by amplitude dependent contributions from the terms proportional to \( a_2, b_1 \) and \( c_2 \) while the terms proportional to \( a_1, b_2 \) and \( c_1 \) provide cross coupling between the modes. Eqs. (3.13-3.14) are still difficult to solve analytically. We have therefore examined their equilibrium states and studied the stability of these states by a detailed numerical local bifurcation analysis. Our results are presented and discussed in the next section.

3.4 Bifurcation Analysis

Here we have applied the standard bifurcation analysis method to find out the possible time asymptotic states when parameters are close to their marginal values. We are mainly interested in local Hopf bifurcations [96] in which case the stability changes because eigenvalues are pure imaginary at criticality. If the stability of the bifurcated solution changes this leads to a secondary Hopf bifurcation.

Now setting \( Z_i = r_i e^{i\theta} \), eqs. (3.13-3.14) can be reduced to three equations for the variables \( r_1, r_2, \phi \), the amplitudes of the modes and \( \phi = 2\theta_1 - \theta_2 \), the relative phase between them. These equations admit three different equilibrium states, namely, the origin \( r_j = 0 \) (often called the “death” state), \( r_1 = 0, r_2 \neq 0 \) (a single island state or “semi-death” state) and \( r_1, r_2 \neq 0 \) (a mixed mode state). We have studied the stability of these states in the functional space of the various coefficients of Eqs. (3.13-3.14) and our results are displayed in the form of phase diagrams in selected parameter regions. The linear stability of the origin is determined by the eigenvalues \( \lambda_j \).

In Fig. 3.2(a) we have shown a phase diagram in the space of Real(\( \lambda \)) and Imag(\( \lambda \)) keeping other co-
Figure 3.2: Bifurcation diagrams
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efficients constant. Physically this corresponds to the space of the parametric distance from the marginal state and the amount of flow induced Doppler frequency shift. The origin loses its stability by a Hopf bifurcation to yield a single island state (an \( m = 4, n = 2 \) island) which for higher values of \( \text{Real}(\lambda_1) \) goes over to a mixed state. Note that this mixed state is characterized by two regions - the \( 1 : 2 \) frequency locked state where \( \phi = 0 \) and an 'incoherent' region where the individual modes continue to oscillate independently leading to a modulated or oscillating island state. The oscillating island states are purely a flow induced state. This is also seen very clearly in Fig3 2(b) where we have obtained a phase diagram in terms of the imaginary coefficient of one of the nonlinear cross coupling terms, namely \( \text{Imag}(c_i) \). The incoherent region vanishes for values of \( \text{Imag}(c_i) \) below a critical value. Fig3 2(c) summarizes our findings for the influence of the real part of the cross coupling term, \( \text{Real}(b_2) \) on the stability diagram. Note that the incoherent region now has no direct access from the single island state but is always mediated by the \( 1:2 \) locked region. The transition from the locked region to the incoherent region also shows interesting 'frequency jump' phenomena as shown in Fig3.2(d). The time average of \( \phi \) taken over several periods shows quantum jumps over intervals of \( \text{Real}(b_2) \) with nearly constant frequency steps. Similar behavior is also observed with the variation of \( \text{Real}(c_1) \) in the incoherent region.

3.5 Summary and Discussion

In this chapter we have investigated the possible nonlinear saturated states of neoclassical tearing modes in the presence of equilibrium sheared flows. We have started with the nonlinear reduced generalized MHD equations with bootstrap current contributions and equilibrium sheared flows. A center manifold reduction method has been used to reduce these MHD equations to a set of amplitude equations in the circumstance when the system parameters are close to their marginal values. Then we have carried out a bifurcation analysis of these amplitude equations to deduce their possible time asymptotic states. Our results show interesting time asymptotic nonlinear states like single saturated magnetic islands, frequency locked states and oscillating magnetic island states. These solutions could represent possible saturated
states of the neoclassical tearing modes in the presence of sheared flows. Oscillating island states have been observed numerically for neoclassical tearing modes in the presence of differential rotation [101]. Experimental evidence of a frequency jump phenomenon also exists during NTM activity apparently due to the torque inserted by the resonant particles [102]. Our results appear to be in accord with such observations.

The basic conclusion from the above studies is that the presence of sheared flows can bring about significant changes in the nonlinear final state of an NTM and lead to interesting physical effects. It should be mentioned however that the present results are a mathematical abstraction obtained from a very general analysis of a set of model equations that reflect the symmetry properties of tearing modes in the presence of flow. For a more realistic assessment of flow induced effects it is necessary to carry out a detailed time evolution study of the modes using a physically relevant set of model equations and to examine the properties of the nonlinear saturated states. The next two chapters are devoted to such an investigation.
3.6 Appendix

Expressions for the Coefficients

To calculate the coefficients of the amplitude equations (3.13, 3.14), let us consider the linear operator matrix \( L \) of the eqn (3.5). Near its marginal value \( L_c \), it can be written as \( L = L_c + \Delta L \).

Here, \( \Delta L = \left( \frac{\partial L}{\partial Z_c} \right)_c Z_c \)

Now substituting (3.11) and (3.12) in (3.5) and equating terms of \( O(|\zeta|) \) we can write the linear equations as,

\[ L_c R_{sc} - i \omega_s R_{sc} = 0 \]  \hspace{1cm} (3.15)

Similarly the adjoint linear equations can be written as,

\[ L_c^A R_{sc}^A - i \omega_s R_{sc}^A = 0 \]  \hspace{1cm} (3.16)

Here the superscript \( A \) refers to the adjoint functions.

Equating terms of \( O(|\zeta|^2) \) we have,

\[ \sum_{0 \leq i \leq \leq 4} Z_i Z_i \left[ L_c R_{sc} - i (\omega_s + \omega_s) R_{sc} \right] = R_{ish} \]  \hspace{1cm} (3.17)

Here the conjugate of \( Z_1 \) and \( Z_2 \) are conveniently written as \( Z_3 \) and \( Z_4 \) respectively and \( R_{ish} = (\phi_{ish}, \psi_{ish}, \psi_{ish})^T \) is given by,

\[
\phi_{ish} = \left[ \left( \frac{\lambda_i}{Z_0} Z_0 Z_1 + \alpha_i Z_1 Z_2 \right) \psi_{ie} + \left( \frac{\lambda_j}{Z_0} Z_0 Z_2 + \alpha_j Z_2 Z_1 \right) \psi_{ic} + \text{c.c.} \right] + \sum_{1 \leq s \leq \leq 4} Z_s Z_s \frac{1}{1 + \delta_s} \\
\times \{ \phi_{rc} \psi_{ie} + \phi_{ec} \psi_{ir} \} + \{ \phi_{ec} \psi_{ie} + \phi_{rc} \psi_{ir} \} - \frac{1}{Z_0} \frac{\partial L}{\partial Z_0} Z_m Z_0 \psi_{rc} \]  \hspace{1cm} (3.18)

\[
\psi_{ish} = \left[ \left( \frac{\lambda_i}{Z_0} Z_0 Z_1 + \alpha_i Z_1 Z_2 \right) \psi_{ie} + \left( \frac{\lambda_j}{Z_0} Z_0 Z_2 + \alpha_j Z_2 Z_1 \right) \psi_{ic} + \text{c.c.} \right] + \sum_{1 \leq s \leq \leq 4} Z_s Z_s \frac{1}{1 + \delta_s} \\
\times \{ \phi_{rc} \psi_{ie} + \phi_{ec} \psi_{ir} \} - \frac{1}{Z_0} \frac{\partial L}{\partial Z_0} Z_m Z_0 \psi_{rc} \]  \hspace{1cm} (3.19)

\[
p_{ish} = \left[ \left( \frac{\lambda_i}{Z_0} Z_0 Z_1 + \alpha_i Z_1 Z_2 \right) p_{ie} + \left( \frac{\lambda_j}{Z_0} Z_0 Z_2 + \alpha_j Z_2 Z_1 \right) p_{ic} + \text{c.c.} \right] + \sum_{1 \leq s \leq \leq 4} Z_s Z_s \frac{1}{1 + \delta_s} \\
\times \{ \phi_{rc} p_{ie} + \phi_{ec} p_{ir} \} - \frac{1}{Z_0} \frac{\partial L}{\partial Z_0} Z_m Z_0 p_{rc} \]  \hspace{1cm} (3.20)
where, $\delta_{zt} = \int \int (\phi_{zc}^A \nabla_z^2 \phi_{zc} + \psi_{zc}^A \psi_{zc} + p_{zc}^A p_{zc}) \, dx \, dy$

Using Fredholm solvability condition with terms related to $Z_0 Z_1$ and $Z_0 Z_2$, we can have,

$$\lambda_{zt} = Z_0 \int \int \left( \phi_{zc}^A \frac{\partial \phi_{zc}}{\partial z_0} + \psi_{zc}^A \frac{\partial \psi_{zc}}{\partial z_0} + p_{zc}^A \frac{\partial p_{zc}}{\partial z_0} \right) \, dx \, dy,$$

For $Z_1 Z_2$ terms we can have,

$$Z_1 Z_2 \left[ L_n R_{32} - i \omega_1 R_{32} \right] = \left[ i (2 \omega_1) R_{32} + a_1 R_{1e} + \cdots \right] Z_1 Z_2$$

As $R_{32} + R_{1e}$ is also a solution of the above equation so we can write

$$(\phi_{1e}^A, \nabla_1^2 \phi_{32}) + (\psi_{1e}^A, \psi_{32}) + (p_{1e}^A, p_{32}) = 0$$

Using Fredholm solvability condition to the above equation we get the coefficients $a_1$ and $a_2$ as,

$$a_1 = \int \int \left[ \phi_{1e}^A \left( -\{ \phi_{1e}, \nabla_1^2 \phi_{2e} \} - \{ \phi_{2e}, \nabla_1^2 \phi_{1e} \} - \{ \psi_{1e}, \nabla_1^2 \psi_{2e} \} - \{ \psi_{2e}, \nabla_1^2 \psi_{1e} \} \right) + \psi_{1e}^A \left( -\{ \phi_{1e}, \psi_{2e} \} - \{ \phi_{2e}, \psi_{1e} \} \right) + p_{1e}^A \left( \{ \phi_{1e}, p_{2e} \} + \{ \phi_{2e}, p_{1e} \} \right) \right] \, dx \, dy + O(\omega). \quad (3.21)$$

$$a_2 = \int \int \left[ \phi_{2e}^A \left( \{ \phi_{1e}, \nabla_1^2 \phi_{1e} \} + \{ \psi_{1e}, \nabla_1^2 \psi_{1e} \} + \psi_{1e}^A \{ \phi_{1e}, \psi_{1e} \} \right) - p_{2e}^A \{ \phi_{1e}, p_{1e} \} \right] \, dx \, dy + O(\omega). \quad (3.22)$$

Similarly we can have other coefficients by equating terms of order $O(|Z|^n)$ as,

$$b_1 = \int \int \left[ \phi_{1e}^A \left( \{ \phi_{1e}, \nabla_1^2 \phi_{3e} \} + \{ \phi_{1e}, \nabla_1^2 \phi_{2e} \} + \{ \psi_{1e}, \nabla_1^2 \psi_{3e} \} + \{ \psi_{1e}, \nabla_1^2 \psi_{2e} \} \right) + \psi_{1e}^A \left( \{ \phi_{1e}, \psi_{3e} \} + \{ \phi_{1e}, \psi_{2e} \} \right) + p_{1e}^A \left( -\{ \phi_{1e}, p_{2e} \} - \{ \phi_{1e}, p_{3e} \} \right) \right] \, dx \, dy, \quad (3.23)$$
\[ b_2 = \int \int [\psi_{i0c}^3 (\{\phi_{12}, \nabla_1^2 \phi_{2c}\} + \{\phi_{23}, \nabla_1^2 \phi_{1c}\} + \{\phi_{13}, \nabla_1^2 \phi_{2c}\} + \{\psi_{12}, \nabla_1^2 \psi_{3c}\}) \\
+ (\psi_{13}, \nabla_1^2 \psi_{2c}) + (\psi_{23}, \nabla_1^2 \psi_{1c}) + \psi_{i0c}^3 (\{\phi_{12}, \psi_{3c}\} + \{\phi_{23}, \psi_{1c}\} + \{\phi_{13}, \psi_{2c}\}) \\
+ p_{i0c}^2 (-\{\phi_{12}, p_{3c}\} - \{\phi_{23}, p_{1c}\} - \{\phi_{13}, p_{2c}\})] \, dx \, dy. \quad (3.24) \]

\[ c_1 = \int \int [\phi_{i0c}^3 (\{\phi_{12}, \nabla_2^2 \phi_{2c}\} + \{\phi_{24}, \nabla_2^2 \phi_{1c}\} + \{\phi_{14}, \nabla_2^2 \phi_{2c}\} + \{\psi_{12}, \nabla_2^2 \psi_{1c}\}) \\
+ (\psi_{24}, \nabla_2^2 \psi_{1c}) + (\psi_{14}, \nabla_2^2 \psi_{1c}) + \psi_{i0c}^3 (\{\phi_{12}, \psi_{4c}\} + \{\phi_{24}, \psi_{4c}\} + \{\phi_{14}, \psi_{2c}\}) \\
+ p_{i0c}^2 (-\{\phi_{12}, p_{4c}\} - \{\phi_{24}, p_{1c}\} - \{\phi_{14}, p_{2c}\})] \, dx \, dy \quad (3.25) \]

and

\[ c_2 = \int \int [\phi_{i0c}^3 (\{\phi_{22}, \nabla_1^2 \phi_{4c}\} + \{\phi_{24}, \nabla_1^2 \phi_{2c}\} + \{\psi_{22}, \nabla_1^2 \psi_{4c}\} + \{\psi_{24}, \nabla_1^2 \psi_{2c}\}) \\
+ \psi_{i0c}^3 (\{\phi_{22}, \psi_{4c}\} + \{\phi_{24}, \psi_{2c}\}) + p_{i0c}^2 (-\{\phi_{22}, p_{4c}\} - \{\phi_{24}, p_{2c}\})] \, dx \, dy \quad (3.26). \]