CHAPTER V

MODULATIONAL INSTABILITY IN THE PRESENCE OF LANGMUIR TURBULENCE

5.1 Introduction:

The study of some exact nonlinear forms of waves and their envelopes, brought about by the balance of nonlinearity and dispersive effects (in the collisionless plasma) is a topic of much interest. A knowledge of these nonlinear states can provide an appropriate description for the plasma in the turbulent state. The underlying philosophy of such a representation is to look upon these localized entities as forming the 'basis' states of the system with a weak residual interaction between them.

It has been shown by Vedenov, Gordeev and Rudakov\(^1\) that a 'cold' plasmon gas (i.e. a plasmon gas for which the
mean square of the spread in the group velocities is zero) tends to break up into blobs, provided the wave number \( k \) of the perturbation exceeds the wave number of the plasmon. In the case of a 'warm' plasmon distribution of width \( \Delta \) in K-space, there is a threshold for this break up given by \( \frac{\varepsilon_T}{\varepsilon_{kin}} > \Delta^2 \lambda_D^2 \)

where \( \varepsilon_T \) is the energy density of the turbulent waves, \( \varepsilon_{kin} \) is the kinetic energy density of the particles and \( \lambda_D \) is the electron Debye length. The applicability of weak turbulence theory breaks down when \( \frac{\varepsilon_T}{\varepsilon_{kin}} > (k \lambda_D)^2 \). This is because the characteristic rate of nonlinear interactions \( \delta \omega \sim \omega_p \frac{\varepsilon_T}{\varepsilon_{kin}} \)

becomes greater than the frequency spread due to thermal effects \( \delta \omega_k \sim \omega_p (k \lambda_D)^2 \) \( (2) \). For the case \( \Delta/k \ll 1 \), this instability is identified with the decay instability or at higher amplitudes with the oscillating - two stream instability. In the opposite case \( \Delta/k \sim 1 \) since the resonant condition cannot be satisfied for the entire set of \( k \), only the modulational instability can exist. We shall be interested more in the latter case.

Another modulational instability that has attracted much attention is that of an electromagnetic mode due to transverse perturbations. Kaw, Schmidt and Wilcox \( (3) \) have investigated the stability of a large amplitude electromagnetic mode in an unmagnetized plasma to transverse perturbations. The nonlinearity responsible for the existence of this instability is provided by the ponderomotive force exerted on the plasma by the electromagnetic wave. It is shown that a plane electromagnetic wave is unstable against modulation in a direction perpendicular to the direction of propagation. Furthermore due to the saturating nature of the nonlinearity the final steady state consists of light filaments from which the plasma has been expelled, in equilibrium with the surrounding plasma pressure.
An alternative way of looking at the filamentation instability is as a coherent four-wave interaction. Drake$^4$, et. al have synthesised the electromagnetic instabilities (Raman scattering and filamentation, Brillouin scattering, Compton scattering and modulational instability) by deriving a general dispersion relation in terms of the susceptibility functions of the unmagnetized plasma and studying it in various limits. We shall approach the problem along the same lines.

It is recognized that the modulational and filamentation instabilities may play an important role in laser plasmas. These instabilities may drastically modify the backscattering instabilities in the underdense region of the plasma by modulating the plasma density. It may also facilitate the decay of the electromagnetic mode to electrostatic modes. It has been shown by Langdon and Lasinski$^5$ that as a result of self-focusing or filamentation of the light beam, strong plasma heating can occur in a wider range of densities than is usually expected. Previously the anomalous heating mechanism was considered to occur near the critical density surface where the local electron plasma frequency matches the laser frequency $\omega_c$. However, in a plasma which has undergone filamentation, the density changes are significant and this leads to an extension of the region in which frequency matching for parametric processes can occur. The authors$^5$ have specifically considered the $2\omega_p$ instability, a decay of the laser wave into two Langmuir plasmons at the quarter critical density. When a filament forms in a higher density region, the density depression establishes the frequency matching conditions necessary for the $2\omega_p$ instability. These localized conversions lead to strong plasma heating. Therefore it is expected that filamentation may introduce such modifications in the plasma density so as to facilitate heating by parametric processes.
The problem investigated by us involves both these instabilities. We wish to study how the presence of Langmuir plasmons affects the filamentation instability of the electromagnetic wave and vice-versa. The plasmons are assumed to be governed by the wave kinetic equation in the adiabatic approximation i.e. in the absence of resonant wave-wave and wave-particle interactions. The wave vector of the perturbation has to be less than that of the plasmons to justify the use of the adiabatic behaviour. The background plasma in general can be inhomogeneous (as we have discussed for the cases of Raman and Brillouin scattering). However, it is known that the existence of an inhomogeneity does not affect the absolute nature of the four-wave interaction. Therefore we will restrict our analysis to the case of a homogeneous turbulent plasma.

5.2 Basic Equations and General Dispersion Relation:

The basic set of equations representing the turbulent plasma are given by

$$\frac{\partial n_{\sigma}}{\partial t} + \nabla \cdot (n_{\sigma} \vec{v}_{\sigma}) = 0$$  \hspace{1cm} (4.1)

$$m_{\sigma} \left( \frac{\partial \vec{v}_{\sigma}}{\partial t} + \vec{v}_{\sigma} \nabla \cdot \vec{v}_{\sigma} \right) = e_{\sigma} q_{\sigma} \vec{E} + \frac{e_{\sigma} q_{\sigma}}{c} \vec{v}_{\sigma} \times \vec{B}$$  \hspace{1cm} (4.2)

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$  \hspace{1cm} (4.3)

$$\nabla \times \vec{B} = +\frac{1}{c} \frac{\partial \vec{E}}{\partial t} + 4\pi \vec{J}$$  \hspace{1cm} (4.4)

$$\nabla \cdot \vec{E} = 4\pi q_{\sigma} \sum_{\sigma} n_{\sigma} \epsilon_{\sigma}$$  \hspace{1cm} (4.5)

where \( \vec{J} = \sum_{\sigma} n_{\sigma} q_{\sigma} \vec{v}_{\sigma} \epsilon_{\sigma} \), \( \epsilon_{+} = -\epsilon_{-} = 1 \)
Here $\mathcal{O}^-$ denotes the species (electrons and ions) and the rest of the notations are standard.

In the equilibrium state we assume the existence of a large amplitude electromagnetic wave (plane polarized)

$$\bar{E}_0 = \mathbf{k} \cdot \mathbf{E}_0 \cos \left( \mathbf{k} \cdot \mathbf{x} - \omega_0 t \right) = E_{0\uparrow} + E_{0\downarrow} \tag{4.6}$$

propagating in a homogeneous turbulent medium. We assume $(\omega_0, k_0)$ satisfy the usual linear dispersion relation

$$\omega_0^2 = \omega_p^2 + k_0^2 c^2 \tag{4.7}$$

Physically one would expect the short wavelength turbulence to affect the dispersion relation of the electromagnetic wave. However it is obvious that these enter through the thermal correction which is indeed very small. The validity of the use of the linear dispersion relation is $\alpha = c E_0/m \omega_0 c \ll 1$ so that the relativistic mass corrections can be neglected.

The equilibrium is now perturbed by considering a density perturbation going as $\exp \left( i \mathbf{k} \cdot \mathbf{x} - i \omega t \right)$ and this perturbation may be due to some normal electrostatic mode of the system. Due to the time and space dependent equilibrium state, currents at $\omega \pm \omega_0$ and $k \pm k_0$ will be induced in the system (here $l$ is an integer). These side band modes (which may be mixed e.s. and e.m modes in general) interact with the pump wave field and shape out an effective potential through the ponderomotive force which leads to the amplification of the initial perturbation. This in turn enhances the side band amplitude and this bootstrap effect leads to the simultaneous amplification of the side bands and the initial perturbation.
For the case $\alpha \ll 1$ it is only necessary to consider the lowest order coupling. The Fourier transformed wave equation for the side band modes $\omega_\pm = \omega \pm \omega_0$, $k_\pm = k \pm k_0$ may be written as

$$\left[ \left( \frac{k_\pm^2 - \omega_\pm^2}{c^2} \right) \bar{k}_\mp \bar{k}_\pm - \frac{\bar{k}_\pm}{k_\pm^2} \right] \bar{E}_\pm = \frac{4\pi i \omega_\pm j_\pm}{c^2} \bar{E}_\mp$$

(4.8)

where II denotes the unit dyadic and $\bar{E}_\pm$ represents $E(\omega_\pm, k_\pm)$. The total current can be written as

$$\bar{J}_\pm = \bar{J}_{\text{linear}} + \bar{J}_{\text{nonlinear}} = \sigma_\pm \bar{E}_\pm + n_c(k, \omega) v_0 \bar{E}_\pm$$

(4.9)

where $\sigma_\pm = i \omega_\pm (\epsilon_\pm - 1)/4 \pi$ is the linear conductivity and $\epsilon_\pm$ is the linear dielectric function. $v_0 = \omega / m \omega_0$ and $n_c(k, \omega)$ is the perturbed electron density. In view of the fact that the side bands are high frequency modes ($\omega_\pm > \omega_p$) the contribution to the current comes only from the electrons because the ions fail to respond to high frequency fields due to their inertia. Inverting the expression for the side bands $\bar{E}_\pm$ after plugging in the expression for the current $\bar{J}_\pm$ we get

$$\bar{E}_\pm = -\omega_\pm^2 n_c(k, \omega) \left\{ \left[ \left( \frac{\bar{k}_\pm}{k_\pm^2} \right) \bar{D}_\pm - \frac{\bar{k}_\pm k_\pm^2}{\omega_\pm^2 k_\pm^2 \omega_0^2} \right] \bar{E}_\mp \right\}$$

(4.10)

where

$$\bar{D}_\pm = \frac{k_\pm^2 c^2}{\omega_\pm^2} = c^2 k_\pm^2 \pm \frac{2 \bar{k}_\pm k_0 c^2}{\omega_\pm^2} + 2 \omega_\pm \omega_0 - \omega^2$$

(4.11)

The equation of motion for the electrons in the presence of Langmuir Turbulence becomes

$$\frac{\partial \bar{v}_e}{\partial t} + \bar{v}_e \bar{v}_e + \frac{\bar{E}}{m} = -\nabla \left( \bar{v}_0 \cdot \bar{v}_e \right) - \frac{\omega_p}{2m n_0} \nabla \sum_k N_k$$

(4.12)

where, as discussed in the last chapter, the plasmon density $N_q$ is conserved in phase space so that
\[ \frac{dN_q}{dt} = \frac{\partial N_q}{\partial t} + \frac{\partial}{\partial x} \frac{\partial v_q}{\partial x} - \frac{\partial}{\partial y} \frac{\partial N_q}{\partial y} = 0 \]  
(4.13)

\[ \frac{d\tilde{x}}{dt} = \frac{\partial \tilde{v}_q}{\partial t} - \frac{\partial \tilde{v}_q}{\partial x} \]  
(4.14)

Since the plasmon density is modulated by a long scale length density perturbation, the perturbed distribution is given by

\[ \tilde{N}_q = -\omega p \frac{n_e}{2n_0 \tilde{\nu}_q} \frac{\partial N_q^{(e)}}{\partial \tilde{v}_q}, \quad \tilde{v}_q = \frac{e^2}{m_0} \]  
(4.15)

Hence

\[ \frac{\partial \tilde{v}_q}{\partial t} + \frac{1}{4n_0} \left[ k \tilde{v}_q - \epsilon \tilde{v}_q \frac{\partial N_q^{(e)}}{\partial \tilde{v}_q} \right] \frac{\partial \tilde{v}_e}{\partial \tilde{v}_q} + eV_{\text{ref}} = -\nabla(\tilde{v}_q \tilde{v}_q) \]  
(4.16)

Therefore we see that the turbulence provides an effective pressure which modifies the thermal pressure exerted by the electrons on the ions. In fact under the present circumstances we can have a negative temperature system when the plasmon pressure exceeds the thermal pressure.

Hence the equation of motion becomes

\[ \frac{\partial \tilde{v}_e}{\partial t} + \frac{T_e}{m_0} \frac{\partial v_e}{\partial \tilde{v}_q} + eE = -\nabla(\tilde{v}_q \tilde{v}_q) \]  
(4.17)

where

\[ T_e = T_e \left( 1 - \frac{E_{\text{exc}}}{k_\text{kin} \Delta^3 \Delta_0} \int q \frac{q^2}{e^2} d\tilde{v}_q \right) \]  
(4.18)

where \( E_{\text{exc}} / E_{\text{kin}} = \Sigma \left| E_q^2 \right| / 16 m_0 \Delta T \) defines the ratio of the plasmon energy density to the particle energy density and

\[ \Delta = \omega p / k V_e \cos \theta, \quad \cos \theta = \frac{\tilde{v}_q \cdot \tilde{K}}{\tilde{v}_q \cdot \tilde{K}} \]
denotes the direction and $\Delta$ denotes the spectral bandwidth of the turbulence.

Using Equations (4.1), (4.2) and (4.5) we get

$$
\left( \frac{\partial^2}{\partial t^2} - \frac{T}{m} \nabla^2 + \omega_p^2 \right) n_e - \omega_p^2 n_e = \n_0 \nabla^2 (\vec{v}_0 \cdot \vec{v}) \tag{4.19}
$$

and

$$
\left( \frac{\partial^2}{\partial t^2} - \frac{T}{m} + \omega_p^2 \right) \vec{n}_i - \omega_p^2 \vec{n}_e = 0 \tag{4.20}
$$

Eliminating $\vec{n}_i$ between equations (4.19) and (4.20) we get

$$
\left[ \left( \frac{\partial^2}{\partial t^2} - \frac{T}{m} \nabla^2 + \omega_p^2 \right) \left( \frac{\partial^2}{\partial t^2} - \frac{T}{m} \nabla^2 + \omega_p^2 \right) - \omega_p^2 \right] n_e

= \n_0 \left[ \frac{\partial^2}{\partial t^2} - \frac{T}{m} \nabla^2 + \omega_p^2 \right] \nabla^2 (\vec{v}_0 \cdot \vec{v}) \tag{4.21}
$$

Taking

$$
n_e = \bar{n}_e \ e^{i k \cdot \vec{x} - i \omega t} \quad \vec{v}_i = \vec{v}_i \ e^{i k \cdot \vec{x} - i \omega t} + \vec{v}_i \ e^{i k \cdot \vec{x} - i \omega t} \quad \vec{v}_0 = \vec{v}_0 \ e^{i k \cdot \vec{x} - i \omega t} + \vec{v}_0 \ e^{i k \cdot \vec{x} - i \omega t}
$$

Hence equation (4.21) becomes

$$
\left[ (\omega^2 - k^2 v_e^2 - \omega_p^2) (\omega^2 - k^2 v_i^2 - \omega_p^2) - \omega_p^2 \omega v_i^2 \right] \frac{\bar{n}_e}{n_0} =

\frac{e^2 k^2}{m^2 \omega_o} \left( \omega^2 - k^2 v_i^2 - \omega_p^2 \right) \left( \bar{E}_+ \bar{E}_+ + \bar{E}_- \bar{E}_- \right) \tag{4.22}
$$

where we have used $\vec{v}_0 = \pm \frac{e E_o}{i m \omega_o}$ and $\omega \ll \omega_o$

Substituting the expressions $\bar{E}_+$ and $\bar{E}_-$ we get the required dispersion relation.
\begin{equation}
\frac{1}{\chi_+^2} + \frac{1}{\chi_{i+1}^2} = k^2 \left[ \frac{\left| \frac{k \cdot \nu_o}{k_+ \nu_o} \right|^2 - \left| \frac{k \cdot \nu_o}{k_+ \xi_+} \right|^2}{k_+^2 \xi_+ \omega^2} + \frac{\left| \frac{k \cdot \nu_o}{k_- \xi_-} \right|^2 - \left| \frac{k \cdot \nu_o}{k_- \xi_-} \right|^2}{k_-^2 \xi_- \omega^2} \right]
\end{equation}

(4.23)

where

\begin{equation}
\chi_j = - \omega_p j \left/ \left( \omega - k v_j \right) \right.
\end{equation}

(4.24)

This is the general dispersion relation derived by Drake et al. using the Vlasov equation. This dispersion relation describes the parametric coupling of a low frequency electrostatic wave at \((\omega, k)\) and two high frequency mixed electromagnetic-electrostatic side bands at \((\omega \pm \omega_o, k \pm k_o)\). The effect of turbulence enters through the electron susceptibility function and it is the effect of this turbulence that we wish to investigate.

5.3 Dispersion Relation for Modulational Instability in Presence of Langmuir Turbulence:

For the case of modulational instability the coupling terms involving \(D_+\), \(D_-\) will be considered and the electrostatic contributions from \(\xi_+\) and \(\xi_-\) will be neglected. We investigate the excitation of long wavelength instabilities with \(k \ll 2k_o\). Since \(\omega \ll \omega_p\) and \(\omega \ll kc\), equation (4.23) can be written as

\begin{equation}
\frac{1}{\chi_+^2} + \frac{1}{\chi_{i+1}^2} = -2 \frac{v_o^2 \delta^2}{c^2} \left[ \left( \omega - k \cdot \nu_q \right)^2 - \delta^2 \right]^{-1}
\end{equation}

(4.25)

where \(\nu_q = c^2 k_o / \omega_o\), \(\delta = c^2 k^2 / 2 \omega_o\)

Hence

\begin{equation}
\omega = \nu_q + \delta \left[ 1 - \frac{2 v_o^2 k_o^2}{k^2 c^2 \left( \frac{\xi \tau_c}{\xi k \omega} \right)} \right]\end{equation}

(with, \(k_o\) = Debye wave-number)

(4.26)
References


5. A. Bruce Langdon and Barbara F. Lasinski, Phys. Rev. Letts. 34, 934 (1975)
