CHAPTER II

ION ACOUSTIC SOLITARY WAVES IN A TWO-ELECTRON TEMPERATURE PLASMA

II.1 Introduction

Plasmas with electron velocity distribution that can be represented by the superposition of two Maxwellians are not infrequent under experimental conditions. For example, hot turbulent plasmas of thermonuclear interest interest often have a high energy tail (Utlaut and Cohen 1971, Sipler and Biondi 1972, Kruer et al. 1970, Kruer and Dawson 1972); strong electron beam-plasma interactions also result in such distributions (Sudan 1970) and more often, ordinary hot cathode discharge plasmas also have double electron-temperature distribution (Jones et al. 1974). Though the presence of a group of electrons with lower temperature, in an otherwise hot plasma, was known to the experimental plasma physicists for quite some time (Oleson and Pound 1949), the interest has
only recently been spurred in studying the propagation of ion acoustic waves in such a plasma (Jones et al. 1975). This study of Jones et al. has revealed some new aspects of propagation of ion acoustic waves in such a plasma namely, when a bunch of relatively cold electrons is present in an otherwise hot plasma, the ion acoustic speed is dominantly governed by the lower temperature. This result has some very interesting practical implications. For example, when one wants to utilize ion acoustic waves as a diagnostic tool for determining plasma parameters (e.g., calculating electron temperature from the measurement of ion acoustic speed), one has to be very careful if the plasma contains a colder electron component. Another possible interesting application suggested by the above result is the utilization of the ion acoustic waves to heat ions in a plasma by Landau damping. By introducing a small amount of relatively cold electrons, as the ion acoustic speed can be appreciably reduced, an otherwise undamped wave can also possibly be driven damped, thereby transferring energy to the plasma ions.

II.2 Linear Theory

When the physical processes that produce the two types of electrons with two different temperatures, have time scales much shorter than the relevant ion time scale, the two electron components can be treated as two fluids. Thus
describing the plasma by one dimensional multispecies fluid equations together with the Poisson's equation and assuming that the first order quantities go as \( \exp (i(kx - \omega t)) \), one obtains the linear dispersion relation (Jones et al. 1975).

\[
1 = \frac{\omega_{p1}^2}{\omega^2 - k^2 \lambda_{i}^2} + \frac{\omega_{p1}^2}{\omega^2 - k^2 \lambda_{e}^2} + \frac{\omega_{peh}^2}{\omega^2 - k^2 \lambda_{eh}^2},
\]

where \( \omega \) and \( k \) are the characteristic frequency and wave number, and \( \omega_p \) is the plasma frequency. \( \lambda_e(i) \) is the thermal velocity for the electrons (ions). The suffixes \( e \) and \( i \) refer to electrons and ions respectively whereas \( l \) and \( h \) refer to lower and higher temperature electron components respectively. For ion acoustic waves,

\[
\lambda_i \ll \omega/k \ll \lambda_{el}, \lambda_{eh},
\]

in which case, Eq.(2.1) can be simplified to give

\[
\omega \approx k C_{seff} \left[ 1 - \frac{1}{2} k^2 \lambda_{deff}^2 \right],
\]

where \( C_{seff} = (T_{eff}/m_i)^{1/2} \) and \( \lambda_{deff}^2 = T_{eff}/4 \bar{m} \bar{e}^2 \)

with

\[
T_{eff} = \frac{n_0 T_{eh} T_{el}}{(n_{oh} T_{el} + n_{oi} T_{ei})}
\]

Eqs.(2.3) and (2.4) show that the linear dispersion relation for the ion acoustic waves in a two-electron temperature plasma is similar to the one for a plasma with single electronic component, with the difference that the ion acoustic speed
(C_{seff}) and the Debye length (\lambda_{Deff}) are now characterized by the 'effective temperature' (T_{eff}). In other words, the restoring force for the ion acoustic waves, in this case, is given by an electron pressure which has to be defined in terms of an 'effective temperature' given by Eq.(2.4). From Eq.(2.4), we notice that the effective temperature is a function of both the temperatures and the fractional densities of the two components. It can also be seen from Eq.(2.4) that as the difference of temperatures between the two components increases, the effective temperature and hence the propagation characteristics of ion acoustic waves becomes dominantly governed by the lower temperature. For example, for a plasma with T_{eh} \sim 3 T_{el} (T_{el} = 1 eV) and n_{oh} = n_{ol}, T_{eff} = 1.5 eV. As the temperature of the two components becomes further apart, the relative importance of the high temperature component becomes even smaller. Consider a hypothetical case, where T_{eh} \to \infty and n_{ol}/n_{oh} is finite, in which case T_{eff} = (n_{oh}/n_{ol}) T_{el}. Thus, even if the cold component make up only 10 per cent of the total electron density, T_{eff} cannot be greater than 10 T_{el}, no matter how hot the other 90 per cent of the electrons are. These results have been verified in an experiment by Jones et al. (1975).
II.3 Nonlinear Theory

As mentioned in Chapter 1, it is well known (Washimi and Taniuti 1966, Davidson 1972) that the weakly nonlinear propagation of ion sound disturbances travelling near the ion sound speed can be described by a K-dV equation. Moreover, this equation possesses a stationary solitary wave solution which results due to an exact balance between nonlinear and dispersive effects. The strength of dispersion which is proportional to $\lambda_d^2$, will be quite different for a two electron temperature plasma compared to a plasma with single electronic component. Hence, the propagation characteristics of an ion acoustic solitary wave is expected to get modified in such a plasma (Goswami and Buti 1976). To determine quantitatively, how exactly the propagation characteristics of an ion acoustic solitary wave in such a plasma is modified, we proceed as follows.

The one dimensional basic set of equations governing the system consists of the ion continuity equation, the momentum transfer equations for ions and the two types of electrons and the Poisson's equation, namely

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i u_i) = 0 \quad , \quad (2.5)$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial n_i}{\partial x} + \frac{\partial \phi}{\partial x} = 0 \quad , \quad (2.6)$$
\[ n_{el} \frac{\partial}{\partial x} \left[ \frac{T_{eff}}{T_{el}} \Phi \right] - \frac{\partial n_{el}}{\partial x} = 0 \quad , \quad (2.7) \]
\[ n_{eh} \frac{\partial}{\partial x} \left[ \frac{T_{eff}}{T_{eh}} \Phi \right] - \frac{\partial n_{eh}}{\partial x} = 0 \quad , \quad (2.8) \]
and
\[ \frac{\partial^2 \Phi}{\partial x^2} = n_{eh} + n_{el} - n_i \quad . \quad (2.9) \]

In Eqs. (2.5) - (2.9) \( n_i \) and \( n_e \) are the ion and electron densities normalized to equilibrium value \( n_0 \), \( v_i \) is the ion fluid velocity normalized to \( C_{seff} \) and \( \Phi \) is the potential normalized to \( T_{eff} / e \). Moreover, lengths are normalized to \( \lambda_{Deff} \) and time is normalized to ion plasma period \( \omega_i^{-1} \) \( (\omega_i^{-1} = 4\pi n_0 m_i s^2 / e_i) \). Eqs. (2.7) and (2.8) can be immediately integrated to give

\[ n_{el} = n_{ol} \exp \left[ \frac{T_{eff}}{T_{el}} \Phi \right] \quad , \quad (2.10) \]

and
\[ n_{eh} = n_{oh} \exp \left[ \frac{T_{eff}}{T_{eh}} \Phi \right] \quad . \quad (2.11) \]

Now, on combining Eqs. (2.10) and (2.11) with Eq. (2.9) and on retaining terms up to \( \Phi^2 \), we get

\[ \frac{\partial^2 \Phi}{\partial x^2} = \Phi + \frac{\Phi^2}{2} - \tilde{n}_i \quad . \quad (2.12) \]

where \( \tilde{n}_i \) is the perturbed part of the ion density and

\[ \Delta = \left[ \frac{n_{eh}}{T_{eh}} + \frac{n_{ol}}{T_{el}} \right] T_{eff}^2 = \frac{n_{oh} (T_{el} / T_{eh})^2 + n_{ol}}{[n_{oh} (T_{el} / T_{eh}) + n_{ol}]} \quad . \quad (2.13) \]
In writing Eq. (2.12), we have made use of the charge neutrality condition i.e., \( n_0 = n_{oh} + n_{ol} \). Now let us introduce the stretched variables (Davidson 1972), \( \zeta = c^{1/2} (x - t) \) and \( \zeta' = c^{3/2} t' \) and for weak nonlinearities, use the following perturbation expansions:

\[
\begin{align*}
\hat{n}_i &= \epsilon \hat{n}^{(1)}_i + \epsilon^2 \hat{n}^{(2)}_i + \cdots , \\
\Phi &= \epsilon \Phi^{(1)} + \epsilon^2 \Phi^{(2)} + \cdots , \\
\text{and} \quad \psi_i &= \epsilon \psi_i^{(1)} + \epsilon^2 \psi_i^{(2)} + \cdots .
\end{align*}
\]

To the lowest order (i.e., \( O(c^{3/2}) \)) Eqs. (2.5), (2.6) and (2.12) give \( n_1^{(1)} = \Phi^{(1)} = \psi_i^{(1)} \). This means that, in the linear approximation, the propagation characteristics of the ion acoustic wave remains same as in a plasma with single electronic component, except that it is now propagating with velocity \( C_{\text{eff}} \). To the next order (i.e., to order \( c^{5/2} \)) Eqs. (2.5), (2.6) and (2.12) give

\[
\begin{align*}
- \frac{\partial^2 n^{(2)}_i}{\partial \zeta'^2} + \frac{\partial n^{(2)}_i}{\partial \zeta'} + \frac{\partial}{\partial \zeta'} \left( n^{(2)}_i \psi^{(2)}_i \right) + \frac{\partial \psi^{(2)}_i}{\partial \zeta'} &= 0 , \quad (2.14) \\
- \frac{\partial^2 \psi^{(2)}_i}{\partial \zeta'^2} + \frac{\partial \psi^{(2)}_i}{\partial \zeta'} + \psi_i^{(1)} \frac{\partial \psi^{(2)}_i}{\partial \zeta'} + \frac{\partial \Phi^{(2)}}{\partial \zeta'} &= 0 , \quad (2.15) \\
\text{and} \quad \frac{\partial^2 \Phi^{(2)}}{\partial \zeta'^2} &= \Phi^{(2)} + \Delta \left[ \frac{\Phi^{(2)}}{2} \right]^2 - n^{(2)}_i . \quad (2.16)
\end{align*}
\]

Eliminating \( n^{(2)}_i, \Phi^{(2)} \) and \( \psi_i^{(2)} \), we obtain
\[
\frac{\partial n_i^{(1)}}{\partial \tau} + \frac{1}{2} \left( 3 - \Delta \right) n_i^{(1)} \frac{\partial n_i^{(1)}}{\partial \xi} + \frac{1}{2} \frac{\partial^3 n_i^{(1)}}{\partial \xi^3} = 0
\]  
\tag{2.17}

In deriving Eq. (2.17), we have made use of the relation \( n_i^{(1)}(s(t)) = \xi(t) = v_i^{(1)}(s(t)) \). Eq. (2.17) is the K-dV equation which describes the weakly nonlinear propagation of ion acoustic waves in a plasma with two types of electrons having different temperatures. Now, let us look for a solution of Eq. (2.17) such that \( n_i^{(1)} \) depend on \( \xi \) and \( \tau \) only through \( \chi = \xi - v \tau \). Eq. (2.17), can then be integrated w.r.t. \( \chi \) under the boundary condition \( \gamma_i^{(1)}, \frac{d^2 n_i^{(1)}}{d \chi^2} \to 0 \) as \( |\chi| \to \infty \); this gives
\[
\frac{d^2 n_i^{(1)}}{d \chi^2} = 2 \left( \gamma_i^{(1)} - \frac{3 - \Delta}{2} \left[ n_i^{(1)} \right]^2 \right).
\tag{2.18}
\]

Multiplying both sides of Eq. (2.18) by \( 2 \frac{d n_i^{(1)}}{d \chi} \) and then integrating it again we get
\[
\frac{d n_i^{(1)}}{d \chi} = \gamma_i^{(1)} \left( 2 \left( \gamma_i^{(1)} - \frac{3 - \Delta}{2} n_i^{(1)} \right) \right)^{1/2} \tag{2.19}
\]

Finally integrating Eq. (2.19), we get
\[
\gamma_i^{(1)} = 3 \left( \frac{2}{3 - \Delta} \right) \frac{3 \cosh^2 \left[ \left( \frac{U}{2} \right)^{1/2} \left( \xi - v \tau \right) \right]}{2}
\tag{2.20}
\]

Eq. (2.20) is the solitary wave solution of the K-dV Eq. (2.17).

Alternatively, using a scaling of the space co-ordinate \( \xi \to \xi / (3 - \Delta) \), the stationary solitary wave solution of Eq. (2.17) can also be written as
\[
\gamma_i^{(1)} = 3 \frac{U}{\xi} \cosh^2 \left[ \left( \frac{3 - \Delta}{2} \right)^{1/2} \left( \frac{U}{2} \right)^{1/2} \left( \xi - \frac{3 - \Delta}{2} U \tau \right) \right]
\tag{2.21}
\]
II.4 Discussion

In Eqs.(2.20) and (2.21), \( U \) and \( U' \) are the velocities of the soliton which in general are arbitrary but of the order of \( C_{\text{seff}} \). For \( T_{\text{eh}} = T_{\text{el}} \), from Eq.(2.13) we get, \( \Delta = 1 \) and Eq.(2.20) or Eq.(2.21) then represents the solitary wave solution for ion acoustic waves in a plasma with single electronic component.

The solitary solutions of Eq.(2.17) given either by Eq.(2.20) or by Eq.(2.21) are valid only for \( \Delta < 3 \). For \( \Delta > 3 \), a solitary solution to Eq.(2.17) does not exist. This result may be understood as follows:

For \( \Delta \) to be \( \gtrsim 3 \) one needs large ratios of \( T_{\text{eh}}/T_{\text{el}} \) and \( n_{\text{oh}}/n_{\text{ol}} \). For example, for \( T_{\text{eh}}/T_{\text{el}} = 12 \) and \( n_{\text{oh}}/n_{\text{ol}} = 9 \), \( \Delta \approx 3.47 \). When \( T_{\text{eh}}/T_{\text{el}} \) is large, the effective temperature is mostly governed by \( T_{\text{el}} \) and an appreciable reduction in the strength of dispersion (which is proportional to \( \Delta^2 \)) takes place. Corresponding to \( \Delta \gtrsim 3 \), the strength of dispersion is so weak that a balance between the nonlinearity and dispersion can no longer take place. Hence, no solitary wave.

Referring to the work of Jones et al. (1975), we note that, in their experiment \( T_{\text{eh}}/T_{\text{el}} \) ranged from less than 2 to 5 while \( n_{\text{oh}}/n_{\text{ol}} \) ranged from about 1/6 to 3. This gives a variation of \( \Delta \) from 1.033 to 1.875. For example, when \( T_{\text{eh}}/T_{\text{el}} = 5 \) and \( n_{\text{oh}}/n_{\text{ol}} = 3 \), \( \Delta = 1.875 \).
Eq. (2.20) shows that, for a given width, as long as the solitary solution is maintained, the amplitude increases by a factor of $2/(3 - \Delta)$. In other words, for a given amplitude, the width of a solitary wave decreases by a factor of $((3 - \Delta)/2)^{1/2}$ (Eq. (2.21)). When a small fraction of relatively cold electrons is present in an otherwise hot plasma, $T_{\text{eff}} < T_{\text{eh}}$ and $\Delta > 1$, which implies an increase in amplitude of a solitary wave for a given width; this can be understood as follows: As $T_{\text{eff}}$ decreases, the strength of dispersion also decreases. Hence, a larger amplitude is necessary to produce sharper gradients so that the dispersion effects are sufficient to produce the soliton with the given width.

Lastly, we would like to emphasize that, in order to estimate the quantitative increase in the amplitude of an ion acoustic soliton for a given set of values of $T_{h}/T_{l}$ and $n_{oh}/n_{ol}$, we must take into account the decrease in $C_{\text{seff}}$ as well. The amplitude of the soliton is proportional to $U$ which in turn is proportional to $C_{\text{seff}}$ and thus $U$ goes as $T_{\text{eff}}^{1/2}$. Since for $T_{h}/T_{l} > 1$, $T_{\text{eff}} < T_{h}$ and $\Delta > 1$; the amplitude of the soliton will be determined by the net balance between the decrease in $U$ and the increase in the factor $2/(3 - \Delta)$. Let us take a specific example: $T_{h}/T_{l} = 5$ and $n_{oh}/n_{ol} = 3$ (Jones et al. 1975) for which case, $\Delta = 1.875$ and $2/(3 - \Delta) = 1.778$. Moreover, in this case $T_{\text{eff}} = 2.5 T_{l} = 0.5 T_{h}$; so that, $C_{\text{seff}} = 0.707 C_{s}$. Hence, the amplitude of the soliton is proportional to $3Ux(1.778 \times 0.707) \approx 3Ux(1.257)$, where $U$
is now proportional to \( U_s \). This shows that, for this set of parameters, there is a net increase of about 25 per cent in the amplitude of the soliton compared to one in a plasma with single electronic component having a temperature equal to \( T_n \).

II.5 Conclusions

Due to the presence of a relatively cold electron component in a plasma, the ion acoustic solitary wave of a given width has a larger amplitude. When the temperature differences between two electron components is sufficiently large, the strength of dispersion is reduced to such an extent that a solitary solution is no longer possible.