CHAPTER 4

Analysis and Comparison of Some Retrial Inventory Models

4.1. Introduction

Retrial queues (queues with repeated calls, returning customers *etc.*) are a type of network with re-servicing after blocking. Inventory systems in which arriving customers who find all items are out of stock, may retry for the items after a period of time, are called retrial inventory. Artalejo, Krishnamoorthy and Lopez-Herrero [9] were the first to attempt to study inventory policies with positive lead time and retrial of customer who could not get the items during their earlier attempts. In 2007, Parthasarathy and Sudheesh [56] obtained transient solution using continued fraction approach to a single server retrial queue in which arrival and retrial rates are state dependent.

This chapter is an extension of the last chapter. Here we introduce retrial of unsatisfied customers into the models discussed in chapter 3, with the assumption that there is no waiting space for the customers at the service station except the one under going service. Customers arrive to a single server system according to a Poisson process with rate $\lambda$ and service times are exponentially distributed with parameter $\mu$. One unit of item is demanded by each customer. An order for replenishment of $Q = S - s$ quantity of goods is placed when the inventory level depletes to $s$. The lead time follows an exponential distribution with parameter $\beta$. An arriving customer who finds the server busy, proceeds to an orbit of infinite capacity and tries its luck to access the server from there. The inter-retrial times follow an exponential distribution with constant rate $\theta$. In Model I, customers do not join the orbit when the inventory level is zero. In Model II, customers join the orbit even when the inventory level is zero. In Model III and IV it is assumed that a local purchase of one unit and $s$ units of the item, respectively, at a higher cost if a customer (orbital customer or primary customer) enters for service when the inventory level is zero. In Model V, under the same situation a local purchase
of $S$ units of the item is made cancelling the existing order. The time required to make a local purchase is assumed to be negligible. Local purchase is made to decrease the waiting time of the customers thereby earning the goodwill of the customers.

4.2. Mathematical Formulation of Model I

Problem I is described as follows: Arrival of customer to a single server system forms a Poisson process with rate $\lambda$. Service times are identically and independently distributed exponential random variables with parameter $\mu$. When the inventory level depletes to $s$ due to demands, an order for replenishment for $Q = S - s$ quantity is placed where $S$ is the maximum capacity of the system. The lead time is exponentially distributed with parameter $\beta$. An arriving customer, who finds the server busy, proceeds to an orbit of infinite capacity and tries its luck from there. Customers do not join the orbit when the inventory level is zero. The inter retrial times follow an exponential distribution with parameter $\theta$. It is assumed that retrial rate is the same, independent of the number of customers in the orbit. This is possible, for example, by assuming that a queue of customers is formed in the orbit (see Gomez-Corral [20])

Let $N(t)$ be the number of customers in the orbit, $I(t)$ be the inventory level and $C(t)$, the server state at time $t$.

Here $C(t) = \begin{cases} 1 & \text{if the server is busy} \\ 0 & \text{if the server is idle} \end{cases}$

Then $\{(N(t), C(t), I(t)), t \geq 0\}$ is a Continuous Time Markov Chain (CTMC) on the state space $\{(i, 0, j), 0 \leq j \leq S\} \cup \{(i, 1, j), 1 \leq j \leq S\}, i \geq 0$. The above model can be studied as Linearly Independent Quasi-Birth and Death (LIQBD) process. The infinitesimal generator $\overline{Q}$ of the process has the following form
\[ \bar{Q} = \begin{bmatrix} A_{10} & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_{11} & A_0 & 0 & 0 & \cdots \\ 0 & A_2 & A_{11} & A_0 & 0 & \cdots \\ 0 & 0 & A_2 & A_{11} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \] (4.2.1)

where \( A_{10}, A_{11}, A_0, A_2 \) are square matrices of order \((2S + 1)\) and they are given by

\[ A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda I_S \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & E_1 \\ 0 & 0 \end{bmatrix} \]

where 
\[ E_1 = \begin{bmatrix} 0 \\ \theta I_S \end{bmatrix}_{(S+1) \times S} \]

\[ A_{1i} = \begin{bmatrix} M_1 & 0 & M_2 & M_3 & 0 & 0 \\ 0 & M_4 & 0 & 0 & M_5 & 0 \\ 0 & 0 & M_6 & 0 & 0 & M_7 \\ M_8 & 0 & 0 & M_9 & 0 & M_{10} \\ M_{11} & M_{12} & 0 & 0 & M_{13} & 0 \\ 0 & M_{14} & M_{15} & 0 & 0 & M_{16} \end{bmatrix} \quad i = 0, 1 \] (4.2.2)

where \( M_1 \) is a square matrix of order \((s + 1)\) whose non-zero entries are given by 
\( M_1(1, 1) = -\beta \) and \( M_1(j, j) = -(\lambda + \beta + i\theta), \) \( j = 2 \) to \( s + 1, \)

\( M_2 \) is a square matrix of order \((s + 1)\) whose non-zero entries are given by \( M_2(j, j) = \beta, \)
\( j = 1 \) to \( s + 1, \)

\( M_3 \) is of order \((s + 1) \times s\) whose non-zero entries are given by \( M_3(j + 1, j) = \lambda, \) for 
\( j = 1 \) to \( s, \)

\( M_4 \) is a square matrix of order \((S - 2s - 1)\) where the non-zero entries are given by 
\( M_4(j, j) = -(\lambda + i\theta), \) \( j = 1 \) to \( S - 2s - 1, \)

\( M_5 \) is a square matrix of order \((S - 2s - 1)\) whose non-zero entries are given by 
\( M_5(j, j) = \lambda, \) \( j = 1 \) to \( S - 2s - 1, \)

\( M_6 \) is a square matrix of order \((s + 1)\) where the non-zero elements are given by
$M_6(j, j) = -(\lambda + i\theta),$

$M_7$ is a square matrix of order $(s+1)$ whose non-zero entries are given by $M_7(j, j) = \lambda,$

$M_8$ is of order $s \times (s+1)$ whose non-zero elements are given by $M_8(j, j) = \mu,$ $j = 1$ to $s,$

$M_9$ is a square matrix of order $s$ whose non-zero entries are given by $M_9(j, j) = -\left(\lambda + \mu + \beta\right),$ $j = 1$ to $s,$

$M_{10}$ is of order $s \times (s+1)$ where non-zero entries are given by $M_{10}(j, j+1) = \beta,$ $j = 1$ to $s,$

$M_{11}$ is of order $(S - 2s - 1) \times (s+1)$ whose non-zero entries are given by $M_{11}(1, s+1) = \mu,$

$M_{12}$ is a square matrix of order $(S - 2s - 1)$ where non-zero elements are given by $M_{12}(j+1, j) = \mu,$ $j = 1$ to $S - 2s - 2,$

$M_{13}$ is a square matrix of order $(S - 2s - 1)$ whose non-zero entries are given by $M_{13}(j, j) = -(\lambda + \mu),$ $j = 1$ to $S - 2s - 1,$

$M_{14}$ is of order $(s+1) \times (S - 2s - 1)$ where non-zero elements are given by $M_{14}(1, S - 2s - 1) = \mu,$

$M_{15}$ is a square matrix of order $(s + 1)$ whose non-zero entries are given by $M_{15}(j+1, j) = \mu,$ $j = 1$ to $s,$

$M_{16}$ is a square matrix of order $(s + 1)$ where non-zero entries are given by $M_{16}(j, j) = -(\lambda + \mu),$ $j = 1$ to $s + 1.$

4.3. Mathematical Formulation of Model II

The only difference of this model from the first one is that customers join the orbit even when the inventory level is zero. Here also $\{(N(t), C(t), I(t)), t \geq 0\}$ is a CTMC on the state space $\{(i, 0, j), 0 \leq j \leq S\} \cup \{(i, 1, j), 1 \leq j \leq S\}, i \geq 0.$ Then the generator has the form (4.2.1) where $A_{10}, A_{11}, A_0, A_2$ are square matrices of order $(2S + 1)$ and they are given by $A_0 = \begin{bmatrix} E_2 & 0 \\ 0 & \lambda I_s \end{bmatrix}$ where $E_2 = \begin{bmatrix} \lambda e_1 \\ 0 \end{bmatrix}_{(S+1) \times (S+1)},$

where $e_j$ is a row vector with 1 in the $j$th place and zeros elsewhere. $A_2$ is the same
as in the problem I, and $A_{1i}, i = 0, 1$ have the form of (4.2.2) where all the sub-
matrices, except $M_1$ is same as in the first model and here $M_1(1, 1) = -(\lambda + \beta)$ and
$M_1(j, j) = -(\lambda + \beta + i\theta), j = 2 \text{ to } s + 1$ and $M_1$ is of order $(s + 1) \times (s + 1)$

4.4. Analysis of Models I and II

4.4.1. System stability. Define the generator matrix $A$ (for each model) as $A = A_0 + A_{11} + A_2$ and
$\pi = (\pi(0, 0), \pi(0, 1), \cdots, \pi(0, S), \pi(1, 1), \pi(1, 2), \cdots, \pi(1, S))$ where
$\pi$ is a the steady state probability vector of $A$. From the relation $\pi A = 0$ we get the
following solution:

$$
\pi(1, k) = \begin{cases}
\left(\frac{\lambda + \beta + \theta}{\mu}\right)^{k-1} \left(\frac{\mu + \beta}{\lambda + \theta}\right)^{k-1} \frac{\beta}{\mu} \pi(0, 0), & \text{for } k = 1, 2, \ldots, s \\
\left(\frac{\lambda + \beta + \theta}{\mu}\right)^{s} \left(\frac{\mu + \beta}{\lambda + \theta}\right)^{s} \frac{\beta}{\mu} \pi(0, 0), & \text{for } k = s + 1, \ldots, Q
\end{cases}
$$

$$
\pi(1, Q + k) = \pi(1, Q) - \pi(1, k), \ k = 1, 2, \ldots, s
$$

$$
\pi(0, k) = \begin{cases}
\left(\frac{\lambda + \beta + \theta}{\mu}\right)^{k-1} \left(\frac{\mu + \beta}{\lambda + \theta}\right)^{k} \frac{\beta}{\mu} \pi(0, 0), & \text{for } k = 1, 2, \ldots, s \\
\left(\frac{\lambda + \beta + \theta}{\mu}\right)^{s} \left(\frac{\mu + \beta}{\lambda + \theta}\right)^{s} \frac{\beta}{\mu} \pi(0, 0), & \text{for } k = s + 1, \ldots, Q
\end{cases}
$$

$$
\pi(0, Q + k) = \frac{\mu}{\lambda + \theta} \pi(1, Q) - \pi(0, k), \ k = 1, 2, \ldots, s
$$

$\pi(0, 0)$ can be obtained from $\pi e = 1$

**Theorem 4.4.1.** The system in model I is stable if and only if $\lambda^2 < \theta(\mu - \lambda)$. The
system in model II is stable if and only if

$$
\lambda \left(\frac{\mu}{\lambda + \beta + \theta}\right)^s < Q \left(\frac{\beta + \mu}{\lambda + \theta}\right)^s \beta \left(\frac{\theta}{\lambda + \theta} - \frac{\lambda}{\mu}\right).$
$$

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PROOF. For the positive recurrence of $\overline{Q}$ we must have $\pi A_0 e < \pi A_2 e$ (see Neuts [53]). Simplifying this we get the indicated results.

4.4.2. Steady-state analysis. Let $X = (x(0), x(1), \ldots)$ be the steady state probability vector of $\overline{Q}$. Then $X \overline{Q} = 0$ together with $X e = 1$ result in $x(i)$ having the matrix geometric solution:

$$x(i) = x(1) R^{i-1} \text{ for } i \geq 2$$  \hspace{1cm} (4.4.1)

where $R$ is the minimal non negative solution of the matrix equation $A_0 + RA_{11} + R^2 A_2 = 0$. $x(0)$ and $x(1)$ are calculated from the equations

$$x(0)A_{10} + x(1)A_2 = 0$$  \hspace{1cm} (4.4.2)

$$x(0)A_0 + x(1)(A_{11} + RA_2) = 0$$  \hspace{1cm} (4.4.3)

subject to the normalizing condition $X e = 1$,

that is, $x(0) e + x(1)(1 - R)^{-1} = 1$.

Having found, $x(1)$ we can find $x(i), i \geq 2$ from (4.4.1).

4.5. System Performance Measures

Let $X = (x(0), x(1), \ldots)$ be the steady-state probability vector of $\overline{Q}$ (for each model) and $x(i), i \geq 0$, be partitioned as

$$x(i) = (y_{i00}, y_{i01}, \ldots, y_{i0S}, y_{i11}, y_{i12}, \ldots y_{i1S})$$

Then we have the following performance measures.

(1) Expected number of customers in the orbit EC is given by

$$\text{EC} = \sum_{i=1}^{\infty} ix(i) e$$

(2) Expected inventory level EI is given by

$$\text{EI} = \sum_{i=1}^{\infty} \sum_{j=1}^{S} j(y_{i0j} + y_{i1j})$$
(3) Expected re-order rate ER is given by

\[ ER = \mu \sum_{i=1}^{\infty} y_{i,1,s+1} \]

(4) Overall retrial rate OR is given by

\[ OR = \theta \sum_{i=1}^{\infty} x(i) e \]

(5) Successful retrial rate SR is given by

\[ SR = \theta \sum_{i=1}^{\infty} \sum_{j=1}^{S} y_{i,0,j} \]

(6) Probability that the server is busy is given by

\[ P(B) = \sum_{i=1}^{\infty} \sum_{j=1}^{S} y_{i1j} \]

Model I

(7) Expected waiting time EW is given by EW = \( \frac{EC}{\lambda} \).

Model II

(8) Expected waiting time EW is given by EW

\[ = \frac{EC}{\lambda[1 - \sum_{i=0}^{\infty} y_{i00}]} \]

Model I

(9) Expected number of customers EJ not joining the orbit when the inventory level is zero, is given by

\[ EJ = \lambda \sum_{i=0}^{\infty} y_{i00} \]

4.6. Cost Function

To construct cost function we define the costs as follows:

\[ C' = \text{fixed ordering cost} \]
$C_1 = \text{procurement cost/unit}$

$C_2 = \text{holding cost of inventory/unit/unit time}$

$C_3 = \text{shortage cost of inventory/unit/unit time}$

The total expected cost function $ETC$ is given as follows:

**Model I**

$ETC = [C + QC_1]ER + C_2EI + C_3EJ$

**Model II**

$ETC = [C + QC_1]ER + C_2EI$

### 4.7. Mathematical Formulation of Model III

In addition to assumptions in problem II, here a local purchase of one unit of the commodity is made if a customer enters for service when the inventory level is zero. Let $N(t)$ be the number of customers in the orbit, $I(t)$ be the inventory level and $C(t)$ be the server state at time $t$.

$$C(t) = \begin{cases} 
1 & \text{if the server is busy} \\
0 & \text{if the server is idle}
\end{cases}$$

Then $\{ (N(t), C(t), I(t)), t \geq 0 \}$ is a CTMC on the state space.

$$\{(i, 0, j), 0 \leq j \leq S\} \cup \{(i, 1, j), 1 \leq j \leq S\}, \ i \geq 0.$$ 

Then the generator has the form (4.2.1) where $A_{10}, A_{11}, A_0, A_2$ are square matrices of order $(2S + 1)$ and they are given by

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda I_S \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & E_3 \\ 0 & 0 \end{bmatrix}, \quad \text{where } E_3 = \begin{bmatrix} \theta & 0 & \cdots & 0 \\ \theta & 0 & \cdots & 0 \\ 0 & \theta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta \end{bmatrix}_{(S+1) \times S}.$$
$A_{i1}, i = 0, 1$ is given by (4.2.2) in which all the sub-matrices except $M_1$ and $M_3$ are same and they are given as follows: $M_1$ a square matrix of order $(s + 1)$ whose non-zero entries are given by $M_1(j, j) = -(\lambda + \beta + i\theta), j = 1$ to $s + 1$; $M_3$ is of order $(s + 1) \times s$ where non-zero entries are given by $M_3(1, 1) = \lambda$ and $M_3(j + 1, j) = \lambda, j = 1$ to $s$.

**4.8. Mathematical Formulation of Model IV**

In this model we make a local purchase of $s$ units of inventory if a customer enters for service when the inventory level is zero. Here also $\{(N(t), C(t), I(t)), t \geq 0\}$ is a CTMC with the state space

$\{(i, 0, j), 0 \leq j \leq S\} \cup \{(i, 1, j), 1 \leq j \leq S\}, i \geq 0$.

The infinitesimal generator $\overline{Q}$ has the form of (4.2.1) where $A_{10}, A_{11}, A_0, A_2$ are square matrices of order $(2S + 1)$ and they are given by

$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda I_s \end{bmatrix}, A_2 = \begin{bmatrix} 0 & E_4 \\ 0 & 0 \end{bmatrix}$ where

$E_4 = \begin{bmatrix} 1 & 2 & \cdots & s & \cdots & S \\ 0 & 0 & \cdots & \theta & \cdots & 0 \\ 1 & \theta & 0 & \cdots & 0 & \cdots \\ 2 & 0 & \theta & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S & 0 & 0 & \cdots & 0 & \cdots \theta \end{bmatrix}_{(S+1)\times S}$

$A_{i1}, i = 0, 1$ is given by (4.2.2), where all sub-matrices except $M_1$ and $M_3$ are same in the first model and they are given as follows.

$M_1$ is a square matrix of order $(s + 1)$ whose non-zero entries are given by

$M_1(j, j) = -(\lambda + \beta + i\theta), j = 1$ to $s + 1$. 

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$M_3$ is of order $(s + 1) \times s$ whose non zero entries are given by

$M_3(1, s) = \lambda, M_3(j + 1, j) = \lambda, j = 1$ to $s$.

### 4.9. Mathematical Formulations of Model V

The main difference of this model from third and fourth model is that here we make a local purchase of maximum capacity of inventory $S$ units, if a customer enters for service while the inventory is zero, which results in the cancellation of the existing order, as the maximum capacity of the inventory is $S$. The infinitesimal generator $\overline{Q}$ has the form of (4.2.1) where $A_{10}, A_{11}, A_2, A_0$ are square matrices of order $(2S + 1)$ they are given by

$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda I_S \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & E_5 \\ 0 & 0 \end{bmatrix}$

where

$E_5 = \begin{bmatrix} 0 & 0 & \ldots & \theta \\ \theta & 0 & \ldots & 0 \\ 0 & \theta & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \theta \end{bmatrix}_{(S+1) \times S}$

$A_{0i} = \begin{bmatrix} M_1 & 0 & M_2 & M_3 & 0 & M_{17} \\ 0 & M_4 & 0 & 0 & M_5 & 0 \\ 0 & 0 & M_6 & 0 & 0 & M_7 \\ M_8 & 0 & 0 & M_9 & 0 & M_{10} \\ M_{11} & M_{12} & 0 & 0 & M_{13} & 0 \\ 0 & M_{14} & M_{15} & 0 & 0 & M_{16} \end{bmatrix}$

$i = 0, 1.$

where all the sub matrices $M_2$ to $M_{16}$ is same as those in Model I. $M_1$ and $M_{17}$ are given as follows:

$M_1$ is a square matrix of order $(s + 1)$ where the non zero entries are given by

$M_1(j, j) = -(\lambda + \beta + i\theta), j = 1$ to $s + 1$. 

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$M_{17}$ is also a square matrix of order $(s + 1)$ whose only non zero entry is given by $M_{17}(1, s + 1) = \lambda$.

4.10. Analysis of Models III, IV and V

4.10.1. System stability. Define the generator matrix $A$ (for each model) as $A = A_0 + A_{11} + A_2$ and $\pi = (\pi(0, 0), \pi(0, 1), \cdots, \pi(0, S), \pi(1, 1), \pi(1, 2), \cdots, \pi(1, S))$ where $\pi$ is the steady state probability vector of $A$. From the relation $\pi A = 0$ we get the following values for each model.

Model III.

$$\pi(1, 1) = \frac{\lambda + \beta + \theta}{\mu} \pi(0, 0),$$

$$\pi(1, k) = \begin{cases} \left(\frac{\lambda + \beta + \theta}{\mu}\right)^{k-1} \left(\frac{\mu + \beta}{\lambda + \theta}\right)^{k-1} \left[\left(\frac{\mu + \beta}{\lambda + \theta}\right) \left(\frac{\lambda + \beta + \theta}{\mu}\right) - 1\right] \pi(0, 0), & k = 2, \ldots, s \\
\left(\frac{\lambda + \beta + \theta}{\mu}\right)^s \left(\frac{\mu + \beta}{\lambda + \theta}\right)^{s-1} \left[\left(\frac{\mu + \beta}{\lambda + \theta}\right) \left(\frac{\lambda + \beta + \theta}{\mu}\right) - 1\right] \pi(0, 0), & k = s + 1, \ldots, Q \end{cases}$$

$$\pi(1, Q + k) = \pi(1, Q) + \frac{\lambda + \theta}{\mu} \pi(0, 0) - \pi(1, k)$$

$$k = 1, 2, \ldots, s,$$

$$\pi(0, k) = \begin{cases} \left(\frac{\lambda + \beta + \theta}{\mu}\right)^{k-1} \left(\frac{\mu + \beta}{\lambda + \theta}\right)^{k-1} \left[\left(\frac{\mu + \beta}{\lambda + \theta}\right) \left(\frac{\lambda + \beta + \theta}{\mu}\right) - 1\right] \pi(0, 0), & k = 1, 2, \ldots, s \\
\left(\frac{\lambda + \beta + \theta}{\lambda + \theta}\right)^s \left(\frac{\mu + \beta}{\mu}\right)^{s-1} \left[\left(\frac{\mu + \beta}{\lambda + \theta}\right) \left(\frac{\lambda + \beta + \theta}{\mu}\right) - 1\right] \pi(0, 0), & k = s + 1, \ldots, Q \end{cases}$$

$$\pi(0, Q + k) = \frac{\mu}{\lambda + \theta} \pi(1, Q) + \pi(0, 0) - \pi(0, k), \ k = 1, 2, \ldots, s - 1$$

$$\pi(0, S) = \frac{\mu}{\lambda + \theta} \pi(1, Q) - \pi(0, s)$$
Model IV.

\[
\pi(1, k) = \begin{cases} 
\left(\frac{\lambda+\beta+\theta}{\mu}\right)^k \left(\frac{\mu+\beta}{\lambda+\theta}\right)^{k-1} \pi(0, 0), \\
\text{for } k = 1, 2, \ldots, s \\
\left(\frac{\lambda+\beta+\theta}{\mu}\right) \left[\left(\frac{\mu+\beta}{\lambda+\theta}\right)^s \left(\frac{\lambda+\beta+\theta}{\mu}\right)^s - 1\right] \pi(0, 0), \\
\text{for } k = s + 1, \ldots, Q 
\end{cases}
\]

\[
\pi(1, Q + k) = \pi(1, Q) + \frac{\lambda + \theta}{\mu} \pi(0, 0) - \pi(1, k), \text{ for } k = 1, \ldots, s
\]

\[
\pi(0, k) = \left(\frac{\lambda + \beta + \theta}{\mu}\right)^k \left(\frac{\mu + \beta}{\lambda + \theta}\right)^k \pi(0, 0), \text{ for } k = 1, \ldots, s - 1
\]

\[
\pi(0, s) = \left[\left(\frac{\mu + \beta}{\lambda + \theta}\right)^s \left(\frac{\lambda + \beta + \theta}{\mu}\right)^s - 1\right] \pi(0, 0),
\]

\[
\pi(0, k) = \frac{\lambda + \beta + \theta}{\lambda + \theta} \pi(0, s), \text{ for } k = s + 1, \ldots, Q
\]

\[
\pi(0, Q + k) = \frac{\mu}{\lambda + \theta} \pi(1, Q) + \pi(0, 0) - \pi(0, k), \text{ for } k = 1, \ldots, s - 1
\]

\[
\pi(0, S) = \frac{\mu}{\lambda + \theta} \pi(1, Q) - \pi(0, s)
\]

Model V.

\[
\pi(1, k) = \begin{cases} 
\left(\frac{\lambda+\beta+\theta}{\mu}\right)^k \left(\frac{\mu+\beta}{\lambda+\theta}\right)^{k-1} \pi(0, 0), \\
\text{for } k = 1, \ldots, s \\
\left(\frac{\lambda+\beta+\theta}{\mu}\right)^{s+1} \left(\frac{\mu+\beta}{\lambda+\theta}\right)^s \pi(0, 0), \\
\text{for } k = s + 1, \ldots, Q 
\end{cases}
\]

\[
\pi(1, Q + k) = \pi(1, Q) + \frac{\lambda + \theta}{\mu} \pi(0, 0) - \pi(1, k), \text{ for } k = 1, \ldots, s
\]

\[
\pi(0, k) = \begin{cases} 
\frac{\mu}{\lambda + \theta} \left(\frac{\lambda+\beta+\theta}{\mu}\right)^k \pi(0, 0), \text{ for } k = 1, \ldots, s \\
\left(\frac{\lambda+\beta+\theta}{\mu}\right)^{s+1} \left(\frac{\mu+\beta}{\lambda+\theta}\right)^s \pi(0, 0), \\
\text{for } k = s + 1, \ldots, Q 
\end{cases}
\]
\[ \pi(0, Q + k) = \frac{\mu}{\lambda + \theta} \pi(1, Q) + \pi(0, 0) - \pi(0, k), \ k = 1, \ldots, s - 1 \]
\[ \pi(0, S) = \frac{\mu}{\lambda + \theta} \pi(1, Q) - \pi(0, s) \]

We can find \( \pi(0, 0) \) from the equation \( \pi e = 1 \). Here \( Q = S - s \).

**Theorem 4.10.1.** The systems in models 3 to 5 are stable if and only if

\[ \lambda^2 < \theta(\mu - \lambda) \]

**Proof.** For the positive recurrence of \( \overline{Q} \) we must have \( \pi A_0 e < \pi A_2 e \) (see Neuts [53]). Simplifying this leads to the above condition. \( \square \)

**4.10.2. Steady-state analysis.** Let \( X = (x(0), x(1), \ldots) \) be the steady state probability vector of \( \overline{Q} \) (for each model). Then \( X \overline{Q} = 0, \ X e = 1 \) and \( x(i) \) are given by

\[ x(i) = x(1) R^{i-1} \text{ for } i \geq 2 \]  

(4.10.1)

where \( R \) is the minimal non negative solution of the matrix equation \( A_0 + RA_{11} + R^2 A_2 = 0 \). \( x(0) \) and \( x(1) \) are calculated from the equation

\[ x(0) A_{10} + x(1) A_2 = 0 \] 

(4.10.2)

\[ x(0) A_0 + x(1) (A_{11} + RA_2) = 0 \] 

(4.10.3)

subject to the normalizing condition \( X e = 1 \),

That is, \( x(0) e + x(1) (1 - R)^{-1} e = 1 \). Then we can find \( x(i), \ i \geq 2 \) from (4.10.1)

**4.11. System Performance Measures**

Let \( X = (x(0), x(1), \ldots) \) be the steady-state probability vector of \( \overline{Q} \) (for each model) and \( x(i), \ i \geq 0 \) partitioned as

\[ x(i) = (y_{i00}, y_{i01}, \ldots, y_{i0S}, y_{i11}, y_{i12}, \ldots, y_{i1S}) \].

Then we have the following performance measures:
(1) Expected number of customers $EC$ in the orbit is given by

$$EC = \sum_{i=1}^{\infty} ix(i)e$$

(2) Expected inventory level $EI$ is given by

$$EI = \sum_{i=1}^{\infty} \sum_{j=1}^{S} j(y_{i0j} + y_{i1j})$$

(3) Expected re-order rate $ER$ is given by

$$ER = \mu \sum_{i=1}^{\infty} y_{i1,s+1}$$

(4) Overall retrial rate $OR$ is given by

$$OR = \theta \sum_{i=1}^{\infty} x(i)e$$

(5) Successful retrial rate $SR$ is given by

$$SR = \theta \sum_{i=1}^{\infty} \sum_{j=0}^{S} y_{i0j}$$

(6) Probability that the server is busy is given by

$$P(B) = \sum_{i=1}^{\infty} \sum_{j=1}^{S} y_{i1j}$$

(7) Expected waiting time $EW$ is given by $EW = \frac{EC}{\lambda}$.

(8) Expected rate of local purchase $EL$ is given by

$$EL = \lambda \sum_{i=0}^{\infty} y_{i00} + \theta \sum_{i=1}^{\infty} y_{i00}$$
4.12. Cost Function and Numerical Examples

To construct the cost function we define the cost as follows:

\[ C = \text{fixed ordering cost} \]
\[ C_1 = \text{procurement cost/unit} \]
\[ C_2 = \text{holding cost of inventory/unit/unit time} \]

\[(1 + k)C_1 z E L = \text{total cost of local purchase of } z \text{ units of inventory with a hike of } k \text{ times } C_1/\text{unit.}\]

In model V as we make a local purchase of \( S \) units and thus cancelling the existing order, the system loses the ordering cost already paid and \((ER - EL) = \text{the remaining rate of ordering inventory.}\)

The total expected cost function ETC is given as follows.

**Model III**
\[ \text{ETC} = [C + QC_1]ER + C_2EI + (1 + l)C_1 EL \]

**Model IV**
\[ \text{ETC} = [C + QC_1]ER + C_2EI + (1 + m)C_1 \times s \times EL \]

**Model V**
\[ \text{ETC} = C' ER + QC_1[ER - EL] + C_2EI + (1 + n)C_1 \times S \times EL, \text{ where } l, m, n \text{ are proper fractions and } l > m > n > 0, \text{ when the local purchase is made in higher quantity the hike in price decreases.} \]
### Table 4.1. Variations in Maximum inventory level $S$. $\lambda = 1, \mu = 1.7, \beta = .2, \theta = 3, s = 10$

<table>
<thead>
<tr>
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<th>EI</th>
<th>ER</th>
<th>EW</th>
</tr>
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<td>29</td>
<td>I</td>
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### Table 4.2. Variations in re-order level $s$. $\lambda = 1, \mu = 1.7, \beta = .2, \theta = 3, S = 40$

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Table 4.3. Variations in replenishment rate $\beta$. $\lambda = 1$, $\mu = 1.7$, $\theta = 3$, $s = 10$, $S = 25$.

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</table>

Table 4.4. Variations in arrival rate $\lambda$. $\lambda = 1$, $\mu = 1.7$, $\beta = 0.2$, $s = 10$, $S = 25$. 

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4.12.1. Interpretations of the Numerical Results.

1. **Effect of the maximum inventory level \( S \) on various performance measures:**
   
   As \( S \) increases in all models considered, expected inventory level increases. The number of customers and hence the waiting time decrease in model I and II. As more inventory is with the system the time interval to reach the re-order level increases, so re-order rate decreases in model I and II. Due to the same reason local purchases in models III, IV and V decreases. The number of customers in III, IV and V is same due to local purchase. (see table 4.1)

2. **Effect of the re-order level \( s \) on various performance measures.**
   
   From table 4.2 one may conclude that the behaviour of system performance measures as \( s \) increases, is similar to that of \( S \), except that the re-order rate increases, the time interval to reach the re-order point decreases and so more orders are placed.

3. **Effect of the replenishment rate \( \beta \) on various performance measures.**
   
   As we expect when \( \beta \) increases the inventory level increases in all models. In models III, IV and V, as replenishment takes place at a higher rate the rate of local purchase decreases. Due to local purchase the waiting time of customers do not increase. The number of customers who do not join when the inventory level is zero, also decreases in model I. The number of customers and their waiting time decreases in models I and II as \( \beta \) measures (see table 4.3)

4. **Effect of the arrival rate \( \lambda \) on various performance measures.**
   
   Table 4.4 shows that when the arrival rate increases the number of customers and their waiting time increases in all models. Reorder rate increases in models I and II. The number of customers who do not join when the inventory level is zero also increases in model I. Due to more arrivals, the rate of local purchase also increases in models III, IV and V.
Figure 4.1. $\lambda = 1$, $\mu = 1.7$, $\beta = .2$, $\theta = 3$, $s = 10$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 7$, $l = .75$, $m = .5$, $n = .25$
Re-order level versus ETC.

Figure 4.2. $\lambda = 1$, $\mu = 1.7$, $\beta = .2$, $\theta = 3$, $S = 40$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 7$, $l = .75$, $m = .5$, $n = .25$
Figure 4.3. $\lambda = 1, \mu = 1.7, \theta = 3, s = 10, S = 25, C = 100, C_1 = 20, C_2 = 1, C_3 = 7, l = .75, m = .5, n = .25$
Arrival rate versus ETC.

Figure 4.4. $\lambda = 1$, $\mu = 1.7$, $\beta = .2$, $s = 10$ $S = 25$, $C = 100$, $C_1 = 20$, $C_2 = 1$, $C_3 = 7$, $l = .75$, $m = .5$, $n = .25$
4.12.2. **Interpretation of the graphs.** In order to find the most profitable model, we compute the expected total cost per unit time for each model by varying the parameters one at a time keeping others fixed.

Figure 4.1 shows that as the maximum inventory level increases the total expected cost increases, this is primarily due to the increase in the holding cost of inventory. When the re-order level increases then also the cost increases as the inventory level increases (see 4.2). As the replenishment rate increases, here also cost increases due to the same reason (see figure 4.3). Figure 4.4 shows that the cost function is directly proportional to the arrival rate.

4.13. **Conclusion**

From all the graphs we understand that comparing models I and II, the cost is less in model I. Comparing models III, IV and V the cost involved in model III is least. That is local purchase by one unit is profitable. Among all the models the cost is least for model III. So model III is the best with the given cost function and given values of parameter. However, the input parameters do influence the total expected cost. Hence the models are sensitive to input parameters.