Chapter 10

Electron-acoustic solitary waves in an inhomogeneous plasma

10.1 Introduction

First time the idea of electron-acoustic (EA) waves have been originated for the explanation of numerical solutions of the linear electrostatic Vlasov dispersion equation in an unmagnetized, homogeneous plasma in the mind of Fried & Gould (1961). Later, the EA waves were observed in the laboratory (Derfler & Simonen (1969); Henry & Treuier (1972); Ikezawa & Nakamura (1981)) when the unmagnetized plasma is composed of two electron populations, described by two Maxwellian distribution functions with different temperature and density. These two populations will be referred to as cold and hot electrons.

The wave is analogous to ion-acoustic wave. The cold electrons in EA mode play the role of cold ions in the ion-acoustic mode. The ions dynamics plays no role in EA solitary wave. They are simply required for charge neutralization (Berthomier et al. (2000)). The propagation characteristics of EA waves have
been studied by a number of authors. The role of hot ions was considered for investigation (Yu & Shukla (1983)). However, both these models are restrictive because the large ion-electron temperature ratio required for weak Landau damping do not commonly occur.

Gary & Tokar (1985) performed a parameter survey and found conditions for the existence of $EA$ waves. The most important condition is $T_c << T_h$, where $T_c (T_h)$ is the temperature of cold(hot) electrons. Another condition for the existence of this wave is that the hot electron population should represent a significant fraction of the total electron density (probably more than 20%).

As plasmas with two electrons population, are known to occur frequently both in laboratory experiments and space, $EA$ wave plays an important role in these environments (Berthomier et al. (2000)). In the earth’s bow shock, particularly in the upstream region, the electron-acoustic waves have been suggested as a possible source of broadband electrostatic noise ($BEN$). They are also of potential importance in interpreting $BEN$ observed in cusp of terrestrial magnetosphere in auroral region and in geomagnetic tail (Schriver & Ashour-Abdalla (1989); Singh & Lakhina (2001); Tokar & Gary (1984)). The $EA$ mode has also been used to explain various wave emissions in different regions of the Earth’s magnetosphere. Furthermore, $EA$ mode has been applied to interpret the hiss observed in the polar cusp region.

Dubouloz et al. (1993) rigorously studied the $BEN$ observed in the day side of auroral zone and explained short duration burst of $BEN$ in terms of $EA$ solitary waves. They considered a one dimensional unmagnetized collisionless plasma consisting of cold electrons, Maxwellian hot electrons and stationary ions. A study
of nonlinear properties of larger amplitude necessarily useful for understanding. BEN was pointed by Mace et al. (1991). In the presence of large electron beam energy, the nonlinear effects combine with the dispersive properties of the EA wave which leads to formation of EA solitons (Berthomier et al. (2000)). This leads to the existence of new electron-acoustic solitons with velocity related to beam velocity. Berthomier et al. (2000) pointed out that the positive potential structure is very important from the point of view of the interpretation of various electrostatic structures observed in the auroral region at intermediate altitude by FAST, at higher altitudes by POLAR and in geomagnetic tail by GEOTAIL. Mace & Hellberg (2001) studied the effect of a magnetic field on electron-acoustic solitons. They derived KdV-ZK equation for weakly nonlinear EA waves and discussed its solitonic solutions.

Further, the effect of inhomogeneity on the nonlinear waves propagation is an important aspect. The non-uniformity in the plasma plays a crucial role in characterizing the physics of the nonlinear waves (Das et al. (1996); Duan & Zhao (1999); Li et al. (2004); Malik et al. (1994a); Rao & Varma (1979)). As a consequences of inhomogeneity, the KdV equation describing the soliton behavior is modified either by varying the coefficients or by some additional terms and it is generally treated as modified KdV equation. Under actual conditions we encounter the inhomogeneous plasma. So, it will be of great interest to analyze the electron-acoustic waves in an inhomogeneous plasma. We present here an investigation on nonlinear electron-acoustic waves in an inhomogeneous plasma consisting of cold electrons, hot electrons obeying the Boltzmann's distribution and stationary ions. KdV equation is derived with additional terms that governs
the propagation of electron-acoustic solitons in the presence of density gradient in a plasma. The modified $KdV$ equation is also solved for constant density gradient to obtain characteristics of electron-acoustic solitary waves.

### 10.2 Fluids equations

Let us consider a collisionless unmagnetized inhomogeneous plasma consisting of a cold electron fluid, hot electrons obeying a Maxwellian distribution and stationary ions. The dynamics of $EA$ waves is governed by

\begin{align}
\frac{\partial n_c}{\partial t} + \frac{\partial (n_c u_c)}{\partial x} &= 0 \quad (10.1) \\
\frac{\partial u_c}{\partial t} + u_c \frac{\partial u_c}{\partial x} &= \alpha \frac{\partial \phi}{\partial x} - \frac{3\sigma \alpha}{n_c} \frac{\partial n_c}{\partial x} \quad (10.2) \\
\frac{\alpha}{1 + \alpha} \frac{\partial^2 \phi}{\partial x^2} &= n_c + n_h - 1 \quad (10.3)
\end{align}

where $n_c(n_h)$ is cold (hot) electron number density normalized by its equilibrium value $n_0$, $u_c$ is the cold electron fluid speed normalized by $c_e = (\frac{k_B T_e}{\alpha m_e})^{1/2}$ and $\phi$ is the electrostatic wave potential normalized by $\frac{k_B T_e}{\epsilon}$. $m_e$ is the electron mass, $e$ is the magnitude of the electron charge and $k_B$ is the Boltzmann’s constant. The time and space variables are in the units of the cold electron plasma period $\omega_{pe}^{-1} = \left(\frac{m_e}{4\pi n_0 e^2}\right)^{1/2}$ and the hot electron Debye length $\lambda_{Dh} = \left(\frac{k_B T_h}{4\pi n_h e^2}\right)^{1/2}$ respectively.

The hot electrons are assumed to be Maxwellian distributed and associated number density is given by

\[\frac{\alpha}{1 + \alpha} n_h = e^\phi, \quad (10.4)\]

where $\sigma = T_e/T_h$ and $\alpha = n_{h0}/n_{e0}$. 

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10.3 Derivation of KdV equation

We carry out a reductive perturbation analysis of Eqs. (10.1 - 10.3) to obtain the KdV equation which governs the behavior of the one-dimensional small amplitude electron-acoustic solitary waves in the collisionless plasma. To determine the soliton behavior by carrying out a perturbation expansion based on the assumption that soliton width is small compared with the scale length of the plasma inhomogeneity. Under that condition, the soliton retains its identity and further its amplitude, width and speed are slowly varying functions of the position. In this analysis, we use a set of stretched coordinates, which is appropriate for spatially inhomogeneous plasma, along with the zeroth order fluid velocities.

In the reductive perturbation method, the independent variables are scaled according to the following stretching coordinates (Li et al. (2004); Rao & Varma (1979)):

\[ \xi = \epsilon^{1/2} \left( \int \frac{dx}{\lambda_0(x)} - t \right) \quad (10.5) \]
\[ \eta = \epsilon^{3/2} t \quad (10.6) \]

where \( \epsilon \) is small \((0 < \epsilon < 1)\) expansion parameter characterizing the strength of the inhomogeneity and \( \lambda_0 \) is the phase velocity of the electron-acoustic soliton. In the uniform plasma \( \lambda_0 \) is a constant. However, in a nonuniform plasma \( \lambda_0 \) is a function of the slow variable \( \eta \). Since the basis for the perturbation expansion is that the scale length be sufficiently large, it follow that \( \epsilon \) can be taken as formal expansion parameter. The condition \( \epsilon << 1 \) implies that the plasma dimension must be much large than the Debye length, which is satisfied in the most cases of
interest. Since \( \lambda_0, n_{c0} \) and \( n_{h0} \) are the function of \( x \) only, we have the following:

\[
\frac{\partial \lambda_0}{\partial t} = \frac{\partial n_{c0}}{\partial t} = \frac{\partial n_{h0}}{\partial t} = 0
\]  \quad (10.7)

The dependent variables are expanded in powers of \( \epsilon \) for the reductive perturbation as follow:

\[
\begin{align*}
    n_c &= n_{c0} + \epsilon n_{c1} + \epsilon^2 n_{c2} + \ldots \\
    u_c &= u_{c0} + \epsilon u_{c1} + \epsilon^2 u_{c2} + \\
    \phi &= \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \ldots
\end{align*}
\]  \quad (10.8)

Substituting Eqs. (10.5), (10.6) and (10.8) in the system of equations (10.1 - 10.3) and equating coefficients of same powers of \( \epsilon \), we get the following set of equations:

\[
\begin{align*}
    n_{c0} + n_{h0} - 1 &= 0 \quad (10.9) \\
    -\frac{\partial n_{c1}}{\partial \xi} + \frac{1}{\lambda_0} \frac{\partial}{\partial \xi} (n_{c0} u_{c1} + n_{c1} u_{c0}) + \frac{\partial}{\partial \eta} (n_{c0} u_{c0}) &= 0 \quad (10.10) \\
    -n_{c0} \frac{\partial u_{c1}}{\partial \xi} + \frac{u_{c0} n_{c0}}{\lambda_0} \frac{\partial u_{c1}}{\partial \xi} + u_{c0} n_{c0} \frac{\partial u_{c0}}{\partial \eta} - \frac{\alpha n_{c0} \phi_1}{\lambda_0} \frac{\partial \phi_0}{\partial \xi} + \frac{3 \sigma \alpha}{\lambda_0} \frac{\partial n_{c1}}{\partial \xi} + 3 \sigma \alpha \frac{\partial n_{c0}}{\partial \eta} &= 0 \quad (10.11)
\end{align*}
\]

Eq. (10.11) is integrated under the boundary conditions that \((n_{c1}, u_1, \phi_0) \to 0\) as \(|\xi| \to \infty\). This yields

\[
\phi_1 = -\frac{\xi (P n_{c0} T + R)}{\alpha ((1 + 3 \sigma \alpha) - P^2 n_{c0}^2)} \quad (10.12)
\]

\[
P = \frac{\lambda_0 - u_{c0}}{n_{c0}}, \quad R = \lambda \left( u_{c0} \frac{\partial u_{c0}}{\partial \eta} + \frac{3 \sigma \alpha}{n_{c0}} \frac{\partial n_{c0}}{\partial \eta} + \alpha \frac{\partial \phi_0}{\partial \eta} \right) \quad (10.13)
\]

\[
\text{and} \quad T = \frac{\lambda_0}{n_{c0}} \frac{\partial}{\partial \eta} (n_{c0} u_{c0})
\]
Since a first order term cannot be determined by the zeroth order terms, the right hand side of the above equation is made indeterminate by putting separately the numerator and denominator equal to zero. This procedure gives the following relations:

\[(\lambda_0 - u_{c0})^2 = (1 + 3\sigma\alpha)\]

\[PTn_{c0} + R = 0\]

It is clear from the relation (10.14) that phase velocity depends upon the \(\sigma (= T_c/T_h)\) and \(\alpha (= n_{ho}/n_{c0})\). It is decreases with the increase of the hot electrons temperature and increase with the increase of hot electrons density.

To derive the KdV equation, the various equations of higher order of \(\epsilon\) in the perturbation technique are summarized below.

\[-\frac{\partial n_{c2}}{\partial \xi} + \frac{1}{\lambda_0} \frac{\partial}{\partial \xi} (n_{c0} u_{c2} + n_{c1} u_{c1} + n_{c2} u_{co}) + \frac{\partial}{\partial \eta} (n_{c0} u_{c1} + n_{c1} u_{co}) = 0\]  
(10.16)

\[-n_{c0} \frac{\partial u_{c2}}{\partial \xi} - n_{c1} \frac{\partial u_{c1}}{\partial \xi} + \frac{1}{\lambda_0} \left( n_{c0} u_{c0} \frac{\partial u_{c2}}{\partial \xi} + n_{c0} u_{c1} \frac{\partial u_{c1}}{\partial \xi} + n_{c1} u_{co} \frac{\partial u_{c1}}{\partial \xi} \right)\]

\[+ n_{c0} u_{c1} \frac{\partial u_{c1}}{\partial \eta} + u_{c0} n_{c1} \frac{\partial u_{co}}{\partial \eta} + n_{c0} u_{c1} \frac{\partial u_{co}}{\partial \eta} = 0\]  
(10.17)

\[\frac{1}{\lambda_0^2} \frac{\partial^2 \phi_1}{\partial \xi^2} = \left( \frac{1 + \alpha}{\alpha} \right) n_{c2} + \left( \frac{1 + \alpha}{\alpha} \right) n_{ho} \phi_2 + \frac{1}{2} \phi_1^2 \left( \frac{1 + \alpha}{\alpha} \right) n_{ho}\]  
(10.18)

Differentiating Eq. (10.18) w.r.t. \(\xi\) and using Eq. (10.16) and (10.17), we obtain the following modified KdV equation:

\[\frac{\partial \phi_1}{\partial \eta} + A\phi_1 \frac{\partial \phi_1}{\partial \xi} + B \frac{\partial^3 \phi}{\partial \xi^3} + C \phi_1 - D \frac{\partial \phi_1}{\partial \xi} = 0,\]  
(10.19)

where

\[A = - \frac{(n_{co} + 2\alpha(1 + 3\sigma\alpha + \alpha))}{(2\lambda_0^2(1 + 3\sigma\alpha)^{1/2})}\]  
(10.20)
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\[ B = \frac{1}{2\lambda_0^2 n_0 (1 + 3\sigma a)^{1/2}(1 + \alpha)} \]

\[ C = \frac{1}{2n_0} \frac{\partial n_0}{\partial \eta} \]

and

\[ D = \frac{\xi}{\lambda_0} \frac{\partial n_0}{\partial \eta} \] (10.21)

Here it may be noted that last terms arise due to the density gradient in the plasma and these are the additional terms to the KdV equation.

10.4 Discussion

The modified KdV Eq. (10.19) has been obtained in the inhomogeneous plasma without any approximation or condition of weak and strong density gradient. We follow the approach and conditions of Singh et al. (2005) for study the characterization of solitons in inhomogeneous plasma. In order to obtain the solitary wave solution, we simplify the equation to a standard form. For this, we use a new transformation

\[ \phi_1 = g(\eta) \Phi_1 \] (10.22)

where

\[ g(\eta) = \exp \left( - \int \frac{\partial n_0}{\partial \eta} \, d\eta \right) \]

Using this transformation in Eq. (10.19), we get the well known form of KdV equation

\[ \frac{\partial \phi_1}{\partial \eta} + gA\phi_1 \frac{\partial \phi_1}{\partial \xi} + B \frac{\partial^3 \phi_1}{\partial \xi^3} = 0 \] (10.23)
Figure 10.1: Soliton profile for different values of cold electron density \(n_{c0}\), with fixed value of \(\alpha = 4, \sigma = 0.8, V = 1.01\) and \(u_{c0} = 0.5\). Solid curve corresponds to \(n_{c0} = 0.2\), dotted to \(n_{c0} = 0.3\) and dashed to \(n_{c0} = 0.4\).

In order to obtain the solution of Eq. (10.23), we use the transformation \(\tau = \xi - V\eta\), where \(V\) is a constant. The solution is given as:

\[
\phi_1(\tau) = \phi(\tau) = \Phi_m sech^2 \left\{ \frac{\tau}{W} \right\}
\]

(10.24)

where, the peak amplitude \(\Phi_m\) and width \(w\) of the electron-acoustic solitons are given as

\[
\Phi_m = \frac{3V}{gA}
\]

(10.25)

and

\[
W = \sqrt{\frac{4B}{V}}
\]

(10.26)

It is observed that the nonlinear coefficient \(A\) depends upon the the temperatures and the number densities of hot and cold electrons. It is also clear from
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Figure 10.2: Soliton profile for different values of speed ($u_{c0}$), with fixed value of $\alpha = 4$, $\sigma = 0.8$, $V = 1.01$ and $n_{c0} = 0.2$. Solid curve corresponds to $u_{c0} = 0.2$, dotted to $u_{c0} = 0.4$ and dashed to $u_{c0} = 0.6$.

Figure 10.3: Soliton profile for different values of temperature ratio of cold to hot electrons ($\sigma$), with fixed value of $\alpha = 4$, $u_{c0} = 0.5$, $V = 1.01$ and $n_{c0} = 0.2$. Solid curve corresponds to $\sigma = 0.6$, dotted to $\sigma = 0.7$ and dashed to $\sigma = 0.8$. 
the expression that the nonlinear coefficient is negative, so it is deduced that only rarefactive electron-acoustic solitons exist. To study the effect of temperatures and densities of hot and cold electrons on the amplitude, the numerical computation is done. To see the effect of the number density of cold electrons on the amplitude and width of the solitons, we plot the solitons profile at different values of the cold electrons density ($n_c$) as shown in Figure 10.1. It is observed that the amplitude remains constant but width decreases with increase in the cold electrons density. It means that higher is the cold electrons component in plasma, smaller will be the width of the solitons.

Figure 10.2, shows the profile of the soliton at different value of initial speed of the solitons. It is evident from the figure that when we increase the $u_c$, amplitude of the soliton increases, while opposite the case for width, i.e., width decreases with increase in the speed of the soliton. Similar behavior is observed when we plot soliton profile for different temperature ratio of cold to hot electrons ($\sigma$) (see Figure 10.3).

10.5 Conclusions

In this chapter, we have studied the role of densities, velocity and temperature of cold and hot electrons on electron-acoustic solitons in collisionless inhomogeneous plasma. We have derived the modified KdV equation using reductive perturbation method. Only rarefactive solitons are observed to exist in such plasma system. It is observed that in all the cases width decreases while, amplitude is affected only by speed and temperature of the solitons.
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