CHAPTER III

NONLINEAR DRIFT DISSIPATIVE ION ACOUSTIC WAVES

III.1 Introduction

Having discussed the case of nonlinear propagation of ion acoustic waves in a collisionless plasma in the last chapter, we shall go over to study the nonlinear propagation of a different class of low-frequency waves namely, the drift waves in a collisional plasma in this and in the next chapter. In particular, in this chapter we shall investigate the weakly nonlinear propagation of the drift-dissipative ion acoustic mode in the presence of ion viscosity.

The drift waves derive their importance from their possible causal relation to the enhanced particle losses observed in low-$\beta$ plasmas. In this connection, collisional drift waves are of significant importance because, they are known to have large instability growth rates ($\text{Im} \omega \sim \text{Re} \omega$). Earlier theoretical
work on drift waves in resistive plasmas was done by Moiseev and Sagdeev (1963) and Chen (1964). Drift dissipative instabilities are observed, among other experiments, in magnetically confined alkali metal plasmas in Q-machines (Hendel et al. 1969, Ivanov et al. 1968). In the present chapter we investigate the propagation of weakly nonlinear ion acoustic wave in an inhomogeneous and strongly collisional plasma in which both the parallel resistivity and perpendicular viscosity are present. We consider the plasma in which the electrons are magnetized \((\omega < \Omega'_e)\) and ions are unmagnetized \((\omega >> \Omega'_i)\); \(\omega, \Sigma'_{e,i}\) being the characteristic wave frequency and electron and ion cyclotron frequencies respectively. We also assume that the electron mean free path, is much smaller than the longitudinal wavelength, so that the diffusion approximation is valid for the longitudinal motion of the electrons. Such conditions do exist in a positive column with a very low neutral pressure (Kadomtsev 1965) and in certain Q-machine experiments (Buchal'nikova 1968).

Assuming the magnetic field to be in z-direction and density gradients in the negative x-direction it has been shown (Kadomtsev 1965) that the high frequency \((\omega >> \Omega'_i)\) ion acoustic mode \((\omega \sim k_y c_s)\) in the presence of collisions is unstable for \(k_y >> k_z\) (drift dissipative instability); this is a negative energy mode and the instability is due to a phase difference between the electric field and density fluctuations introduced by the collisions and due to finite
$k_z$. The ion-viscosity and finite-Larmor radius corrections are known to have stabilizing effects on this instability (Hendel et al. 1968, Coppi 1964).

In a collisionless plasma, the propagation of a weakly nonlinear ion acoustic wave can be described by the K-dV equation (Washimi and Taniuti 1966, Davidson 1972). Since drift dissipative ion acoustic wave has a linear dispersion relation similar to that of ion acoustic wave itself, it is interesting to examine if, at least up to certain stage of nonlinearity, this wave can also be described by a K-dV type equation. We have taken into account the ion viscosity and by using reductive perturbation method derived a set of two coupled partial differential equations describing the propagation of nonlinear drift dissipative ion acoustic mode. We have looked for a special solution and shown that if a perturbation has a long wave length sinusoidal variation along the magnetic field direction, the propagation, along a direction transverse to both magnetic field and density gradients, is governed by a modified K-dV equation. When the stabilizing effects due to ion viscosity dominates over the destabilizing effects due to the collisions, this equation allows a shock solution.

In order to explore the region of interest when the destabilizing effects due to collisions are stronger than the stabilizing effects due to ion viscosity, we have numerically solved the equation. It is seen that the equation still
permits solitary wave solutions as long as the net stabilizing or destabilizing effects are not too strong (i.e. the linear decay or growth rates are small compared to the amplitude of the initial perturbation). These solitary waves are found to either grow or decay with time depending on whether the viscous effects are weaker or stronger compared to the resistive effects.

III.2 General Theory

The basic equations governing our system are the ion and the electron continuity equations, ion equation of motion, electron parallel equation of motion and the Poisson's equation, namely

\[ \frac{\partial n_i}{\partial t} + \frac{\partial}{\partial y} (n_i v_i) = 0 \]  \hfill (3.1)

\[ \frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial y} + \frac{\partial \Phi}{\partial y} = \frac{\mu}{c_s \lambda_0 m_i n_n n_i} \frac{\partial^2 v_i}{\partial y^2} \]  \hfill (3.2)

\[ \frac{n_e}{m_i} \frac{1}{\omega_{ri} \tau_e} \frac{\partial n_e}{\partial t} + \frac{n_e}{\tau_e} \frac{\partial \Phi}{\partial y} + \frac{\partial}{\partial z} (n_e \frac{\partial \psi}{\partial z}) = 0 \]  \hfill (3.3)

\[ - \frac{1}{n_e} \frac{\partial n_e}{\partial z} + \frac{\partial \Phi}{\partial z} - \frac{\partial \psi}{\partial z} = 0 \]  \hfill (3.4)

and

\[ \frac{\partial^2 \Phi}{\partial y^2} = n_e - n_i \]  \hfill (3.5)

with
\[ \frac{\partial \psi}{\partial z} = \frac{m_e}{T_e \zeta_e} \psi_{ez} \]  

(3.6)

In this set of equations, \( n_{e,i} \) are the electron and ion densities, \( V_i \) is the ion velocity, \( C_s = (T_e/m_i)^{1/2} \) is the sound speed and \( \lambda_D = (T_e/4\pi n_0 e^2)^{1/2} \) is the electron Debye length. Eqs. (3.1) - (3.5) are written in terms of normalized quantities; densities are normalized to equilibrium value \( n_0 \), lengths to Debye length, time to ion plasma period \( \omega_p^{-1} (\omega_p^2 = 4\pi n_0 e^2/m_i) \), potential to \( T_e/e \) and \( V_i \) to ion acoustic speed. In Eq. (3.2), \( \chi = -\frac{dr}{d\xi} / n_0 \), \( \Omega_c = eB_0/m_e C \), \( e \) being the electronic charge and \( \zeta_e \) is the mean collision time between electrons and ions. The quantity \( \psi \) appearing in Eq. (3.4) is a velocity potential introduced through Eq. (3.6) where \( \psi_{ez} \) is the z-component of electron velocity. It is to be noted that the electron perpendicular equation of motion is not written down along with Eqs. (3.1) - (3.5), but use of which has been made to write Eq. (3.3) under local approximations (Kadomtsev, 1965). Since the mode under consideration is an electrostatic mode and the propagation is nearly perpendicular to the magnetic field, the motion of the ions along the direction of the magnetic field has been neglected. The term on the right hand side of Eq. (3.2) represents the viscous force with \( \chi = \eta_0/3 + \eta_1 \) and \( \eta_0 = C_0 n_i T_i / \nu_{ii} \) and \( \eta_1 = C_1 n_i T_i \nu_{ii} / \Omega_i^2 \), \( C_0 \) and \( C_1 \) being constants and \( T_i \) and \( \nu_{ii} \) being ion temperature and ion-ion collision frequency.
respectively. The constants as obtained by Braginskii (1965) are $C_0 = 0.95$ and $C_1 = 0.3$. In writing Eq. (3.2) the ion pressure term is neglected because the ion temperature $T_1$ is assumed to be much smaller than the electron temperature $T_e$ and the strength of the ion pressure term compared to the ion viscosity term goes as $O(\omega_i^2/\nu_{ii} k y C_s)$. For $k y C_s \sim \omega >> \omega_i$, the strength of the ion pressure term is even smaller than that of ion-viscosity term if $\nu_{ii} \gg \omega_i$.

Let us write the ion density $n_1$ as $n_1 = 1 + \tilde{n}_1$, where $\tilde{n}_1$ is the perturbed part of ion density. Now, integrating Eq. (3.4) w.r.t. $z$ we get $n_e = \exp (\Phi - \Psi)$, the integration constant is put equal to unity in view of the fact that the equilibrium value of $n_e$ is also equal to unity. Substituting the expression for $n_e$ in Eq. (3.5), we get

$$\frac{\partial \Phi}{\partial y} \sim \Phi - \Psi + \frac{1}{2} (\Phi - \Psi)^2 - \tilde{n}_1. \quad (3.7)$$

We now introduce the stretched variables $\Xi = \epsilon^{1/2}(y-z)$ and $\tau = \epsilon^{3/2} t$. The perturbed quantities can then be expanded as

$$\tilde{n}_1 = \epsilon \bar{n}_1 + \epsilon^2 \bar{n}^{(2)}_1 + \ldots$$

$$\Phi = \epsilon \bar{\Phi} + \epsilon^2 \bar{\Phi}^{(2)} + \ldots$$

$$\Psi = \epsilon \bar{\Psi} + \epsilon^2 \bar{\Psi}^{(2)} + \ldots$$

$$\bar{\Psi} = \epsilon \bar{\Psi}^{(1)} + \epsilon^2 \bar{\Psi}^{(2)} + \ldots$$
The last expansion, namely that for \( \Psi \) is rather crucial in our theory which follows from Eq.(3.4). In the absence of collisions the electron density fluctuations are governed only by the potential fluctuations namely \( n_e = \exp(\Phi) \). The collisional term \( \delta \Psi / \delta z \) in Eq.(3.4) is treated as a correction to the potential fluctuations \( \delta \Phi / \delta z \). Thus the term \( \delta \Psi / \delta z \) is taken to be one order smaller than the term \( \delta \Phi / \delta z \). Hence the above expansion for \( \Psi \).

The smallness parameter \( \epsilon \) is chosen in such a way that,

to the lowest order Eq.(3.3) gives

\[
\frac{\partial^2 \Psi^{(1)}}{\partial z^2} = -\frac{\kappa}{\Omega_e \tau_e} \frac{\partial \Phi^{(1)}}{\partial \xi} \tag{3.8}
\]

This requirement demands that, \( (\kappa/\kappa_0^2 \Omega_e \tau_e)^{1/2} \sim \epsilon \) and

\[
\left( m_e/m_i \right) \left( \omega_p \tau_e \right)^{-1} (\kappa/\Omega_e \tau_e)^{-1} \sim \epsilon.
\]

To the lowest order Eqs.(3.1), (3.2) and (3.7) give

\( n_i^{(1)} = v_i^{(1)} = \Phi^{(1)} \). To the next higher order these equations can be written as

\[
-\frac{\partial n_i^{(2)}}{\partial \xi} + \frac{\partial v_i^{(2)}}{\partial z} + \frac{\partial n_i^{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi} \left( n_i^{(1)} v_i^{(1)} \right) = 0, \tag{3.9}
\]

\[
-\frac{\partial v_i^{(2)}}{\partial \xi} + \frac{\partial v_i^{(1)}}{\partial z} + v_i^{(1)} \frac{\partial v_i^{(1)}}{\partial \tau} + \frac{\partial \Phi^{(1)}}{\partial \xi} = \frac{\mu}{n_0 m_i} \frac{\partial^2 \Psi^{(2)}}{\partial \xi^2} \tag{3.10}
\]

and

\[
\frac{\partial^2 \Phi^{(2)}}{\partial \xi^2} = \Phi^{(2)} + \left[ \Phi^{(1)} \right]^2/2 - \Psi^{(1)} - n_i^{(2)}, \tag{3.11}
\]

On eliminating \( n_i^{(2)}, \Phi^{(2)} \) and \( v_i^{(2)} \) and on using the relation \( n_i^{(1)} = v_i^{(1)} = \Phi^{(1)} \), Eqs.(3.9) - (3.11) can be simplified to
\[
\frac{\partial \tilde{\eta}^{(n)}}{\partial \zeta} + \tilde{\eta}^{(n)} \frac{\partial \tilde{\eta}^{(n)}}{\partial \zeta} + \frac{1}{2} \tilde{\eta}^{(n)} \frac{\partial \tilde{\eta}^{(n)}}{\partial \zeta} + \frac{1}{2} \tilde{\phi}^{(1)} \frac{\partial \tilde{\phi}^{(1)}}{\partial \zeta} - \frac{2}{\gamma_0 m \lambda \sigma_s} \frac{\partial \tilde{\eta}^{(n)}}{\partial \zeta^2} = 0,
\]

Eqns. (3.8) and (3.12) constitute a set of two coupled differential equations describing the propagation of nonlinear drift dissipative ion acoustic mode. We now look for a special solution for this set of equations. We assume that the \(z\)-dependence of any perturbation is sinusoidal in nature (Maxworthy and Redekopp 1976), namely \(n^{(1)}(z, \zeta, \gamma) = \tilde{n}(\zeta, \gamma)\) \(\sin_k z\) and \(\bar{\phi}^{(1)}(z, \zeta, \gamma) = \bar{\psi}(\zeta, \gamma)\) \(\sin_k z\). With such a prescription Eq. (3.8) can be solved for \(\bar{\phi}^{(1)}\) which is then substituted in Eq. (3.12). The \(z\)-dependence from this equation is then removed by integrating over \(dz\) from 0 to \(\pi/k_z\). The resulting equation for \(\bar{n}\) is

\[
\frac{\partial \bar{n}}{\partial \zeta} + \frac{4}{\pi} \bar{n} \frac{\partial \bar{n}}{\partial \zeta} + \frac{1}{2} \frac{\partial^2 \bar{n}}{\partial \zeta^2} + (\beta' - \alpha') \frac{2 \bar{n}}{\partial \zeta^2} = 0, \tag{3.13a}
\]

where \(\beta' = (\kappa/2 \Omega_e \zeta \sigma C^2)\) and \(\alpha' = (\kappa/2 \nu m \lambda \sigma_s C^2)\). If we drop the nonlinear and viscosity terms and assume that \(\bar{n} = \exp(i(k \zeta - \Omega \zeta))\) Eq. (3.13a) yields the linear dispersion relation for this mode (Kadomtsev 1965). If we had Fourier analysed Eq. (3.8), then instead of Eq. (3.13a) we would have obtained a nonlinear integro-differential equation which can not be solved analytically even for some special cases that we have discussed later. By making a transformation of the space co-ordinate, \(\zeta = \frac{\pi}{4} \zeta\), Eq. (3.13a) can be written as:
\[
\frac{\partial \overline{\gamma}}{\partial \tau} + \overline{\gamma} \frac{\partial \overline{\gamma}}{\partial \xi} + \delta \frac{\partial \overline{\gamma}}{\partial \xi}^3 + (\beta - \alpha) \frac{\partial^2 \overline{\gamma}}{\partial \xi^2} = 0, \quad (3.13)
\]

where \( \delta = (1/2) (\pi/4)^3 \), \( \beta = (\pi/4)^2 \beta' \) and \( \alpha = (\pi/4)^2 \alpha' \). Before we give the numerical solution of Eq. (3.13) we will qualitatively discuss some special cases of Eq. (3.13).

i) When \( \alpha = \beta = 0 \), Eq. (3.13) reduces to a K-dV equation and represents the propagation of an ion acoustic wave in a collisionless plasma.

ii) In the case when \( \alpha > \beta \), the stabilizing effects due to ion viscosity is strong enough to overcome the destabilizing effects due to the collisions. By retaining only the \( \gamma_{10} \) term in the expression for \( \mu \), the condition \( \alpha > \beta \) can be expressed as

\[
K_2 > \left( \frac{\lambda_D}{L} \right)^{1/2} \left( \frac{1}{\Omega_e \tau_e} \right)^{1/2} \left( \frac{\gamma_{10}}{\lambda_D e_s} \right)^{1/2} \left( \frac{T_e}{T_i} \right)^{1/2}, \quad (3.14)
\]

where \( L \) is the scale length of density gradients. In this case Eq. (3.13) becomes a 'modified K-dV equation'. It is well known that this equation possesses a stationary shock solution with either an oscillating or a monotonic profile (Shut'ko 1970, Johnson 1970, Jeffrey and Kakutani 1972, Grad and Hu 1969).

iii) When \( \beta > \alpha \), the ion viscosity effects are not strong enough to quench the instability and hence we cannot look for a steady state solution in this case.
III.3 Numerical Analysis and Discussions

In this section we shall discuss some results of numerical solution of the equation,

\[
\frac{\partial n}{\partial \xi} + \eta \frac{\partial n}{\partial \zeta} + \delta \frac{\partial^3 n}{\partial \zeta^3} = -\gamma \frac{\partial^2 n}{\partial \zeta^2},
\]

(3.15)

where \( \gamma = \beta - \alpha \). Now, for \( \gamma > 0 \), Eq. (3.15) corresponds to the case when the destabilizing effects are stronger while \( \gamma < 0 \) corresponds to the case when stabilizing viscosity effects are stronger. Although in our case \( \delta = 1/2 \), by an appropriate scaling of \( n \) and \( \gamma \), \( \delta \) can be made as small as we like. This reduces the computer time necessary to obtain the solitary solutions and only necessitates a change in the normalization of the initial perturbation. For all the calculations to be presented here we have used \( \delta = 5 \times 10^{-4} \). The difference equation that approximates the modified K-dV equation, i.e. Eq. (3.15), is

\[
\gamma_{i+1}^j = \gamma_i^j - \frac{1}{3} \left( \Delta \gamma / \Delta \zeta \right) \left( \gamma_{i+1}^j - \gamma_{i-1}^j \right) \left( \gamma_{i+1}^{j+1} + \gamma_{i+1}^{j-1} + \gamma_{i-1}^{j+1} + \gamma_{i-1}^{j-1} \right)
- \delta \left( \Delta \gamma / \Delta \zeta^3 \right) \left( \gamma_{i+2}^j - 2\gamma_{i+1}^j + 2\gamma_{i-1}^j - \gamma_{i-2}^j \right)
- 2\gamma \left( \Delta \gamma / \Delta \zeta^2 \right) \left( \gamma_{i+1}^j - \gamma_i^j - \gamma_i^{j-1} + \gamma_{i-1}^j \right),
\]

(3.16)

where \( n_{i+1}^j \equiv n(i \Delta \xi, j \Delta \zeta) \) and \( \Delta \xi \) and \( \Delta \zeta \) are appropriate step lengths. In Eq. (3.16) we have used the DuFort and Frankel's scheme (Richtmyer and Morton 1967) to replace the second derivative term in Eq. (3.15) by appropriate differences.
Integration is performed with 200 steps in $\zeta$. In order that the numerical solution of Eq. (3.15) does not become numerically unstable, we have calculated the amplification matrix corresponding to Eq. (3.16) and the numerically stable region is determined for the initial perturbation having a normalized amplitude which is less than or of order unity.

First, we have taken the initial conditions as
\[ n_1^0 = \cos(n\Delta\zeta) \] with periodic boundary conditions and amplitude normalized to unity. In this case we have used a mesh with $\Delta\zeta = 0.01$ and $\Delta\zeta = 2 \times 10^{-4}$. The integration is performed with $\gamma = 0$ and $\gamma = \pm 10^{-4}$. In all the three cases the integration is carried on till the solitons are fully developed. In table (3.1), we have shown the amplitudes of the solitons at $\zeta = 0.9$ for these three cases. Decrease of amplitude when $\gamma < 0$ and increase of amplitude when $\gamma > 0$, is observed. This result can be understood as follows:

Linearly, the second derivative term in Eq. (3.15) with $\gamma < 0$, represents a growth or damping of the initial perturbation. Since $\gamma < 1$ (the amplitude of the initial perturbation being normalized to unity) this term will simply result in an increase or decrease by a small amount compared to $\gamma = 0$ case:

Next interesting case is to see the development of an initial pulse whose amplitude is comparable to $\gamma$. For this, we take $n(\zeta, 0) = 0.03 \text{sech}^2(5\zeta)$ and observed the development of the pulse till $\zeta = 0.3$ with $\gamma = \pm 0.02$. At $\zeta = 0.3$,
TABLE 3.1

Amplitudes of the solitons at $\gamma = 0.9$ obtained from the numerical integration of Eq. (3.15) with $n(\xi, 0) = \cos(\eta \xi)$ and $\delta = 5 \times 10^{-4}$. In the spatial region of 200 space steps 6 solitons are observed and the serial numbers are the numbers attached to these solitons from left to right.

<table>
<thead>
<tr>
<th>Sr. No.</th>
<th>$\gamma = 0$</th>
<th>$\gamma = -1 \times 10^{-4}$</th>
<th>$\gamma = +1 \times 10^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.783</td>
<td>1.662</td>
<td>1.785</td>
</tr>
<tr>
<td>2</td>
<td>2.128</td>
<td>2.042</td>
<td>2.174</td>
</tr>
<tr>
<td>3</td>
<td>2.504</td>
<td>2.427</td>
<td>2.571</td>
</tr>
<tr>
<td>4</td>
<td>0.351</td>
<td>0.317</td>
<td>0.387</td>
</tr>
<tr>
<td>5</td>
<td>0.810</td>
<td>0.762</td>
<td>0.864</td>
</tr>
<tr>
<td>6</td>
<td>1.282</td>
<td>1.225</td>
<td>1.342</td>
</tr>
</tbody>
</table>
the amplitude of the initial pulse increases to 0.0547 for the case \( \gamma = 0.02 \) and decreases to 0.024 for \( \gamma = -0.02 \). This is shown in Fig. 3.1. It is to be noted that the increase in amplitude when \( \gamma = +0.02 \) is much larger than the decrease in amplitude when \( \gamma = -0.02 \). From Fig. (3.1) we also observe that the increase in amplitude is associated with decrease in width and vice versa. But this increase and decrease of amplitude do not follow the amplitude width relationship for a soliton. This is because of the fact that, to start with our initial pulse itself was not a pure soliton. In Fig. 3.1 we have shown the development of the initial pulse only up to \( \tau = 0.3 \). Beyond \( \tau = 0.3 \) the solution seems to be unreliable due to accumulation of round-off errors. The reason for this may be the following: The way the second derivative term is replaced by finite differences in Eq. (3.16) introduces round-off error proportional to \((\Delta \tau / \Delta \xi)^2 (\delta^2 / \delta \xi^2)^n\). The rate of change of the solution being quite rapid in this case, the accumulation of round-off error is also expected to be large.

We have also tried to examine the development of an initial pulse of sufficiently large amplitude, viz. \( n(\xi, 0) = 3.0 \text{ sech}^2(\xi + 1) \), so that shock structure is produced within a reasonable amount of computer time. But in this case problems of numerical instability do not allow us to take sufficiently large values of \( \gamma (~3) \). So, we have to restrict ourselves to values as small as \(|\gamma| = 10^{-2}\). At any given
FIGURE 3.1 Time evolution of an initial pulse
\( n(\xi, \tau = 0) = 0.3 \text{ Sech}^2 (5\xi) \) as given by Eq. (3,16) is shown. Curve - I is the initial pulse, Curves - II and III are \( n(\xi, \tau = 0.3) \)
for \( \delta = 5 \times 10^{-4} \) and \( \gamma = -0.02 \) and + 0.02 respectively.
time (observed up to $\gamma = 1.0$) the shock structure for these two cases (with $\gamma = \pm 10^{-4}$) are found to be almost identical (the changes in the amplitude as well as in the width are found to be less than 1 per cent).

### III.4 Conclusions

A modified K-dV equation governs the propagation of the nonlinear drift dissipative ion acoustic mode in the presence of ion viscosity, in a direction perpendicular to both magnetic field and density gradient directions when the propagation along the magnetic field direction is sinusoidal. The equation allows a stationary shock solution when the stabilizing ion viscosity effects dominates over the destabilizing effects due to collisions. It has been shown from numerical solution of the equation that it still permits solitary wave solutions if the net stabilizing and destabilizing effects are not too strong. The solitons thus obtained are found to either grow or decay with time depending on whether the viscous effects are weaker or stronger compared to the resistive effects. We would like to emphasize that the actual two dimensional perturbation has to be constructed by superimposing on this solution the sinusoidal variation along the z-direction.

Even though the ion viscosity effects are not sufficient to quench the drift dissipative instability, the instability
cannot make the wave grow indefinitely. Other nonlinear effects become important and finally saturates the instability. One such effect is the ion-trapping (Karatzas et al., 1975). Hence, one can modify the present theory to include this effect in order to enable one to look for a steady state solution.