Chapter 5

Nonlinear Dynamical System Analysis

In the previous chapters we have used spectral analysis technique with the concept of modes in linear and nonlinear theories. However in recent years new techniques of analysing the time series with the concept of modes substituted by quantities such as dimension, Lyapunov exponent and the Kolmogrov entropy have developed. These are few tools of nonlinear dynamical system analysis. Given a times series, on embedding it in phase space one can get information about the above mentioned quantities using the nonlinear time series analysis. They have similar analog in plasma physics: Correlation analysis of Arter and Edward 1986, Sawley et al., 1987, Ströhlein and Piel 1989 suggest that number of degrees of freedom in a way represent the number of competing waves. Similarly Lyapunov exponent which determines the divergence (for positive Lyapunov exponent) or convergence (for negative Lyapunov exponent) of neighboring trajectories exponentially in a time series. In the language of plasma waves we speak of unstable (growing) and damped (decaying) waves.
The above discussion implies that if few modes are excited one would expect a low dimensional (described latter) attractor and attractor dimensionality increase with increase in number of waves excited. Dimensionality analysis of turbulent fluctuations (continuous power spectrum) in fusion devices (Sawely et al., 1989, Prado and Fiedler-Ferrari 1991) and even in laboratory devices (Ströhlein and Piel 1989) arrive at similar conclusion. However, Osborne and Provenzale (1989) have shown that stochastic time series with continuous power spectrum of $f^{-\alpha}$ power law decay have a correlation dimension of $2/(\alpha-1)$ where $1 < \alpha < 3$. The analysis of AE and AL index data of magnetosphere system possessing a power law of $f^{-2.5}$ (Vassiliadis et al., 1990) in low frequency range arrive at dimension close to that predicted by Osborne and Provenzale (1989). This type of power spectra is most commonly observed in many physical system.

The spectral characteristic study presented in chapter 3 and 4 suggests that the density and potential fluctuations makes a transition from coherent multimode state (exhibiting a considerable nonlinearity as seen from the bicoherence spectrum) to a turbulent state with increase in either magnetic field or pressure. In the turbulent state the wavenumber spectrum exhibits a similar power law decay of $k^{-2.6}$. One would be tempted to know the dimension, the Kolmogrov entropy and Lyapunov exponent of the observed time serie as a function of different parameters. In this chapter we will apply these tools on the time series used for the analyses in chapter 3 and 4. As these studies requires lot of CPU time we have restricted study only to two times series of density fluctuations of type–I experiment, one at coherent state i.e., at 200 gauss and one at 600 Gauss when system makes a
transition to the turbulence. The details of data analysis is given in section 5.1 and results of analysis are given in section 5.2.

5.1 Data Analysis

A system which has infinite degrees of freedom or random phases is known to exhibit Gaussian probability distribution function (PDF). The degree of phase coherence can be checked using higher order spectral technique i.e., bispectral analysis (see chapter 3.2). So one can distinguish stochastic system from deterministic chaotic system using tools such as PDF and bispectral analysis. However they do not indicate about the degree of determinism (degree of freedom). Let us look into the method to compute dimension and the statistics of that.

The state of the system can be described by its position along its phase space trajectory. For a dissipative dynamical system these trajectories are drawn towards a region called the attractor. In a chaotic system attractor has a complicated structure, with nearby trajectories exponentially diverging in some directions, while overall volume decreases. These results are sensitive to initial conditions. If one knows the trajectory of the system in phase space than tools of non-linear dynamical systems analysis can be used to quantify the dynamics of it. One such measure is the fractal dimension which gives an estimate of number of effective degrees of freedom in the system. For a chaotic system Hudsroff's dimension is always non-integer. In experiment one has a time series representing the evolution of a single quantity, which may or may not be one of the dynamical variable of the
system. Packard et al. (1980) and Takens (1981) have demonstrated that it is possible to reconstruct an equivalent phase space from the measured time series of a single variable by the method of delays. One creates a set of $D_E$ dimensional vectors whose components are just time delayed values of original time series: $\mathbf{x}_i = [x(t_i), x(t_i + \tau), \ldots, x(t_i + (D_E - 1)\tau)]$ where $t_i = i\delta t$, $\delta t$ is the sampling time, $\tau$ is the time delay between successive elements of vector, and $D_E$ is the embedding dimension. According to Takens (1981), for an attractor of $D$ dimension the reconstruction will reproduce the dynamical measures of the attractor as long as $D_E > 2D + 1$. However in many cases $D_E > D$ may work.

The dimension $D_q$ can be calculated by different ways of averaging. We prefer to adopt the one given by Grassberger and Proccacia (1983a) as it is easy to calculate and most widely used. It is given as:

$$C_q(r) = \left( \frac{1}{N} \left[ \sum_{i=1}^{N} \frac{1}{N-1} \sum_{j\neq i}^{N} \Theta(r - ||\mathbf{x}_i - \mathbf{x}_j||) \right] \right)^{-\frac{1}{q-1}}$$  \hspace{1cm} (5.1)

where $\Theta$ is called Heaviside function and $|| \cdot ||$ is the Euclidean norm. The value of $q$ will be between 0 to 2, accordingly the dimensions are named as:

$D_0 =$ capacity dimension or fractal dimension

$D_1 =$ information or pointwise dimension

$D_2 =$ correlation exponent

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\[ \Theta(s) = \begin{cases} 
1 & \text{if } s > 0 \\
0 & \text{elsewhere} 
\end{cases} \]

These different dimensions are connected by inequalities: \( D_o > D_1 > D_2 \).

For a chaotic system the correlation dimension can give a non-integer dimension, which will be lower bound to the number of degrees of system. When the value of \( r \) is much lesser than the horizontal extent of the data, but larger than scales where measurement errors or noise are important, it is shown that (Carssberger and Procaccia 1983a)

\[ C(r) \sim r^{D_q} \]

For each embedding dimension \( D_E \), this exponent \( D_q \) can be obtained from the slope of the linear part of a plot of \( \log C(r) \) versus \( \log r \). If \( D_q \) approaches a value independent of \( D_E \) as \( D_E \to \infty \) (usually \( D_E > 2D_q \) is sufficient), this value is defined as capacity or pointwise or correlation dimension depending on the value of \( q \). For a random noise which has infinite degrees of freedom one would expect that the dimension computed should increase with increasing embedding dimension (see figure 5.1). However for sinusoidal wave it saturates at dimension 1.

The time delay \( \tau \) could be chosen arbitrarily provided the amount of data is infinite and noise free. However in reality we will be having finite amount of noise data. Then the choice of \( \tau \) is critical. If \( \tau \) is chosen to be small then the phase space portrait of reconstructed attractor will be compressed in some direction while \( \tau \) is too large, noise effects dominates
Figure 5.1: Fractal of random noise together with dimension computed for sinusoid. Note that the fractal dimension of random noise will never be less than the embedding dimension. In contrast the sinusoid has dimension $\sim 1$.

and dynamical information will be lost. The idea behind choice of $\tau$ is to see that component of the vectors are as independent as possible. Many authors have suggested choosing $\tau$ to be some fraction of autocorrelation time. While in many cases this gives a good choice of $\tau$, it is by no means an optimal approach. However Fraser and Swinney (1986) have proposed a method based on mutual information between the components of the vector. It is defined as follows:

$$I(Y, Z) = \left< \log \left( \frac{P_{YZ}}{P_Y P_Z} \right) \right>$$

where $P_Y$ and $P_Z$ are probability density of two functions and $P_{YZ}$ is joint probability density of them. The best value of $\tau$ for which $I_\tau(t), x(t + \tau)$ is a local minimum. Liebert and Schuster (1989) have related mutual information to the generalised correlation integral. They show that the
minima of mutual information correspond to the minima of \( \log C_1 \):

\[
C_1(r) = \frac{1}{N} \sum_{i=1}^{N} \log \left( \frac{1}{N-1} \sum_{j'=1}^{N} \Theta(r - \| \vec{x}_i - \vec{x}_j \|) \right)
\]  

(5.2)

In case the autocorrelation time is closer to first minima of mutual information then there is a possibility that highly correlated points are included in the Heavisde function. These highly correlated points should not be included in the correlation integral as they are purely an effect of the choice of the sampling time. In most experimental situation it is not possible to decrease the sampling rate as it will unduly reduce the amount of data available for the dimension calculation. Theiler (1986) has proposed a method which can take care of this effect by slightly modifying the correlation integral:

\[
C_2(r) = \frac{2}{(N-W)(N-W+1)} \sum_{j=W}^{N} \sum_{i=1}^{N-j} \Theta(r - \| \vec{x}_i - \vec{x}_j \|)
\]  

(5.3)

For \( W = 1 \) one recovers equation 5.1. He gives a relation to calculate \( W \) as:

\[
W > \tau \left( \frac{2}{N} \right)^{2/D_E}
\]  

(5.4)

The above relation connects embedding dimension, number of points and \( \tau \). Eckmann and Rule (1992) have shown that, at least \( N = 10^{D_E/2} \) data points are necessary to reliably estimate fractal dimension \( D \). These calculations of dimension can be cross checked using the method of surrogated data set suggested by Theiler et al., (1991). In this method one randomises the phases of the fourier transform of the original time series and then inverting the transform. Using these surrogated time series the dimension

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is recalculated (see for details of implementation Theiler et al., 1991). If the results are not significantly different than those of the original time series, the dimension estimate should not be trusted.

The mean rate of loss of information is measured by Kolmogrov entropy. Grassberger and Procaccia (1983b) have proposed an algorithm leading to a lower limit $K_2$ of $K$, which obtained from vertical spacing between the parallel lines (with difference $P$ in embedding dimension between them) of log $C_2$ versus log($r$) plot.

$$K_2 = \frac{1}{P \tau} \log \frac{C_k(r)}{C_{k+n}(r)}$$  \hspace{1cm} (5.5)

Lyapunov exponent measures the average rate of separation of nearby points on an attractor. The number of Lyapunov exponents of a system is same as the number of degrees of freedom. The $i^{th}$ Lyapunov exponent $\lambda_i$ is defined as

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln | \exp(i\lambda_i t)|$$  \hspace{1cm} (5.6)

provided the limit exists. where $\lambda_i$ is the eigen value at equilibrium point. Hence the Lyapunov exponents are equal to the real part of eigen values at equilibrium point. They represent contraction if $\lambda_i < 0$ and vice versa. Thus for a chaotic system, at least one of the Lyapunov exponent should be positive. There are several algorithm proposed to compute largest Lyapunov exponent from a given time series. Of them, the most popular is by Wolf et al.,(1985) and by Eckmann and Ruelle (1985). These techniques can be reliably used to find out largest Lyapunov exponent. Kaplan and Yorke (1979) conjectured that Kolmogrov entropy $K$ is related to Lyapunov
exponent by

$$K = j + \left( \sum_{i=1}^{j} \lambda_i \right)$$

where $j$ is the number of Lyapunov exponents which assures the non-negativeness if the above sum i.e.,

$$\sum_{i=1}^{j} \lambda_i \geq 0$$

Eckmann and Ruelle (1992) have pointed out number of points needed to estimate Lyapunov exponents is about the square of that needed to estimate the dimension. So the data requirement for the calculation of Lyapunov exponent very large specially for dynamical system exhibiting dimension $\geq 4$. 

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5. 2 Results and Discussion

We have used fast algorithm suggested Grassberger and Proccica (1983a) by making use of floating point representation of number used in VAX8810 computer. This drastically reduces the computational time. For more detailed discussion see Grassberger and Proccacia (1983a) and Parker and Chu (1980). We have first tested out our implementation of the above outlined algorithm by analysing time series from mathematical systems with known attractor such as Hénon map. Correlation integral calculated for Hénon map is shown in figure 5.2. In order to find linear scaling region we have used local slope analysis. In this analysis slope of a straight line of every three points of the log C(r) versus log r curves was calculated and plotted as a function of middle point. From figure 5.2b the existence of three different region is clear. Region I, corresponding to the smaller r dominated by poor statistics and noise. Region III, corresponding to large r shows a constant decrease in the value of slope, due to the fact that the hyperspheres associated with these large values of r cover almost all the attractor. In region II we see that the curves for different embedding dimension converges and plateau is seen at 1.22. Linear scaling region begins to shrink as the embedding is increased. Computed dimension and Kolmogrov entropy is same as those obtained by Grassberger and Proccacia (1983a and 1983b).

We then performed the correlation analysis on two data sets, one at low magnetic field (≈ 200 Gauss) where in few marked peaks are observed and one at 600 Gauss where the power spectrum exhibits turbulent character-
Figure 5.2: Correlation integral in (a) and local slope in (b) for Hénon map with $a = 1.4$, $b = 0.3$, $\tau = 1$, and $N = 16000$ points. Note that the linear scaling region begins to decrease as embedding dimension is increased.
istics. The power spectrum obtained at 200 gauss at radial location of 6 cm in type-I experiment at low $10^{-4}$ Torr of operating pressure exhibits three marked peaks. The bicoherence spectrum also exhibit a well developed nonlinear interaction and application of nonlinear dynamical system analysis could be meaningful. The data collected was having a sampling time of 25μsec with 8192 samples. According to Ruelle (1989) a maximum dimension which we could go was <7. Mutual information calculated for the above data is shown in figure 5.3. The first minima of mutual information gives a value of $\tau \sim 40$ (in terms of sampling time). With this we have computed the $C_2(r)$ for different values of embedding dimension $D_E$. The results of analysis is given in figure 5.4. It can be clearly seen that the linear scaling region is small and decrease with increase in embedding dimension. The slopes of $\log C(r)$ versus $\log r$ saturates at dimension $D_2$.
= 4.7. Hence dynamics of the system is limited to 5. But we can see from equation 5.4 that we do not have sufficient data points to realise the same. Value of W calculated from equation 5.4 comes out to be 7.6. This implies that in our earlier calculation we have included autocorrelated points in Heaviside function. The correlation dimension thus obtained could be spurious. Hence we recalculated $C_2(r)$ with a modification given equation 5.3 as suggested by Theiler (1986) and Grassberger (1986). This avoids the effect of autocorrelation and excludes highly correlated points in Heaviside function. Figure 5.5 for $W = 8$ do not show a significant variation hence we can conclude that correlation dimension obtained do not have influence of autocorrelation. Further, we repeated the calculations of $C_2(r)$ with the method of surrogated data set which proved to be the most powerful cross check in many cases (Theiler 1992). The results of the analysis is shown in figure 5.6. It is clear that slope increases with increase in embedding dimension which is typical characteristics of random noise. This then certifies our calculations of correlation dimension and we can conclude that the dynamics of this nonlinear system is $\sim 5$. This is slightly higher than the number of waves excited. The higher dimension could be due to several factors affecting the dimensionality calculation. It could be due to the presence of small amplitude noise which has significant effect on the reconstructed phase space portrait which then contributes to increased dimension. This could also be due to the effect of filtering which has been shown to increase with high pass filtering and decrease with low pass filtering.

Then we computed Kolmogrov entropy from the vertical spacings of $\log C(r)$ versus $\log r$ and they are given in figure 5.7. The metric entropy
Figure 5.4: Correlation integral in (a) and slope in (b) as a function \( r \) for different embedding dimension. During this calculation we have used \( W = 1, \tau = 7 \) and \( N = 8192 \).
Figure 5.5: Same as figure 5.4 with $W = 8$
Figure 5.6: Correlation integral in (a) and slope in (b) as a function $r$ for different embedding dimension using surrogate data with $W = 1$, $\tau = 7$ and $N = 8192$. 

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Figure 5.7: Kolmogrov entropy calculated from the vertical spacing of figure 5.4.

approaches a saturated value of $0.018/\delta$ implying that the system is unstable at least in one dimension or at least one of the modes is unstable. As Kolmogrov entropy is sum of the positive Lyapunov exponents and number of Lyapunov exponent is same as the dimension of phase space so in principle more than one mode can be unstable. In order to estimate Lyapunov exponents of a system of dimension 5 the number of points required is $10^3$. Owing to the statistical limitations (Eckmann and Ruelle 1992) we cannot say anything about how many of these waves are growing.

The time series of density fluctuations at 600 gauss exhibit continuous power spectrum is. The data length is of 32000 point and with a sampling time of 20 $\mu$sec. Mutual information for the is shown in figure 5.8 which gives a value of $\tau \sim 7$. According to Eckmann and Ruelle (1992) the dimensional calculation should be less 9. Substituting $\tau \sim 7$ the and em-
bedding dimension of 9 in equation 5.4, we find that the value of W is <1. It implies that autocorrelation effects are very small in Heaviside function. Figure 5.9 shows the correlation integrals and their slopes for \( \tau = 7, \, W = 1 \). It is can be seen that the slopes converges to 5.6. Again width of linear region decrease with increase in embedding dimension. Further we made the test with surrogated data set the results of the analysis is shown in figure 5.10. It is seen that the dimension did not increase with increase in embedding dimension. Indicating that this saturation could be spurious. This further suggests that the dynamics is of higher dimension and a large number of modes are competing. Our observations are consistent with dimensionality calculation of turbulent fluctuation in fusion devices (Sawely et al.,1987, Prado and Fiedler–Ferrari 1991) and in laboratory plasmas (Ströhlein and Piel 1989). This can be interpreted in the following way: as the magnetic field is increased more and more modes exceed the FLR limit which then generate harmonics. Their non-linear interaction could modify the background each time. This modification once again excite long wavelength modes to larger amplitudes. They inturn generate secondary modes. Hence observation of high dimensionality is not unexpected.
Figure 5.8: Mutual information as a function of $\tau$ at 600 gauss.
Figure 5.9: Correlation integral in (a) and local slope in (b) for different embedding dimension, $\tau = 7$, $N = 32000$ and $W = 1$.  

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Figure 5.10: Same as in figure 5.8 but with $W = 2$. Note that the saturation disappears.
Figure 5.11: Correlation integral and slopes in for surrogated data set with \( W = 1 \), \( r = 7 \) and \( N = 32000 \). Note that the slope did not increase with increase in embedding dimension.