CHAPTER VI

A PERIODIC REVIEW REORDER POINT STOCHASTIC INVENTORY MODEL WITH TWO STORAGE FACILITIES

When the inventory system under considerations does not have sufficient storage capacity in their own warehouse (OW), to store the on-hand inventory, additional units are required to be kept in a rented warehouse (RW), which is assumed to have sufficiently large capacity. In this chapter a periodic review reorder point stochastic model with two storage facilities is studied. The model developed is illustrated with an example.

ASSUMPTIONS AND NOTATIONS

The following assumptions are made for the mathematical model development:

1) The inventory position of the system is reviewed regularly at a period of \( w \) time units. Whenever, the on-hand inventory is less than or equal to the reorder point \( s \), a fixed lot-size of \( q \) units is scheduled for replenishment; \( w \) and \( q \) are prescribed constant and \( s \) is the decision variable. Lead time is zero, so that the ordered lot-size is immediately added to the inventory.
iii) The demand \( x \) during any review period \( W_r \) is a random variable having pdf \( g(x) \), cdf \( G(x) \), \( 0 < x < m \), and mean demand

\[
\mu = \int_0^m x \, g(x) \, dx \quad \quad \text{(6.1)}
\]

We assume that the demand of \( x \) units occur in a uniform pattern during review period \( W_r \).

iii) Shortages, if any, are made up as soon as a fresh procurement arrives.

iv) OW can store up to a maximum \( W \) units only, where we assume that \( W > s \). Additional units, if any, are kept at RW, which has unlimited storage capacity.

v) The Inventory carrying cost is \( H/\text{unit}/\text{time unit} \) at OW, and \( F/\text{unit}/\text{time unit} \) at RW. Shortage cost is \( \pi/\text{unit}/\text{time unit} \) at OW, and \( \pi > F > H \).

OPERATIONS OF THE SYSTEM

Let \( S \) denote the on-hand inventory of the system at the beginning of a review period after a replenishment, if any. Then as shown by Naddor (7,p.234) \( S \) is a random variable (rv) with pdf

\[
h(S) = \frac{1}{\eta_p} \quad ; \quad s < S < s+\eta_p
\]

\[
= 0 \quad , \quad \text{otherwise} \quad \quad \text{(6.2)}
\]
In this situation two cases as given below may arise.

i) \( s \leq S \leq W \) — in this case storage at RW is not required.

ii) \( W \leq S \leq s + \sigma \) — storage at RW is needed.

Total cost of the system per time unit for these two cases may be obtained.

**Case 1**: For any \( S \in [s, W] \) as shown in the Naddor [7, p. 244] the total average cost per time unit is

\[
T_1(S) = (H + \pi) R(S) + \pi(\mu/2 - S) \quad \cdots \cdots \cdots \cdots \ (6.3)
\]

where

\[
R(S) = \int_0^S M(Z) \, dZ \quad \cdots \cdots \cdots \cdots \ (6.4)
\]

\[
M(Z) = \int_0^Z m(y) \, dy \quad \cdots \cdots \cdots \cdots \ (6.5)
\]

\[
m(y) = \int_y^\infty g(x)/x \, dx \quad \cdots \cdots \cdots \cdots \ (6.6)
\]

Hence for any \( S \in [s, W] \), total average expected cost of the system is given by

\[
ET_1(s) = \int_s^W T_1(S) \, h(S) \, dS
\]

\[
= \frac{(H + \pi)}{\sigma} \left[ V(W) - V(s) \right]
\]

\[
+ \left( \frac{\pi(W - s)/2}{\sigma} \right) [\mu - \gamma - \delta] \quad \cdots \cdots \cdots \cdots \ (6.7)
\]

where

\[
V(x) = \int_0^x R(S) \, dS \quad \cdots \cdots \cdots \cdots \ (6.8)
\]
Case II: In this case, mainly focus is given on the reorder point under the condition of $s < W$ and $S > W$; i.e., $s < W < S < s + q_p$. Whenever, $S > W$, $W$ units will be kept at OW and $(S-W)$ units will be stored at RW. The average expected cost per time unit at RW and OW for a given $S < (W, s+q_p)$ are discussed below.

Initially demands are satisfied from RW till the stocks at RW is completely exhausted. Further, demands are satisfied from OW, till the on-hand inventory level falls below reorder point $s$. Let $t_i(S|x)$ denote the time interval during which the demand of $(S-W)$ units occur; then clearly $t_i = t_i(S|x)$ is a random variable. Since, average demand during $W$ is $\mu$, the average duration for the demand of $(S-W)$ units to occur is approximately

$$C_i(S) = (S-W)W_p/\mu \quad \text{............(6.9)}$$

So, that total average inventory carried at RW during a cycle is

$$I_RW(S) = (S-W)C_i(S)/2W_p$$

$$= (S-W)^2/2\mu \quad \text{............(6.10)}$$

Note that during $(0,C_i(S))$, the inventory at OW is $W$. At OW two consequences may take place as given below:

No shortage will occur if $x \leq S$ and shortage will occur if $x > S$. 
If \( x \leq S \), then average inventory at OW during \((C_1(S), w_1)\) is
\[
I_{11} (x | x \leq S) = \frac{(W + S - x)}{2}
\]  
\[\cdots \cdots \cdots (6.11)\]
and average shortages are
\[
I_{12} (x | x \leq S) = 0
\]  
\[\cdots \cdots \cdots (6.12)\]
If \( x > S \), suppose that the system carries inventory during
\((C_1(S), t_1)\) and runs with shortages during \((t_1, w_p)\), then,
the average inventory per time unit during \((C_1(S), w_p)\) is
\[
I_{21} (x | x > S) = \frac{(W/2)(t_1 - C_1(S))/(w_p - C_2(S))}{W^2/2(x - S + W)}
\]  
\[\cdots \cdots \cdots (6.13)\]
and average shortages per time unit during \((C_1(S), w_p)\) is
\[
I_{22} (x | x > S) = \frac{(x - S)(w_p - t_1)/2(w_p - C_1(S))}{(x - S)^2/2(x - S + W)}
\]  
\[\cdots \cdots \cdots (6.14)\]
From (6.11) and (6.13), the average inventory per time unit
during \((C_1(S), w_p)\) is
\[
I_1 (S) = \int_{S-W}^{S} I_{11} (x | x \leq S) g(x) \, dx + \int_{S}^{\infty} I_{21} (x | x > S) g(x) \, dx
\]
\[
= \int_{S-W}^{S} \frac{(W + S - x)}{2} g(x) \, dx
\]
\[
+ \int_{S}^{\infty} \frac{W^2/(2(x - S + W))}{2} g(x) \, dx
\]
Let \( x - S + W = x' \) then \( dx = dx' \) and
\[
I_1 (S) = \int_{0}^{W} (W - x'/2) g_1(x') \, dx' + \int_{W}^{\infty} (W^2/2x') g_1(x') \, dx'
\]
\[
= \int_{0}^{W} M(Z) \, dZ = K(W)\]  
\[\cdots \cdots \cdots (6.15)\]
where \( g_1(x) \) denotes pdf of \( x' = x - S + W \).

Also average shortages per time unit during \((C_1(S), w)\) is

\[
I_2(S) = \int_{S}^{\infty} \left[ \frac{(x - S)^2}{2(x - S + W)} \right] g(x) \, dx
\]

\[
= \int_{S}^{\infty} \left[ \frac{(x' - W)^2}{2x'} \right] g_2(x') \, dx'
\]

\[
= R(W) + \frac{(\mu - S + W) / 2 - W}{2 - W} \tag{6.16}
\]

Hence, the total cost for the entire system during the review period \( w_p \) is then given by

\[
T_2(S) = \int_{W}^{\infty} F \, I_1(w) + H \, I_2(S) + \pi \, I_4(S)
\]

\[
= F(S - W)^2 / 2\mu - \pi(S - W)/2 + (H + \pi)R(W) + \pi(\mu/2 - W)
\]

\[
\tag{6.17}
\]

For any \( S \in (W, s + q_p) \) the expected total inventory cost per time unit during the review period \( w_p \) is

\[
ET_2(s) = \mathbb{E}[T_2(S)] = \int_{W}^{s + q_p} T_2(S) \, h(s) \, ds
\]

\[
= \int_{W}^{s + q_p} \left( F(S - W)^2/2\mu - \pi(S - W)/2 + (H + \pi)R(W) \right.
\]

\[
+ \pi(\mu/2 - W)) \cdot (1/q_p) \, ds
\]

\[
= \frac{1}{q_p} \left( \frac{F(s + q_p - W)^3/6\mu}{3} - \pi(s + q_p - W)^2/4 \right.
\]

\[
+ ((H + \pi)R(W) + \pi(\mu/2 - W))(s + q_p - W) \right)
\]

\[
\left. \tag{6.18} \right)
\]

The total average expected cost of the entire system in
either case is given by

\[ ET(s) = ET_1(s) + ET_2(s) \]

\[ = (H + \pi)/q_p (V(W) - V(s)) + \pi(W - s)/2 q_p (\mu - W - s) \]

\[ + 1/q_p \left[ F(s + q_p - W)^2 + \pi(s + q_p - W)^2 /4 \right] + \pi s + q_p - W) \]

\[ \text{................. (6.19)} \]

The optimum value of \( s = s_o \) can be obtained by solving the following equation

\[ R(W) - R(s) = (1/(H + \pi)) \left[ \pi(W - s) - F(s + q_p - W)^2 /2 \mu \right. \]

\[ + \left. \pi(s + q_p - W) \right] \text{................. (6.20)} \]

In (6.19) and (6.20), if we take \( W = s + q_p \) and \( F = H \), then we get

\[ ET(s) = (V(s + q_p) - V(s))/q_p + \pi(\mu/2 - s - q_p/2) \]

\[ \text{................. (6.21)} \]

and

\[ \left( R(s + q_p) - R(s) \right)/q_p = \pi / (H + \pi) \text{................. (6.22)} \]

Equations (6.21) and (6.22) are same, as those given by Naddor [9, p.244] for single storage model.

Example 6.1: Consider a system with pdf of demand \( x \) given by

\[ g(x) = \theta^2 x \exp(-\theta x) \quad 0 < x < \infty \]

\[ \text{................. (6.23)} \]

cdf

\[ G(x) = 1 - (1 + \theta x) \exp(-\theta x) \]

\[ \text{................. (6.24)} \]
and mean demand during $W$,

$$\mu = 2/\theta \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{
For the optimum value of $s_o$ is solution of

$$F_0 s^2 / 4 - (H + \pi/2 - 2(q_p - W)) + (H + \pi) \exp(-\Theta s)/\theta$$

$$= \pi(q_p - W)/2 - HW - F_0(q_p - W)^2 / 4 - (H + \pi)\exp(-\Theta W)/\theta$$

\[ \ldots \ldots \ldots \ldots (6.32) \]

Equation (6.32) can be solved by any numerical method.

Substituting $s_o$ value in equation (6.31), the total minimum cost can be obtained.