CHAPTER 3

Ramsey Interferometry with Quantized Field

Entanglement is one of the most interesting phenomena in quantum mechanics. A large number of methods exist for generating various types of entangled states [7, 11, 49, 50]. Many of these states have been implemented in quantum information processing [12, 13, 51, 52]. In chapter 1, we have discussed nonlocal characteristics [53] of entanglement and shown how cavities can be used to generate atom-atom and atom-field entangled states. In this chapter, we discuss how one can generate entangled state of two spatially separated cavities. We note that entanglement between two modes in a single cavity has been reported by Rauschenbeutel et al [54]. Further, in a recent experiment, the entangled spin-state of two macroscopic atomic samples has been realized by passing a polarized light pulse [55].

We consider two spatially separated high quality cavities aligned along their axis and atoms are passed one by one through the cavities. Thus, our arrangement is equivalent to the Ramsey interferometer [56]. We discuss the role of quantum statistics of the fields in Ramsey interferometry and examine the conditions on the fields so that interference fringes are obtained. In the case of fixed number of photons in the cavities, interference does not occur in the excitation probability of a single atom [57]. We show how the interference can be restored by passing successively two atoms through the cavities and measuring atom-atom correlations [58]. We generate various entangled states by passing a single or two atoms through cavities and show entanglement can be transferred from fields to atoms and viceversa.
3.1 Ramsey Interferometry

Ramsey proposed a new technique of doing molecular beam resonance experiments [56]. He used two spatially separated fields instead of using uniform single field as shown in Fig. 3.1. Ramsey technique has advantage of higher precision measurement over an experiment with continuous single field. The higher precision in Ramsey technique is due to the occurrence of sharper resonance curves with two spatially separated fields. It was originally proposed as a technique in the microwave domain [56] which was then extended to studies in optical domain [59]. Achieving higher resolution by using two separated fields in molecular or atomic beam experiments can be understood as follows.

Consider a two level atom having upper level $|e\rangle$ and lower level $|g\rangle$. The state of the atom at anytime $t$ is given by

$$|\psi(t)\rangle = c_e(t)|e\rangle + c_g(t)|g\rangle.$$  \hspace{1cm} (3.1)

The field in each Ramsey zones is $E(t) = E \cos \omega t$. When an atom enters in the field, the atom-field interaction is given by

$$H_I = -(P_{eg}|e\rangle\langle g| + P_{ge}|g\rangle\langle e|)E \cos(\omega t),$$

$$= -\hbar(\Omega_R|e\rangle\langle g| + \Omega_G^*|g\rangle\langle e|) \cos(\omega t).$$  \hspace{1cm} (3.2)

where $P_{eg} = P_{ge}^*$, are transition dipole matrix elements and $|\Omega_R| = |P_{eg}E/\hbar|$ is Rabi frequency. Thus the Hamiltonian of the atom-field system is given by

$$H = \hbar \omega_e|e\rangle\langle e| + \hbar \omega_g|g\rangle\langle g| - \hbar(\Omega_R|e\rangle\langle g| + \Omega_G^*|g\rangle\langle e|)e^{i\omega t} + e^{-i\omega t}).$$  \hspace{1cm} (3.3)
Figure 3.2: Comparison between resonance curves using single field and Ramsey method at two spatially separated fields.

where $\omega_e$ and $\omega_g$ are frequencies corresponding to the higher and the lower energy levels, respectively. Using interaction picture and rotating wave approximation, the Hamiltonian $H$ reduces to

$$H = -\frac{i\hbar}{2}(\Omega g e^{i\Delta t} |e\rangle\langle g| + \Omega h e^{-i\Delta t} |g\rangle\langle e|), \text{ where } \Delta = (\omega_e - \omega_g) = \omega_e - \omega_g.$$ (3.4)

Therefore, the Hamiltonian for Ramsey method is given by

$$H = -\frac{i\hbar}{2}(\Omega g e^{i\Delta t} |e\rangle\langle g| + \Omega h e^{-i\Delta t} |g\rangle\langle e|), \text{ for } 0 < t < \tau, \ T - \tau < t \leq T - 2\tau \quad (3.5)$$
$$H = 0, \text{ for } \tau < t \leq T + \tau.$$ (3.5)

Using atomic state (3.1) and Hamiltonian (3.5), Schrödinger equation takes the form

$$\dot{c}_e(t) = \frac{i\Omega}{2} e^{i\Delta t} c_g(t), \quad \dot{c}_g(t) = \frac{i\Omega}{2} e^{-i\Delta t} c_e(t), \quad \text{in Ramsey zones.} \quad (3.6)$$

$$\dot{c}_e(t) = \dot{c}_g(t) = 0 \quad \text{every where else.}$$

Using differential equation (3.6), the evolution of the atomic state can be calculated. The probability of detecting the atom in a particular state $|\psi_f\rangle$ after total time $T + 2\tau$ is defined
Ramsey interferometry with Quantized Field

as

\[ P_{fa} = \langle \psi_f | \psi_u(T + 2\tau) \rangle \langle \psi_u(T + 2\tau) | \psi_f \rangle. \]  

(3.7)

If an atom is coming in lower state \(|g\rangle\), the probability of detecting the atom in the upper state \(|e\rangle\) is found to be, for spatially separated fields

\[ P_{eg} = \frac{|\Omega R|^2}{\Omega^2} \sin^2 \left( \frac{\Omega\tau}{2} \right) \left\{ \cos^2 \left( \frac{\Omega\tau}{2} \right) \cos^2 \left( \frac{\Delta T}{2} \right) - \frac{\Delta^2}{\Omega^2} \sin^2 \left( \frac{\Omega\tau}{2} \right) \sin^2 \left( \frac{\Delta T}{2} \right) \right\}. \]  

(3.8)

and for single field

\[ P_{eg} = \frac{|\Omega R|^2}{\Omega^2} \sin^2 \left( \frac{\Omega\tau}{2} \right). \]  

(3.9)

Here \( \Omega = \sqrt{\Delta^2 + \Omega_i^2} \). Generally \( \Omega \tau \) is a small number, so we can drop the terms having \( \sin^4(\Omega\tau/2) \) and the expression (3.8) simplifies to

\[ P_{eg} \approx \frac{|\Omega R|^2}{\Omega^2} \sin^2 \left( \frac{\Omega\tau}{2} \right) \cos^2 \left( \frac{\Delta T}{2} \right). \]  

(3.10)

In Fig. 3.2, we show the transition probabilities \( P_{eg} \) for the two separated fields and for the single field, defined by equations (3.9) and (3.10). Clearly, the presence of cosine term in (3.10) modulates the transition probability and the resolution in the resonance curves becomes proportional to \( 1/\Delta T \).

In an interferometer interference occurs when two monochromatic coherent waves travel to the detector through two different ways. In Ramsey method of two separated fields, a particular transition in atomic states occurs through two different ways and interference fringes appear in the transition probability similar to wave interferometer. For the transition of an atom from its lower state \(|g\rangle\) to higher state \(|e\rangle\) in Ramsey interferometer, the atom follows two different paths

\[ |g\rangle \xrightarrow{\text{first zone}} |g\rangle \xrightarrow{\text{second zone}} |e\rangle. \]  

(3.11)

\[ |g\rangle \xrightarrow{\text{first zone}} |e\rangle \xrightarrow{\text{second zone}} |e\rangle. \]  

(3.12)

The existence of fringes in the transition probability has been interpreted as due to quantum interference between the transition amplitudes [56], therefore Ramsey method is a way of doing atomic interferometry.

More recently, Ramsey interferometry has been used very successfully in the studies of quantum entanglement resulting from the interaction of atoms with radiation in a high quality cavity. Haroche and coworkers [7, 60, 61] have detected a variety of cavity-QED
Ramsey Interferometry with Quantized Field

3.2 Ramsey Interferometry with Quantized Fields

These days interference effects at a single photon or few photon levels are becoming quite common [64, 65, 66, 67], it is natural to enquire how the results of Ramsey interferometry would be modified if the coherent field in each Ramsey zone is replaced by a quantized field [68].

We consider a high quality cavity [62] as the Ramsey zone of quantized field. If the number of photons in the cavity is large and the field has a well defined phase, then it would approach to the classical Ramsey interferometry. We thus consider the situation shown in the Fig. 3.3. An atom with two levels \( |e \rangle \) and \( |g \rangle \), with frequency separation \( \omega_0 \), interacts with two single-mode cavities with identical frequencies \( \omega \). Let the annihilation and creation operators in the i-th cavity be denoted by \( a_i \) and \( a_i^\dagger \), respectively. For the situation shown in the Fig. 3.3, the Hamiltonian in the interaction picture is

\[
\begin{align*}
H_1 &= h\left(|e\rangle\langle g|a_1 e^{i\Delta t} + a_1^\dagger |g\rangle\langle e|e^{-i\Delta t}\right) & 0 < t < \tau_1 , \\
H_1 &= 0 & \tau_1 < t \leq T - \tau_1 , \\
H_1 &= h\left(|e\rangle\langle g|a_2 e^{i\Delta t} + a_2^\dagger |g\rangle\langle e|e^{-i\Delta t}\right) & T + \tau_1 < t \leq T + \tau_1 + \tau_2 .
\end{align*}
\]

Here \( \Delta = \omega_0 - \omega \) and \( g_i \) is the coupling constant of the atom with the vacuum in the i-th cavity. Let us consider an initial state with atom in the lower state \( |g\rangle \) and the fields characterized by the state \( \sum_{n,\mu} F_{n,\mu}|n,\mu\rangle \). Here \( |n\rangle\langle \mu| \) represents the Fock state in first (second) cavity and \( F_{n,\mu} \) is photon distribution function. Let \( \phi_e, \phi_g \) be the phase shifts in \( |e\rangle \) and \( |g\rangle \), which we might introduce using some external perturbation between the cavities. Using the interaction Hamiltonian (3.13), the time evolution of the state can be calculated.
Ramsey Interferometry with Quantized Field

The state of the atom and cavity fields is found to be

$$|\psi(T_1 + T_2)| = \sum_{n,\mu} F_{n,\mu}^a C_{n+1}^a \left(\begin{array}{c} 1 \nn \end{array}\right) \exp\left[-\frac{\Delta^2}{4} (n+1) \right]$$

where

$$C_n^a(\tau_1) = \cos \Omega_n \cdot 2 \cdot \frac{\Delta}{\Omega_n} \sin \Omega_n.$$  

$$S_n^a(\tau_1) = \frac{2g_n \alpha + 1}{\Omega_n} \sin \Omega_n.$$  

$$\Omega_n = \sqrt{\Delta^2 + 4g_n^2 \cdot \alpha - 1}.$$  

$$\alpha = n, \mu \text{ and } g_n = g_1, \mu = g_2.$$  

The functions $C_n$ and $S_n$ describe the dynamics of the atom interacting with a single mode cavity with initial state as a Fock state. Note that $C_n(S_n)$ gives the probability amplitude of finding the atom in the excited (ground) state given that it was in the excited state at time $t = 0$.

The structure of the state clearly suggests that a given final state is reached in two different ways. Consider a measurement in which the outgoing atom is found in the excited state. The probability of excitation $P_{eg}$ defined by

$$P_{eg} = Tr_{\text{field}}(|\psi(T + \tau_1 + T_2)\rangle\langle\psi(T + \tau_1 + T_2)|)$$

Figure 3.3: A schematic arrangement for Ramsey interferometry with quantized fields. Each classical Ramsey zone is replaced by a cavity. There is a phase change between two cavities as it equals and $|g\rangle = e^{-i\phi}|g\rangle$. The functions $C_n$ and $S_n$ describe the dynamics of the atom interacting with a single mode cavity with initial state as a Fock state. Note that $C_n(S_n)$ gives the probability amplitude of finding the atom in the excited (ground) state given that it was in the excited state at time $t = 0$.
can be calculated using Eq. (3.14). We find the result

$$P_{eq} = \sum_{n,\mu} \left| F_{n+1,\mu} X_{n+1,\mu} + e^{i\Delta T + \phi} F_{n,\mu+1} Y_{n,\mu+1} \right|^2.$$  \hspace{1cm} (3.17)

where

$$\phi = \phi_e - \phi_q$$

$$X_{n+1,\mu} = S_n^*(\tau_1) C_{n,\mu}^* \tau_2)$$

$$Y_{n,\mu+1} = C_{n-1}^*(\tau_1) S_{n,\mu}^* \tau_2)$$ \hspace{1cm} (3.18)

A similar result is obtained for $P_{ge}$, i.e., the probability of finding the atom in the ground state if initially the atom is in the excited state,

$$P_{eq} = \sum_{n,\mu} \left| F_{n-1,\mu} X_{n,\mu} - e^{i\Delta T_{1/2}} F_{n,\mu} Y_{n,\mu} - \omega \right|^2.$$ \hspace{1cm} (3.19)

Results (3.17) and (3.19) are important for understanding Ramsey interferometry with quantized fields. These give rise to a number of important consequences as far as the fundamentals of atom-field interaction are concerned. For classical fields, result 3.14 can be modified, since probability amplitude functions $F_{n,\mu}$ is peaked around average number of photons $\bar{n}$ and $\bar{\mu}$. So in the summation, we can replace

$$X_{n,\mu} \rightarrow X_{n,\mu},$$

$$Y_{n,\mu} \rightarrow Y_{n,\mu},$$ \hspace{1cm} (3.20)

Further for large $n$ and $\mu$, make the following replacements:

$$F_{n-1,\mu} \rightarrow F_{n,\mu}$$

$$F_{n,\mu+1} \rightarrow F_{n,\mu}.$$ \hspace{1cm} (3.21)

For normalized photon probability amplitude functions $F_{n,\mu}$, Eq. (3.19) reduces to

$$P_{ge} = |X_{n,\mu}^* + e^{-i\Delta T_{1/2}} Y_{n,\mu}^*|^2.$$ \hspace{1cm} (3.22)

Equation (3.22) is the result for classical fields.
Ramsey Interferometry with Quantized Field

3.3 Dependence of the fringes on quantum statistics of the fields

We now examine the consequences of the quantized nature of the field and, in particular, investigate when the interferences are most pronounced. From result (3.17), we see there are two paths which contribute to the amplitude for detecting the atom in excited state:

\[ |g, n, \mu\rangle \rightarrow |e, n - 1, \mu'\rangle \rightarrow e, n - 1, \mu'. \]
\[ |g, n, \mu\rangle \rightarrow |g, n, \mu\rangle \rightarrow e, n, \mu = 1. \]  

(3.23)

The interference between these two paths depends on the nature of the photon statistics, i.e., on the functions \( F_{n, \mu} \). Clearly, if the field in each cavity is in a Fock state \( n, \mu \), then the interference terms in (3.17) drop out and the two paths 3.23 become independent. This happens even for Fock states with large number of photons. Interferences are obtained as long as the photon statistics is such that the cross terms in 3.17 are nonzero. Consider a situation where detuning \( \Delta \) can be ignored while considering evolution in Ramsey zone, i.e., in each cavity. It can be shown that the cross terms in 3.17 are nonzero if the field statistics is such that,

\[ \langle a_1^\dagger a_2^\dagger \rangle = \frac{1}{\sqrt{a_1 a_2}} \sin(g_1 g_2) \sqrt{a_1} \sqrt{a_2} \cos(g_1 \gamma_1 + g_2 \gamma_2) \left( \sin(g_1 \gamma_1) \sin(g_2 \gamma_2) - \cos(g_1 \gamma_1) \cos(g_2 \gamma_2) \right) \]

which for small interaction times reduces to

\[ \langle a_1^\dagger a_2^\dagger \rangle \neq 0. \]  

(3.24)

Thus, the nature of interference depends on the quantum statistics of the fields in the two Ramsey zones. The conditions (3.24) and (3.25) imply that if the cavities are independent, then the field in each cavity must have a well defined phase for interference to occur. The interference would also not occur if one cavity has a definite number of photons and the other has a field in coherent state. However, interference is obtained if fields in the two cavities are entangled even though the field in each cavity does not have a well-defined phase. In Fig. 3.4, results for classical as well as quantized fields are plotted when each Ramsey zone has a coherent field with average number of photons \( \langle n \rangle = 5 \). Interference fringes for classical fields show higher visibility than in the case of quantized fields.
Figure 3.4: Interference fringes in the probability of detecting a single atom in the excited state when the atom is initially in the ground state for quantized (solid lines) and classical (dashed lines) fields. The parameters are: (a) $\theta = \pi$, $\lambda / g = 10$, $\phi = 0$, $V_0 = 0.68$ (b) $\Delta = 0$, $g = g/8$, $V_0 = 0.96$ (c) $\Delta = 0$, $g = g/4$, $V_0 = 0.14E - 01$ and (d) $\Delta = 0$, $g = g/2$, $V_0 = 0.16$. The common parameters for all curves are $\alpha = 0$, $\gamma_1 = \gamma_2$, $g_1 = g_2 = g$, $V_c = 1.00$. $V_c$, $V_0$ are the visibilities for classical and quantized fields.
3.3.1 Ramsey fringes with fields at single photon level

Having shown that Ramsey fringes vanish if each cavity contains one photon, the next question is what happens if the field in each cavity is at single photon level [65]? For this purpose, we consider a case where each cavity is pumped by a weak coherent state so that the initial state of the cavities is

\[ |\psi_{\text{in}}\rangle \approx \frac{1}{\sqrt{1+|\alpha|^2}} |0\rangle + \alpha |1\rangle = |\alpha\rangle. \] (3.26)

In this case, the result (3.17) leads to

\[ P_{eg} = \frac{4|\alpha|^2}{(1+|\alpha|^2)^2} \left[ \frac{g_0}{\Omega_0} \sin(\Omega_0 T_1/2) \left( \cos(\Omega_0 T_2/2) \exp\{i(\Delta T + \pi/2) + \theta \} \right) \right] \] (3.27)

which for \( \Delta = 0 \) and \( g_1 T_1 = \sqrt{2} g_2 T_2 = \pi/2 \) reduces to

\[ P_{eg} = \frac{|\alpha|^2}{(1+|\alpha|^2)^2} \left( 1 + \sin^2(\pi/2) \cos \theta \right). \] (3.28)

This leads to high visibility for the fringes (about 100\%) though the absolute value of the signal is small. It is clear that the interference in Eq. (3.27) arises from the cross terms in Eq. (3.26), as such cross terms lead to the same final state via two different pathways:

\[ |g, 1, 0\rangle \to |\alpha, 0, 0\rangle \to |1, 0, 0\rangle, \]

\[ |g, 0, 1\rangle \to |\alpha, 0, 0\rangle \to |0, 1, 0\rangle. \] (3.29)

The other terms in Eq. (3.26) do not result in interference, since \( 1, 1 \) leads to different final states and \( |0, 0\rangle \) cannot produce excitation. It should be noted that Eq. (3.27) is different from the result obtained for classical fields, i.e., when one ignores the back-action of atoms on the field. Recently, an arbitrary superposition of \( 0 \) and \( 1 \) states \( c_0 \sqrt{1+|\alpha|^2/2} |0\rangle + c_1 |1\rangle \) has been realized [66, 67]; here the parameter \( \alpha \) need not be small. Such a state is highly nonclassical and is quite distinct from a coherent state with very small excitation. This nonclassical state also has the important characteristics that the average value of the field is nonzero and thus the off-diagonal elements of the density matrix are nonzero. For such a nonclassical state and for \( \Delta = 0 \), the Ramsey fringes are given by

\[ P_{eg} = \frac{|\alpha|^2}{(1+|\alpha|^2)^2} \left\{ |\sin(g_1 T_1) \cos(g_2 T_2) + \sin(g_2 T_2) \cos\theta|^2 \right. \]

\[ + \frac{|\alpha|^4}{(1+|\alpha|^2)^2} \left\{ \sin^2(g_1 T_1) \cos^2(g_2 T_2) + \cos^2(g_1 T_1) \sin^2(g_2 T_2) \right\} \] (3.30)
Ramsey Interferometry with Quantized Field

Figure 3.5: The visibility of interference fringes vs. $\alpha$ for weak coherent fields (solid line) and for a nonclassical state (dashed line) $\frac{1}{\sqrt{1+\alpha^2}}(0 + \alpha|1\rangle)$, with parameters $g\tau = \pi/2$ and $\Delta = 0$.

For $g\tau_1 = g\sqrt{2}\tau_2 = \pi/2$, it reduces to the previously derived result 3.28. Remarkably, the visibility from result (3.30) does not depend on $\alpha$. We show in the Fig. (3.5) a comparison of the visibility in the case of coherent field and a nonclassical field. It may be noted that for the state $(0 + \alpha|1\rangle)$, the P-distribution is highly singular. We note that in the context of cavity quantum electrodynamics the state $\psi_n$ can be produced by following scheme [69]. We could consider a resonant cavity in vacuum state. If a two-level atom in the superposition state $\frac{1}{\sqrt{1+\alpha^2}}(0 + \alpha|1\rangle)$ is sent through the cavity with the interaction time adjusted so that the atom can emit one photon in the cavity, then the state of the cavity would be $|\psi_n\rangle$.

3.3.2 Photon-photon interaction mediated by a single atom and quantum entanglement of two cavities

It is well known in nonlinear optics in a macroscopic system that the fields effectively interact and one knows many examples of three wave and four wave interactions in a medium. Such interactions are significant at macroscopic densities of atoms. In this section, we demonstrate a rather remarkable result that a single atom in a high quality cavity can produce photon-photon interaction. For this purpose, consider an atom in the ground state passing through the two cavity system. We calculate the state of the two cavity system.
subject to the condition that the atom at the output is detected in the ground state. Such a conditional field state is found to be

$$|\psi_{c,g}\rangle = \langle g|\psi(T + \tau_1 + \tau_2)\rangle,$$

$$= C_{n-1}(\tau_1)C_{\mu-1}(\tau_2) \exp(-i\Delta(\tau_1 + \tau_2)/2 - i\phi_g)|n,\mu\rangle$$

$$+ S_{n-1}(\tau_1)S_{\mu}(\tau_2) \exp(-i\Delta(\tau_1 + \tau_2 + 2T)/2 - i\phi_w)|n - 1,\mu + 1\rangle.$$  \hspace{1cm} (3.31)

This involves a linear combination of states $|n,\mu\rangle$ and $|n - 1,\mu + 1\rangle$ leading to the entanglement of two cavities. We note that the entanglement of two macroscopically separated cavities was proposed by Meystre [70]. In addition, the passage of one atom transfers one photon from the first cavity to the second cavity. The transfer from one cavity to the other will be complete if $C_{n-1}(\tau_1) = C_{\mu-1}(\tau_2) = 0$. Other entangled states are also possible; for example, if the atom was initially in the ground state and if it is detected in the excited state, then the conditional state of the cavities is,

$$|\psi_{c,e}\rangle = \langle e|\psi(T + \tau_1 + \tau_2)\rangle,$$

$$= S_{n-1}^*(\tau_1)C_{\mu}^*(\tau_2) \exp(i\Delta(\tau_1 + \tau_2)/2 - i\phi_w)|n - 1,\mu\rangle$$

$$+ C_{n-1}(\tau_1)S_{\mu}^*(\tau_2) \exp(i\Delta(\tau_1 + \tau_2 + 2T)/2 - i\phi_w)|n,\mu - 1\rangle.$$  \hspace{1cm} (3.32)

We may also note that the entanglement of two modes in the same cavity has been achieved by Rauschenbeutel et al [54].

### 3.4 Two Atom Interferometry

In previous section, we considered the possibility of producing entanglement between the two cavities by conditional detection of the atomic state [50]. We next examine how such entanglement (3.32) can be detected. From our previous discussion leading to Eq. (3.17) and Eq. (3.24) it is clear that if we send a second atom and measure its excitation probability, then such a probability would exhibit interference fringes.

For a second atom coming in the ground state $|g\rangle$ and detected in the excited state,
Ramsey Interferometry with Quantized Field

following are the possible pathways:

\[
\begin{align*}
|n - 1, \mu\rangle |g\rangle & \rightarrow |n - 1, \mu\rangle |g\rangle \rightarrow |n - 1, \mu - 1, \mu\rangle , \\
|n, \mu - 1\rangle |g\rangle & \rightarrow |n - 1, \mu - 1, \mu\rangle \rightarrow |n - 1, \mu\rangle , \\
|n - 1, \mu\rangle |g\rangle & \rightarrow |n - 2, \mu\rangle |e\rangle \rightarrow |n - 2, \mu, \mu\rangle , \\
|n, \mu - 1\rangle |g\rangle & \rightarrow |n, \mu - 1, \mu\rangle \rightarrow |n, \mu - 2, \mu\rangle . \quad (3.33)
\end{align*}
\]

In summary, the system as a whole starting with an initial state \(\langle n, \mu, g_1, g_2\rangle\) has two different pathways leading to the detection of the atom-cavity system in state \(\langle n - 1, \mu - 1, \mu\rangle\):

\[
\begin{align*}
|n, \mu, g_1, g_2\rangle & \rightarrow |n - 1, \mu, g_1, g_2\rangle \rightarrow |n - 1, \mu - 1, \mu\rangle , \\
|n, \mu, g_1, g_2\rangle & \rightarrow |n, \mu - 1, g_1, g_2\rangle \rightarrow |n, \mu - 1, \mu\rangle . \quad (3.34)
\end{align*}
\]

The joint probability of detecting both atoms in the excited state \(|e\rangle\) can be used for doing atomic interferometry even if each cavity is in Fock state. This is reminiscent of photon-photon correlation measurements with light produced in the process of down conversion. Mandel and coworkers [71] carried out a series of measurements with photons from a down converted source where they reported no interferences in the measurement of mean intensities, whereas photon-photon correlation exhibited a variety of interference phenomena. In the context of Ramsey interferometry with quantized fields, we suggest a measurement of the atom-atom correlation. An explicit form of the joint detection probability can be obtained following Jaynes-Cummings dynamics. A long calculation leads to the following expression for the joint probability if the initial state of the cavities is \(n, \mu\):

\[
P_{\text{joint}}^{\text{two}} = \left| S_{n-1}(\tau_1)S_{n-2}(\tau_1)C_{n\mu}(\tau_2)C_{n\mu}(\tau_2) \right|^2 + \left| C_{n-1}(\tau_1)C_{n-2}(\tau_1)S_{n-1}(\tau_2)S_{n-2}(\tau_2) \right|^2
\]

\[
+ \left| S_{n-1}(\tau_1)C_{n-1}(\tau_1)S_{n-2}(\tau_2)C_{n-2}(\tau_2) - S_{n-1}(\tau_1)C_{n-1}(\tau_1)S_{n-2}(\tau_2)C_{n-2}(\tau_2) \right|
\]

\[
\exp \left[ i(\Delta T' - T) + \phi' - \phi \right] \right|^2. \quad (3.35)
\]

Here, we allow the possibility of different interaction times and phases (denoted by a dash) for the second atom. In the special case where \(\Delta = 0, g_1 = g_2, \tau_1 = \tau_2 = \tau'_1 = \tau'_2\) and \(gT = \pi/4\) and when initially cavities are in state \(|1, 1\rangle\), the joint detection probability has
Ramsey Interferometry with Quantized Field

the form

\[
P_{11/2}^{\pi/2} = \frac{1}{16} + \frac{1}{4} \cos^2(\pi/2) + \frac{1}{4} \cos(\pi/2) \cos(\phi' - \phi)\]

\[
= 0.1118 + 0.1110 \cos(\phi' - \phi). \tag{3.36}
\]

Interference fringes with almost 100% visibility are obtained. Thus, two-atom interferometry could produce perfect visibility in the situations where single-atom interferometry exhibits no interferences. Other joint detection probabilities like finding one atom in the excited state and the other in the ground state also display interference fringes. An interesting situation also corresponds to sending both atoms in the excited state and measuring the final states of the two atoms. In the case when initially cavities are in the state \( | \psi \rangle \) and \( \Delta = 0 \), the expression for the probability of detecting both the atoms in their ground states has the form

\[
P_{11/2}^{\pi/2} = \sin^2\left(g_1 \tau_1 \right) + \sin^2\left(g_2 \tau_2 \right) + \cos\left(\pi/2\right) \cos\left(\phi' - \phi\right)
\]

Consider the case when \( g_1 \tau_1 = g_2 \tau_2 = \pi \) and \( g_1 = g_2 = 4 \). The probability of detecting both atoms in the ground states is given by

\[
P_{11/2}^{\pi/2} = 0.1327 + 0.3835 \cos(\phi' - \phi). \tag{3.38}
\]

The visibility of fringes in two-atom interferometry is quite significant. We next show how two-atom interferometry can be used to produce a variety of entangled states.

3.4.1 Preparation of the entangled state \( \left| \alpha, 2, 0 \right> + \left| \alpha', 0, 2 \right> \)

Consider the situation when two identical atoms are coming in their excited states and each cavity is in vacuum state. The mode of each cavity is in resonance with the atomic transition frequency. If after passing through the cavities both atoms are detected in their
ground states, the state of the field inside the cavities is given by

$$
|\Phi(\tau_1 + T + \tau_2)\rangle = \sin(g_1 \tau_1) \sin(g_1 \sqrt{2} \tau_1') \exp\{-i(\omega_1 + \frac{\omega_2}{2})\} |0,0\rangle
+ \cos(g_1 \tau_1) \cos(g_1 \tau_1') \sin(g_2 \sqrt{2} \tau_2') \exp\{-i\omega_2\} |0,1\rangle
+ \left[ \cos(g_1 \tau_1) \sin(g_2 \sqrt{2} \tau_2') \sin(g_2 \sqrt{2} \tau_2) \exp\{-i\omega_2\} \sin\left(\frac{\omega_2}{2}\right) \right] |1,0\rangle
+ \sin(g_1 \tau_1) \cos(g_2 \sqrt{2} \tau_2') \exp\{i\omega_2\} \sin\left(\frac{\omega_2}{2}\right) |1,1\rangle.
$$

(3.38)

The $|1,1\rangle$ component drops out for $g_1 \tau_1 = g_2 \tau_2 = \pi$ and the cavities will be in the entangled state

$$
|\Phi(\tau_1 + T + \tau_2)\rangle = \sin(g_1 \tau_1) \sin(g_1 \sqrt{2} \tau_1') \exp\{-i\omega_1\} |0,0\rangle
- \cos(g_1 \tau_1) \sin(g_2 \sqrt{2} \tau_2') \sin(g_2 \sqrt{2} \tau_2) \exp\{-i\omega_2\} |0,1\rangle
+ \left[ \cos(g_1 \tau_1) \sin(g_2 \sqrt{2} \tau_2') \sin(g_2 \sqrt{2} \tau_2) \exp\{-i\omega_2\} \sin\left(\frac{\omega_2}{2}\right) \right] |1,0\rangle
+ \sin(g_1 \tau_1) \cos(g_2 \sqrt{2} \tau_2') \exp\{i\omega_2\} \sin\left(\frac{\omega_2}{2}\right) |1,1\rangle.
$$

(3.40)

This entangled state is very interesting; we can change the degree of entanglement by changing the value of $g_1 \tau_1$ and $g_2 \tau_2$. The state will be maximally entangled for $g_1 \tau_1 = \pi$ and $g_2 \tau_2 = \pi/2$. This can be seen as the first atom comes in excited state $|e\rangle$, interacts with the first cavity for a time such that $g_1 \tau_1 = \pi$. The interaction is like an interaction with a $\pi/2$ pulse; the state of the system evolves into

$$
|e,0,0\rangle + \frac{1}{\sqrt{2}} |e,1,0\rangle.
$$

(3.41)

In the second cavity, the interaction is a $\pi$ pulse interaction and then the state of the total system becomes,

$$
\frac{1}{\sqrt{2}} (|e,0,0\rangle - i |g,1,0\rangle) - \frac{i}{\sqrt{2}} (|g,0,1\rangle - |e,0,1\rangle).
$$

(3.42)

Thus, after passing the first atom, the state of the fields in the cavities is

$$
|\psi_1\rangle = -\frac{i}{\sqrt{2}} (|1,0\rangle + |0,1\rangle).
$$

(3.43)

The second atom comes in the excited state $|e\rangle$, interacts with the fields inside the cavities for times $\tau_1'$ and $\tau_2'$ such that $g_1 \tau_1' = g_2 \tau_2' = \pi$, and after passing through the cavities the atom is detected in the ground state $|g\rangle$, so the atom can follow the two paths. The first path is

$$
|e\rangle |1,0\rangle \rightarrow \cos(\sqrt{2}) |e,1,0\rangle - i \sin(\sqrt{2}) |g,2,0\rangle \rightarrow -i \sin(\sqrt{2}) |g,2,0\rangle - \cos(\sqrt{2}) |e,1,0\rangle,
$$

(3.44)
The second path is

$$|e \rangle |0, 1 \rangle \rightarrow -|e, 0, 1 \rangle \rightarrow i \sin(\pi \sqrt{2})|g, 0, 2 \rangle \rightarrow \cos(\pi \sqrt{2})|e, 0, 1 \rangle,$$

(3.45)

Thus, passing both the atoms initially in the excited states and subsequently detecting them in their ground states, we obtain a maximally entangled state

$$|\psi_2 \rangle = \frac{1}{\sqrt{2}} \sin(\pi \sqrt{2}) \left( \exp[-i (\phi_2 + \phi_3)] e^{i (\phi_2 + \phi_3)} \right) |2, 0 \rangle = \exp[-i (\phi_2 + \phi_3)] |2, 0 \rangle,$$

(3.46)

of the fields inside the cavities. The phase terms in Eq. 3.46 come from the phase change in the region between the cavities. Now, if we pass another atom initially in the excited state |e⟩ through the cavities having field in state 3.46 and the atom is detected in its ground state |g⟩ after passing through the cavities, an entangled state of three photons is generated. The degree of entanglement is controlled by the selection of interaction times in the cavities. For a special case, a three photons maximally entangled state

$$|\psi_3 \rangle = \frac{1}{\sqrt{2}} \sin(\pi \sqrt{2}) \sin(\pi \sqrt{3}) \left[ e^{-i (\phi_2 + \phi_3)} e^{i (\phi_2 + \phi_3)} \right] |3, 0 \rangle = \exp[-i (\phi_2 + \phi_3)] |3, 0 \rangle,$$

(3.47)

is generated if we choose interaction times $\tau_1$ and $\tau_2$ for third atom such that $\phi_1 = \phi_2 \tau_2 = \pi$.

3.4.2 Entanglement Transfer from Fields to Atoms

Here we show how entanglement of fields [50, 55] is transferred to the atoms. For this purpose, consider the fields inside the cavities are in an entangled state:

$$|\psi_f \rangle = \alpha |0, 1 \rangle - \beta |1, 0 \rangle.$$

(3.48)

An atom initially in the ground state |g⟩ is passed through the cavities and the fields inside the cavities are in resonance with atomic transition frequency; then the state of the cavity-atom system is

$$|\psi_4 \rangle = \left\{ \alpha \cos(g_1 \tau_1) e^{-i \phi_2} - \beta \sin(g_1 \tau_1) \sin(g_2 \tau_2) e^{-i \phi_2} \right\} |g, 0, 1 \rangle + \beta \cos(g_1 \tau_1) e^{-i \phi_2} |g, 1, 0 \rangle$$

$$- i \left\{ \alpha \sin(g_1 \tau_1) e^{-i \phi_2} + \beta \sin(g_1 \tau_1) \cos(g_2 \tau_2) e^{-i \phi_2} \right\} |e, 0, 0 \rangle.$$

(3.49)

If another atom coming in the ground state |g⟩, interacts with the fields in both the cavities for the times $\tau_1$ and $\tau_2$ such that $g_1 \tau_1 = g_2 \tau_2 = \pi/2$, then the state of the cavity-atom system
\[ |\psi_3\rangle = -i \left( \alpha \cos(g_2 \tau_2) e^{-i(\phi_1 + \phi_0)} + \beta \cos(g_1 \tau_1) \cos(g_2 \tau_2) e^{-i(\phi_1 + \phi_0)} \right) g, 0, 0 \]
\[ - i \left( \alpha \cos(g_2 \tau_2) e^{-i(\phi_1 + \phi_0)} + \beta \sin(g_1 \tau_1) \sin(g_2 \tau_2) e^{-i(\phi_1 + \phi_0)} \right) g, \tau, 0, 0 \]
\[ - \beta \cos(g_1 \tau_1) \cos(g_2 \tau_2) e^{-i(\phi_1 + \phi_0)} (g, \tau, 1, 0). \tag{3.50} \]

If we choose the interaction time for first atom in first cavity such that \( g_1 > \tau \), the state (3.50) becomes
\[ |\psi_3\rangle = -i \left( \alpha \sin(g_2 \tau_2) e^{-i(\phi_1 + \phi_0)} + \beta \cos(g_1 \tau_1) \cos(g_2 \tau_2) e^{-i(\phi_1 + \phi_0)} \right) g, 0, 0, 0 \]
\[ - i \left( \alpha \cos(g_2 \tau_2) e^{-i(\phi_1 + \phi_0)} + \beta \sin(g_1 \tau_1) \sin(g_2 \tau_2) e^{-i(\phi_1 + \phi_0)} \right) g, \tau, 0, 0, 0 \]
\[ - \beta \cos(g_1 \tau_1) \cos(g_2 \tau_2) e^{-i(\phi_1 + \phi_0)} (g, \tau, 1, 0, 0). \tag{3.51} \]

State (3.51) shows that the atoms are now in entangled state and fields are in independent states, so the entanglement of fields has been transferred to the atoms.

### 3.5 Summary

We have discussed in detail the theory of Ramsey interferometry with quantized fields. The interference is very sensitive to the quantum statistics of the fields in the two Ramsey zones. We have derived general conditions for interference to occur. We have shown how an analog of Hanbury-Brown Twiss photon-photon correlation interferometry can be used to discern a variety of interference effects even in situations where the single atom detection probabilities do not exhibit interferences. We have demonstrated atoms acting as a mediator for photon-photon interaction between two cavities and entanglement can be transferred from fields to atoms. We have generated entangled state of two and three photons by passing two and three atoms through the cavities.