CHAPTER 2

EFFECT OF HOT IONS ON ION ACOUSTIC SOLITONS AND HOLES

2.1 Introduction

The coexistence of relatively cold electrons in the bulk of hot electrons is not uncommon in the laboratory as well as in space. Examples of such plasmas can be found in hot cathode discharge plasmas (Oleson and Found, 1949), in the ELMO confinement device (Krall and Trivelpiece, 1973) etc. Turbulent plasmas of thermonuclear interest have high energy tails in the electron distribution; these super-
thermal electrons are produced due to the interaction of localised high frequency fields with the charged particles (Morales and Lee, 1974). IMP 7 and 8 Satellite data have also shown the existence of double Maxwellian electron distribution in the solar wind near 1 AU (Feldman et al., 1975).

When the plasma contains two species of electrons with different temperatures, the system is governed by the effective temperature which depends on the temperature and fractional densities of the two electron components (Joner et al., 1975). They had shown that as the difference of temperature, between the two components, increases, the effective temperature and hence the propagation characteristics of the ion acoustic wave (IAW) is dominantly governed by the colder temperature. In a weakly nonlinear system with two electron components, Goswami and Buti (1976) had shown that the ion acoustic solitary wave has a larger amplitude for a given width compared to the one in a plasma with single electron species. Investigating the exact localised nonlinear IAW in a two electron component plasma with cold ions, Buti (1980) has shown the possible existence of ion acoustic solitons (density humps) as well as holes (density dips).
In a weakly nonlinear plasma the IAW is governed by KdV equation (Washimi and Taniuti, 1966) which predicts the relationship between velocity and amplitude of the KdV soliton. But the experimental value of velocity was found to be larger than the value predicted theoretically (Ikeda et al., 1970). This discrepancy in theoretical and experimental results suggests that finite ion temperature effects should be incorporated into the theory. Tappert (1972) had modified the KdV equation by taking the finite ion temperature into account but the velocity increased only slightly. Instead of restricting the analysis to weakly nonlinear systems, Sakanaka had worked with strongly nonlinear plasmas and had shown that the inclusion of finite ion temperature could predict a more realistic value for velocity. By retaining the full nonlinearities, in this chapter, we have investigated the effect of hot ions \( T_1 \neq 0 \), but \( T_1 < T_1, T_2 \); the subscripts 1 and 2 refer to the hot and warm electron components respectively) on the ion acoustic solitary waves. From the original set of equations a single equation is obtained. This equation is similar to the energy integral of a classical particle of unit mass moving in an effective potential. Our analysis shows that even an extremely small ion temperature can drastically reduce the maximum amplitude for solitons.
2.2 Energy Integral

Let us consider a plasma with ions and two species of electrons with densities \( n_{i0}, n_{10} \) and \( n_{20} \) respectively. The basic equations governing such a system are the ion continuity equation, the momentum transfer equation for ions and both types of electrons and the Poisson equation, namely,

\[
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i v_i) = 0 ,
\]

\[
 m_i \left( \frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} \right) = -e \frac{\partial \phi}{\partial x} - \frac{T_i}{n_i} \frac{\partial v_i}{\partial x} ,
\]

\[
e \frac{\partial \phi}{\partial x} - \frac{T_i}{n_i} \frac{\partial n_i}{\partial x} = 0 ,
\]

\[
e \frac{\partial \phi}{\partial x} - \frac{T_2}{n_2} \frac{\partial n_2}{\partial x} = 0 ,
\]

and

\[
\frac{\partial^2 \phi}{\partial x^2} = -4\pi e \left( n_i - n_1 - n_2 \right) .
\]
In writing Eqs. (2.3) and (2.4), electron inertia is neglected and we have assumed that both the electron components are separately in equilibrium with the potential \( \Phi \). This assumption is justified provided the phase velocity of the wave is much smaller than the thermal velocity of both the electron components separately, i.e., \( T_{\text{eff}}/m_i \ll T_{1,2}/m_e \), where \( T_{\text{eff}}(= n_0 T_1 T_2 (n_1 T_1 + n_2 T_2)^{-1}) \) is the effective temperature of the electrons. Since \( T_1 \ll T_{1,2} \), we have neglected the effect of Landau damping. Furthermore, the plasma is treated as collisionless; this assumption is valid for the solar wind plasma at 1 AU.

Normalising the densities to \( n_0 = n_{10} + n_{20} \), \( z \) to \( \omega_p^{-1} \), \( x \) to \( \nu_{\text{eff}}/\omega_p \), \( \nu \) to \( \nu_{\text{eff}} \) and \( \Phi \) to \( T_{\text{eff}}/e \), the equations (2.1) to (2.5) can be written as,

\[
\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (n \nu) = 0, \tag{2.6}
\]

\[
\frac{\partial \nu}{\partial t} + \nu \frac{\partial \nu}{\partial x} = -\frac{\partial \Phi}{\partial x} - \frac{T_1}{\nu_{\text{eff}}} \frac{\partial}{\partial x} \ln \chi, \tag{2.7}
\]

\[
\rho_1 = \chi_1 \exp \left( \frac{T_{\text{eff}}}{T_1} \Phi \right), \tag{2.8}
\]
\[ n_2 = \alpha_2 \exp\left( \frac{T_{\text{eff}}}{T_2} \phi \right), \quad (2.9) \]

and

\[ \frac{\partial^2 \phi}{\partial x^2} = -n + \alpha_1 \exp\left( \frac{T_{\text{eff}}}{T_1} \phi \right) + \alpha_2 \exp\left( \frac{T_{\text{eff}}}{T_2} \phi \right), \quad (2.10) \]

where \( n, \alpha_1, \) and \( \alpha_2 \) denote the normalized ion and electron densities respectively. Transforming to a frame of reference \( \xi = x - Mt, \) which moves with a constant velocity \( M \) with respect to the \( x-t \) coordinate frame, and eliminating \( v \) from equations (2.6) and (2.7), we can write

\[ \mathcal{N} = M \left[ M^2 \phi - 2 \left( T_i / T_{\text{eff}} \right) \ln n \right]^{-1/2}. \quad (2.11) \]

Eliminating \( \phi \) between (2.10) and (2.11), we obtain

\[ \frac{1}{2} \left( \frac{dn}{d\xi} \right)^2 + \Psi(n, M) = 0, \quad (2.12) \]

where

\[ \Psi(n, M) = (M^2 n^{-3} - n^{-1} T_i / T_{\text{eff}})^{-2} \left[ M^2 (1-n^{-1}) \right. \]

\[ \left. (1-n) T_i / T_{\text{eff}} + \alpha_1 (T_i / T_{\text{eff}}) \left[ 1 - \exp \frac{\xi}{T_{\text{eff}}} (T_{\text{eff}} / T_i) \right. \right. \]

\[ \left. \left. \left[ \ln n + \frac{M^2}{2} (1-n^{-2}) \right] \right] \right] + \alpha_2 (T_2 / T_{\text{eff}})^2 \]

\[ - \exp \left[ \frac{\xi}{T_{\text{eff}}} (T_{\text{eff}} / T_2) \left[ -(T_i / T_{\text{eff}}) \ln n + \frac{M^2}{2} (1-n^{-2}) \right] \right] \right] \right]. \quad (2.13) \]
Eq. (2.12) is the energy integral of a classical particle of unit mass moving in the effective potential \( \psi(n, M) \) (more commonly known as the Sagdeev potential) with the kinetic energy \( \frac{1}{2}(\frac{d\psi}{d\xi})^2 \). In deriving the energy integral, the boundary conditions \( n = 1 \) and \( \frac{d\psi}{d\xi} = 0 \) at \( \xi = \pm \infty \) have been used.

2.3 Solitary Waves

To ascertain the existence of solitary waves, we have analysed the Sagdeev potential. Expanding \( \psi \) around \( n = 1 \), we obtain

\[
\psi(n \sim 1, M) = -\frac{1}{2}(n-1)^2 \left( \frac{M^2 - T_i/T_{eff}}{1 + T_i/T_{eff}} \right) \frac{1}{2} M^2 \left( 1 + T_i/T_{eff} \right)^2.
\]

From Eq. (2.12) it is evident that \( \psi(n, M) \leq 0 \) for real solutions. So \( \psi(n \sim 1, M) \) has to be negative and hence Eq. (2.14) demands that either \( M^2 > \left( 1 + T_i/T_{eff} \right) \) or \( M^2 < T_i/T_{eff} \). But if \( M^2 < T_i/T_{eff} \), the IAW will travel with a velocity smaller than the ion thermal speed which is not physical. So such small values of \( M^2 \) are not allowed. Hence only the supersonic ion acoustic solitary waves are allowed in the system. Further analysis of \( \psi \) shows that \( n = N \) is an extremum provided the Mach number \( M \)
satisfies the relation.

\[
M^2 (1 - \frac{1}{N^2}) + \gamma \frac{\mu + 2}{\mu + 1} (1 - N) + \frac{\mu + 2}{\mu + 1} (N + \frac{1}{2}) - \frac{\mu (\mu + 1)}{(\mu + 2)^2} \exp \left\{ \frac{\mu + 1}{2(\mu + 2)} \right\} \left[ - \frac{\mu + 2}{\mu + 1} \right] \]

\[
\frac{1}{2} M^2 \left( 1 - \frac{1}{N^2} \right) \right\}^2 - \frac{\mu + 2}{2(\mu + 1)} \exp \left\{ \frac{\mu + 1}{2(\mu + 2)} \right\} \left[ - \frac{\mu + 2}{\mu + 1} \right] \]

\[
\frac{\mu + 2}{\mu + 1} \gamma \ln N + \frac{1}{2} M^2 \left( 1 - \frac{1}{N^2} \right) \right\}^2 = 0.
\]

(2.15)

This equation gives the relationship between \( N \) and the maximum amplitude \( N \) of the nonlinear wave. In Eq. (2.15), \( \mu = \alpha_1 / \alpha_2 \), \( \nu = T_1 / T_2 \) and \( \gamma = T_1 / T_1 \). For \( T_1 = 0 \), Eq. (2.15) reduces to that obtained by Suti (1980). The critical Mach number \( M_c \) (corresponding to the maximum potential) can be calculated easily from Eq. (2.15). The results are shown in table 1. Unlike the cold ion case, \( N \) has an upper limit given by

\[
N_U = M_c \gamma^{-1/2} \left[ \frac{1}{(\mu + 1)/((\mu + 2))} \right]^{1/2}.
\]

(2.16)

\( N_u \) for different values of \( \mu, \nu \) and \( \gamma \) are tabulated in table 2.
Expanding $\psi$ around $n = N$, from Eq. (2.13) we obtain

\[
\psi(n \approx N, M) = (n - N) \frac{M^2}{N^3} \left( \frac{M^2}{N^3} \right)
\]

\[
\frac{1}{N} \left( \frac{T_c}{T_{\text{eff}}} \right)^{-2} F(N, M),
\]

where

\[
F(N, M) = N - \frac{N^3}{M^2} \gamma \frac{\lambda + 2\nu}{\lambda + 1} + \left[ \frac{N^2}{M^2} \gamma \right] \frac{(M+2\nu - \frac{\lambda + 2\nu}{\lambda + 1})}{(M+1)^2}
\]

\[
\frac{M(M+2\nu) - \frac{M}{\lambda+1}}{(M+1)^2} \exp \left[ \frac{\gamma}{M+2} \left[ - \frac{\lambda + 2\nu}{\lambda + 1} \gamma \right] \right] \left[ N^2 \gamma \frac{\lambda + 2\nu}{(M+1)^2} \frac{1}{\lambda + 1} \right]^N
\]

\[
\exp \left[ \frac{2\nu(M+1)}{M+2} \left[ - \frac{\lambda + 2\nu}{\lambda + 1} \gamma \right] \right] \left[ N^2 \gamma \frac{\lambda + 2\nu}{(M+1)^2} \frac{1}{\lambda + 1} \right]^N
\]

From Eq. (2.14) one can see that $n = 1$ is a double root of the equation $\psi(n, M) = 0$ which implies that both $\psi$ and $d\psi/dn$ vanish but $d^2\psi/dn^2$ is finite at $n = 1$. Again from Eqs. (2.15) and (2.17) it can be seen that $\psi(n, M)$ vanishes but $d\psi/dn$ is finite at $n = N$. Hence the necessary conditions for the existence of solitary waves (as discussed in chapter 1) are satisfied. Since $\psi$ has
to be negative, Eq. (2.16) requires that $F(N,M) < 0$ for holes ($N < 1$) and $F(N,M) > 0$ for solitons ($N > 1$). Hence in the system under consideration, ion acoustic holes are assured if and only if $F(N,M) < 0$ and $M_1^2 > (1 + \frac{T}{2C_{\infty}})$, where $M = M_1$ satisfies Eq. (2.15).

In an one electron component plasma, i.e., for $\nu = 1$, Eq. (2.17) can be written as

$$F(N, M, \nu=1) = \left(1 - \frac{N^2}{M^2} \frac{T_e}{T_i} \right) \left[ N - \exp\left(1 - \frac{N}{T_i} \right) + \frac{M^2}{2} \left(-\frac{N}{2}\right)^2 \right].$$

Therefore, for $N < 1$, $F(N,M) > 0$ and the condition for the existence of holes is not satisfied. So hole solutions are not possible in a plasma having one electron species. The presence of the second electron species is essential for the existence of holes.

By solving equations (2.15) and (2.18), the allowed regions for the existence of solitons and holes can in principle be obtained. But since these equations are complicated transcendental equations, it is not possible to solve them analytically. So we have solved them numerically. Before discussing the numerical results,
let us first consider the small amplitude case where analytical solutions can be obtained.

2.4 Small Amplitude Limit

In the small amplitude limit, the potential is simplified and it is possible to obtain an analytic solution. On taking \( n = 1 + \delta n \), one can write Eq. (2.19) retaining terms up to \( \delta n^3 \), Eq. (2.20)

\[
\frac{1}{2} \left( \frac{d}{dS} \delta n \right)^2 + X_1 \delta n^2 + X_2 \delta n^3,
\]

where

\[
X_1 = -\frac{1}{2} \left( M^2 - \frac{T_e}{T_{eff}} \right)^{-1} \left[ M^2 - \left( \frac{1}{2} + \frac{T_e}{T_{eff}} \right) \right]
\]

\[
X_2 = \frac{1}{2} \left( M^2 - \frac{T_e}{T_{eff}} \right)^{-1} \left[ 4 \left( M^2 - \frac{1}{3} T_e / T_{eff} \right) \right]
\]

\[
- \left( T_e / T_{eff} \right)^{-1} - 3 \left( M^2 - \frac{1}{3} T_e / T_{eff} \right) - \frac{1}{3} \left( M^2 - \frac{T_e}{T_{eff}} \right)^2.
\]
\[ \Delta = \left( \frac{\chi_1}{T_1^2} + \frac{\chi_2}{T_2^2} \right) \text{Teff}^2. \]

Eq. (2.20) has the solution

\[ Sn = \left( \frac{\chi_1}{\chi_2} \right) \text{Sech}^2 \left( T_1 T_{\text{eff}} \right). \]

Since \( \chi_1 \) is always negative, \( \chi_2 > 0 \) or \( \chi_2 < 0 \), soliton or hole solutions are obtained. For \( M^2 - T_1/T_{\text{eff}} \gg 1 \), we find that

\[ \Delta > \left( 3 + 2 \frac{T_1}{T_{\text{eff}}} \right). \]

Here \( T_1 \) appears as a correction factor and, as such, so in the finite amplitude case as can be seen from numerical computations. In a single electron component plasma, \( \Delta = 1 \) and so the condition (2.25) can never be satisfied.

Hence holes do not occur in a single electron component plasma. From Eq. (2.25), it is apparent that the condition for the existence of holes becomes more stringent for finite \( T_1 \).
2.5 Numerical Results

IMP 7 satellite data had shown that in the solar wind around the orbit of the earth, two species of electrons, a relatively dense component \( (N_1 \approx 20 \text{ cm}^{-3}) \) at temperature \( T_1 \approx 2 \times 10^5 \text{K} \) and a relatively cold component \( (N \approx 5 \text{ cm}^{-3}) \) at \( T_2 \approx 10^4 \text{K} \) are present. In the experiment of Jones et al. (1975), two species of electrons were observed at temperatures \( 3 \times 10^4 \text{K} \) and \( 1 \times 10^5 \text{K} \) with densities \( N = \text{max} \) \( 10^8 \) and \( 10^9 \text{ cm}^{-3} \). For typical values of \( \gamma = 3.5 \) and \( \nu = 0.05 \), we have computed the allowed regions for the existence of solitons and holes. Let \( M = M^* \) be the root of the equation \( F(N,M) = 0 \). From numerical computations, we find that for \( M < M^* \), \( F(N,M) < 0 \) for \( N < 1 \) and \( F(N,M) > 0 \) for \( N > 1 \). So the allowed regions for the existence of holes as well as solitons would be between the lines \( M^2 = M^* \) and \( M^2 = M^2_L \) where \( M^2_L = (1 + T_i/T_{\text{eff}}) \) with the condition that \( M^2_1 < M^* \).

The critical Mach number \( M^2_c \) is calculated for different values of \( \mu, \nu \) and \( \gamma \) and is tabulated in table 1 which shows that \( M^2_c \) decreases as \( \gamma \) increases. Another interesting result due to the finite temperature of ions is that there exists an upper limit on the maximum value of \( N \), whereas, \( N \to \infty \) in the case of cold
ions. In table 2, we have listed the values of \( N_u \) for different values of \( \mu \), \( \nu \) and \( \gamma \). From this table, it is apparent that \( N_u \) decreases as \( \gamma \) increases. So increase in \( \gamma \) causes \( M_c^2 \) and \( N_u \) to decrease. Hence the allowed regions for solitons are decreased by the limits set by \( M_c^2 \) and \( N_u \). In fig.1 we have plotted \( M_c^2 \) versus \( \gamma \) for \( \mu = 3 \), \( \nu = 5 \) and \( \gamma = 0.0, 0.02 \) and 0.05. It is clearly seen that the allowed regions for solitons decrease as \( \gamma \) increases from 0.0 to 0.05. From fig.3 it is apparent that the allowed regions for solitons increase as \( \nu \) increases. However, fig.3 shows that the allowed regions for solitons increase as \( \mu \) increases. This last result is in contradiction with the results obtained in a plasma having cold ions. In such a plasma, with two electron species, the allowed region for solitons decrease as \( \gamma \) increases (Buti, 1980). This discrepancy can be explained as follows. Due to the finite ion temperature, there exists an upper limit for \( N \) which is larger for larger \( \mu \). So, even though \( M_c^2 \) is smaller for larger \( \mu \), \( M_c^2 \) is also smaller and together with larger \( N_u \), the allowed region for solitons increases as \( \mu \) increases.

Let us now discuss about the holes. In fig.4 \( M_c^2 \) is plotted against \( N \) for \( \mu = 3 \), \( \nu = 20 \) and \( \gamma = 0.02 \) which shows that as \( \gamma \) increases from 0.0 to 0.02, the
allowed region for holes is drastically reduced. Moreover, as \( \gamma \) increases to 0.05, holes are destroyed. From fig. 5, it can be seen that for \( \mu = 3 \) and \( \gamma = 0.02 \), holes exist when \( \mathcal{V} = 20 \). But when \( \mathcal{V} \) is reduced to 16.0, holes are forbidden. As a matter of fact, there exists a critical value of \( \mathcal{V} (= \mathcal{V}_c) \), below which holes do not occur. For \( \mu = 3 \) and \( \gamma = 0.02 \), \( \mathcal{V}_c \) is found to be 16. Fig. 6 illustrates allowed regions for holes for different values of \( \mu \). As is seen, for \( \mathcal{V} = 20.0 \) and \( \gamma = 0.02 \), holes are forbidden for \( \mu = 1 \) but when \( \mu \) increases to 3, there exists an allowed region for the occurrence of holes.

2.6 Conclusions

From the numerical results, we find that for the finite amplitude ion acoustic solitary waves, the allowed regions for the existence of solitons and holes decrease as the ion temperature increases. The hotter the ions, more drastic are these effects, so much so that in some cases holes are even forbidden. Finite ion temperature also reduces the critical Mach number. Furthermore, finite ion temperature imposes an upper limit on \( N \) which is inversely proportional to the square root of \( \gamma \).
TABLE 1

$M_c^2$ for various values of $\mu$, $\gamma$, and $\gamma$.

For $\gamma = 0.02$

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<th>10</th>
<th>$M'_c$</th>
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<tr>
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For $\gamma = 0.05$


<table>
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<th>10</th>
<th>( \mu )</th>
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<td>0.32</td>
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\( \gamma = 0.02 \) and \( \gamma = 0.35 \)
Fig. 1: $M^2$ versus $N(N > 1)$ for $\mu = 3, \nu = 5$ and $\gamma = 0.00, 0.02$ and 0.05. The upper allowed value of $M$ is given by either $M_C$ or $M^*$, whichever is the smaller. The regions bounded by the lines $\gamma = 0, \gamma = 0.02$ and $\gamma = 0.05$ are the allowed regions for solitons.
Fig. 2: $M^2$ versus $N$ for $\lambda_i = 3$, $\nu = 0.02$ and $\nu = 5, 10$ and 20. The upper allowed value of $M$ is given by either $M_c$ or $M^*$, whichever is the smaller. The region bounded by the lines $\nu = 5$, $\nu = 10$ and $\nu = 20$ are the allowed regions for solitons.
Fig. 3: $M^2$ versus $N$ for $\gamma = 5$, $\gamma' = 0.02$ and $\mu = 0.1$, 3 and 10. The upper allowed value of $M$ is given by either $M_c$ or $M^*$, whichever is the smaller. The region bounded by the lines $\alpha\alpha\alpha\alpha$ ($\mu = 0.1$), $\alpha\alpha\alpha\alpha$ ($\mu = 3$) and $\alpha\alpha\alpha\alpha$ ($\mu = 10$) are the allowed regions for solitons.
Fig. 4: $M^2$ versus $N$ ($\lambda < 1$) for $\mu = 3$, $\nu = 20$, $\gamma^* = 0.0$ and 0.02. The region between the curves $M = M_1$ and $N = 1$ bounded by the lines $-- (\gamma^* = 0)$ and $-\cdots- (\gamma^* = 0.02)$ are the allowed regions for holes.
Fig. 5: $M^2$ versus $N$ for $\lambda = 3$, $\eta = 0.02$ and $\nu = 18$ at $\nu' = 0$. The region between the curves $M = M_1$ and $M = M^*$ bounded by the lines $-\varepsilon_0(\alpha = 20)$ is the allowed region for holes. Holes are forbidden for $\nu' = 18$. 
Fig. 6: $M^2$ versus $N$ for $V = 20$, $V' = 0.02$ and $\mu = 3.0$ and $3.0$. The region between the curves $M = M_1$ and $M = M_2$ bounded by the lines $\pm \beta$ ($\beta = 3$) is the allowed region for holes. Holes are forbidden for $\mu = 1$. 