In Mathematics we behold the conscious logical activity of the human mind, in its purest and most perfect sense.

- Helmholtz.
(a) Two variable Analogue of Extended Jacobi Polynomials.

(b) Another form of Extended Jacobi polynomial in two variable and some generating functions.
(a) TWO VARIABLE ANALOGUE OF EXTENDED JACOBI POLYNOMIAL:

11a.1 The extended Jacobi polynomial is defined by Thakare [11]

\[ F_n (\alpha, \beta; x) = \frac{\lambda^n (1+\alpha)^n \lambda^n (1+\beta)^n}{n!} \binom{\lambda}{n} _2 F_1 \left[ \begin{array}{c} -n, 2n+\alpha+\beta+1 \\ \beta+\lambda, b-a \end{array} \right] \]

Two variable analogue of generalized Laguerre polynomial is defined by Beniwal [1, p.1]

\[ L_{m,n} (b, b', x, y) = \frac{(b)_m (b')_n}{m! n!} F_2 \left[ \begin{array}{c} b+b'+a, -m-n; b, b'; x, y \end{array} \right] \]

where, \( F_2 \) is Appell's function defined by [3, p.23]

We will define a two variables analogue of extended Jacobi polynomial by means of Kampé de Feriet function [3, p.29] as follows

\[ F_{m,n} (a_1, \ldots, a_A; -m-n, b_1, b_1', \ldots, b_B, b_B'; c_1, \ldots, c_C; d_1, d_1', \ldots, d_D, d_D') \]

\[ = \frac{(d_1)_m (d_1')_n}{m! n!} \left[ \begin{array}{c} A \\ B+1 \\ C \\ D \end{array} \right] \left[ \begin{array}{c} a_1, \ldots, a_A \\ -m-n, b_1, b_1', \ldots, b_B, b_B' \\ c_1, \ldots, c_C \\ d_1, d_1', \ldots, d_D, d_D' \end{array} \right] x, y \]
Particular Case:

(i) Put \( A = D = 1, B = C = 0, a_1 = b + b' + a, \)
\( d_1 = b, d'_1 = b', \lambda = 1, \) then

\[
F_{m,n}(b+b'+a; -m, -n; b, b'; x, y) = \lambda^{m+n}(b+1)_m (\delta+1)_n \frac{m+n}{m! \cdot n!} F \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -m, -n, m+\beta + b, n+\gamma + \delta + 1; x, y \\ \beta + b, \delta + 1 \end{bmatrix}
\]

(ii) Put \( A = C = 0, B = D = 1,\)
\( b_1 = m + \alpha + \beta + 1, \)
\( b'_1 = m + \gamma + \delta + 1, \)
\( d_1 = \beta + 1, d'_1 = \delta + 1, \) then

\[
F_{m,n}(-m, -n, m+\alpha+b+1, n+\gamma+\delta+1; \beta+1, b'_1, x, y)
\]

(iii) Put \( A = C = 0, m = 0\)

\[
F_{0,n}(0, -n, b_1, b'_1, \ldots; b_B, b'_B; d_1, d'_1; \ldots, d_B, d'_B; x, y)
\]

\[
= \lambda^n(d'_1)_n \frac{n!}{n!} F \begin{bmatrix} 0 \\ B+1 \\ D \end{bmatrix} \begin{bmatrix} 0, -n, b_1, b'_1, \ldots; \beta b, b'_B \end{bmatrix} \begin{bmatrix} x, y \end{bmatrix}
\]
(11a.1.6) \[ \frac{\lambda^n(d'_i)^d}{n!} B^{+if} F_D \left[ \begin{array}{c} -n, b'_1, ... , b'_B ; \vdots \end{array} \right] \]

(iv) Put \( A = C = 0, x = 0 \), then

\[ F_{m,n}(-m, -n, b_1, b'_1, ... , b_B, b'_B, d_1, d'_1, ... , d_D, d'_D, 0, y) \]

\[ \frac{\gamma^{m+n} (d_i)_m (d'_i)_n}{m! n!} F \left[ \begin{array}{c} 0 \\ m+n \\ \vdots \\ D \\ \vdots \\ 0 \\ D \\ \vdots \\ m+n \\ \vdots \\ 0 \\ D \\ \vdots \\ m+n \end{array} \right] \]

(11a.1.7) \[ \frac{\gamma^{m+n} (d_i)_m (d'_i)_n}{m! n!} B^{+if} F_D \left[ \begin{array}{c} -n, b'_1, ... , b'_B ; \vdots \end{array} \right] \]

11a.2 The Rodrigues Type formula:

Shankar [10] has proved that if

\[ F = \sum_{\alpha, \beta} a_{\alpha, \beta} x^{\alpha} y^{\beta+q} \]

then

\[ \frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[ x^{b-1} y^{b'-1} F \right] = x^{b-m-1} y^{b'-n-1} x \]

\[ x \sum_{\alpha, \beta} a_{\alpha, \beta} \frac{(b)_{\alpha+\beta}(b')_{\alpha+\gamma}}{(b)_{\alpha+\beta-m}(b')_{\alpha+\gamma-n}} x^{\alpha+\beta} y^{\beta+\gamma} \]
Thus we can write:

\[(\text{la.2.1}) \frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[ x^{d_1+m-1} y^{d'_1+n-1} G \right] \]

\[= (d_1)_m (d'_1)_n x^{d_1-1} y^{d'_1-1} \sum_{\alpha, \lambda = 0}^{\infty} a_{\alpha, \lambda} \frac{(d_1+m)_\alpha (d'_1+n)_\lambda}{(d_1)_m (d'_1)_n} x^\alpha y^\lambda \]

where \( G = \sum_{\alpha, \lambda = 0}^{\infty} a_{\alpha, \lambda} x^\alpha y^\lambda \)

With the help of the relation (\text{la.2.1}) we get

the Rodrigues type formula of the polynomial as:

\[(\text{la.2.2}) P_{m,n} (a_1, \ldots, a_A; -m, -n, b_1, b'_1, \ldots; b_B, b'_B; c_1, \ldots; c_C; d_1, d'_1, \ldots; d_D, d'_D; x, y) \]

\[= \frac{x^{1-d_1} y^{1-d'_1}}{m! n!} \lambda^{m+n} \frac{\partial^{m+n}}{\partial x^m \partial y^n} \left( x^{d_1+m-1} y^{d'_1+n-1} \right) \]

\[\times \left[ A \begin{bmatrix} a_1, & \ldots, & a_A \\ B+1 & -m-1, & b_1, b'_1, \ldots, b_B, b'_B \\ C & c_1, & \ldots, c_C \\ D & d_1+m, & d'_1+n, d_2, d'_2, \ldots, d_D, d'_D \end{bmatrix} \right] \]

\[\cdot \begin{bmatrix} x, y \end{bmatrix} \]

**Particular Case:**

In (\text{la.2.2}) put \( A = D = 1, B = C = 0, a_1 = b + b' + a, d_1 = b, d'_1 = b', \lambda = 1, \)

then in view of (\text{la.1.4}) we get the Rodrigues type formula for \( L_{m,n}^{(a-m-n)} (b, b', x, y) \) [1, p.2]:

\[ L_{m,n}^{(a-m-n)} (b, b', x, y) = \frac{x^{1-b} y^{1-b'}}{m! n!} x \]

\[\times \frac{\partial^{m+n}}{\partial x^m \partial y^n} \left[ x^{b+m-1} y^{b'+n-1} F_2 (b+b'+a, -m, -n; b+m, b'+n, x, y) \right] \]
Hence, we have the generating function for the polynomials as:

\[
\frac{\beta}{\prod_{j=1}^{d} (d_j)^{x} (d'_j)^{d_j}} (\frac{-\lambda x t}{1-\lambda x t})^{x} (\frac{-\lambda y T}{1-\lambda y T})^{d_j}
\]

\[
\sum_{m,n \geq 0} \frac{\Gamma(m) \Gamma(n)}{(d_m) (d'_n)} F_{m,n} (\alpha_1, \ldots, \alpha_n; -m, -n, b_1, b_1', \ldots, b_B, b_B'; c_1, \ldots, c, d_1, d_1', \ldots, d_D, d_D'; x, y) t^m T^n
\]

\[
= (1-\lambda t) (1-\lambda T)^{\lambda} F \left[ \begin{array}{c} A \\ B+1 \\ C \\ D \end{array} \right| \begin{array}{cccc} a_1, \ldots, a_A \\ v, v', b_1, b_1', \ldots, b_B, b_B' \\ c, \ldots, c \\ d_1, d_1', \ldots, d_D, d_D' \end{array} \frac{-\lambda x t}{1-\lambda x t}, \frac{-\lambda y T}{1-\lambda y T} \right]
\]

**Particular Cases:**

(i) In (11a.3.1) put \(A = D = 1, B = C = 0,\)

\[a_1 = b + b', \quad d_1 = b, \quad d_1' = b', \quad \gamma = c,\]

\[\gamma' = c', \quad \lambda = 1,\]

then using the special case of Kampe de Feriet function [3, p.30], we get generating function for the polynomial

\[
L_{m,n}^{(a-m-n)} \left( b, b', x, y \right) \left[ 6, p.9 \right]:
\]

\[
(11a.3.2) \quad (1-t)^{-c} (1-T)^{-c'} F_2 \left[ \begin{array}{c} b+b'+a, \ c, \ c' \end{array}; b, b' \right]
\]

\[
= \sum_{m,n \geq 0} \frac{\Gamma(m) \Gamma(n)}{(b)^m (b')^n} L_{m,n}^{(a-m-n)} \left( b, b', x, y \right) t^m T^n.
\]
(ii) In (11a.3.1) put \( A = C = 0, \ B = D = 1, \)
\[
\begin{align*}
    b_\perp &= m + \alpha + \beta + 1, \\
b^\perp &= n + \gamma + \delta + 1, \\
d_\perp &= \delta + 1,
\end{align*}
\]
and replace \( x \) by \( \frac{b-x}{b-a} \) and \( y \) by \( \frac{b-y}{b-a} \), then we get generating function for extended Jacobi polynomial
\[
\sum_{m,n=0}^{\infty} \frac{(\mu)m(\nu)n}{(\beta+1)m(\delta+1)n} F_m(\alpha,\beta; x) F_n(\nu,\delta; y) t^m T^n
\]
\[
= (1-\lambda t)^{-\mu} (1-\lambda T)^{-\nu} \cdot F \left[ \begin{array}{c}
0 \\
2 \\
0 \\
1
\end{array} \right] \begin{array}{c}
\frac{(x-b)\lambda t}{(b-a)(1-\lambda t)} \\
\frac{(y-b)\lambda T}{(b-a)(1-\lambda T)}
\end{array}
\]

Using the special case of Kampé de Feriet function [3, p.30], we get
\[
(11a.3.3) \sum_{m,n=0}^{\infty} \frac{(\nu)m(\mu)n}{(\beta+1)m(\delta+1)n} F_m(\alpha,\beta; x) F_n(\nu,\delta; y) t^m T^n
\]
\[
= (1-\lambda t)^{-\nu} (1-\lambda T)^{-\mu} \cdot \sum_{j=0}^{\infty} \frac{(x-b)\lambda t}{(b-a)(1-\lambda t)} j 
\]
\[
\cdot \sum_{j=0}^{\infty} \frac{(y-b)\lambda T}{(b-a)(1-\lambda T)} j
\]
Consider the series

\[
\sum_{m,n=0}^{\infty} \frac{(\alpha_1)_{m+n}}{(d_i)_m (d'_i)_n} F_{m,n}(a_1, \ldots, a_A; -m, -n, b_1, b'_1, \ldots, b_B, b'_B; c_1, \ldots, c_C; d_1, d'_1, \ldots, d_D, d'_D; x, y) t^m T^n
\]

\[
= \sum_{m,n=0}^{\infty} \sum_{k,l=0}^{\infty} \frac{\lambda_1^{m+n-\lambda}}{m! n!} \frac{A}{C} \prod_{d=1}^{A} (q_d)_{\lambda+d} \frac{B}{D} \prod_{d=1}^{B} (d_d)_{\lambda+d} \prod_{d=1}^{C} (c_d)_{\lambda+d} x^k y^l t^m T^n
\]

\[
= \sum_{m,n=0}^{\infty} \sum_{k,l=0}^{\infty} \frac{\lambda_1^{m+n+\lambda}}{m! n!} \frac{A}{C} \prod_{d=1}^{A} (q_d)_{\lambda+d} \frac{B}{D} \prod_{d=1}^{B} (d_d)_{\lambda+d} (-x)^k (-y)^l t^m T^n
\]

\[
= \sum_{k,l=0}^{\infty} \frac{\lambda_1^{-\lambda-\delta}}{(1-\lambda t - \lambda T)^\delta} \frac{A}{C} \prod_{d=1}^{A} (q_d)_{\lambda+d} \frac{B}{D} \prod_{d=1}^{B} (d_d)_{\lambda+d} \frac{\lambda_1^{-\lambda}}{\delta!} x^k y^l t^m T^n
\]

\[
= \left( \frac{-\lambda x t}{1-\lambda t - \lambda T} \right)^\delta \left( \frac{-\lambda y T}{1-\lambda t - \lambda T} \right)^\delta
\]
Thus the generating function for the polynomial is given by

\[
(11a.4.1) \quad \sum_{m,n=0}^{\infty} \frac{(\alpha_i)_m}{d_i)_m} \frac{F_{m,n}}{d_i)_n} \left( a_1 \ldots a_A ; -m,-n, b_1, b_2 \ldots b_B, b_B^\prime ;
\right.
\]
\[
c_1, \ldots, c_B^\prime ; d_1, d_2, \ldots, d_D, d_D^\prime ; x, y \right) t^m T^n
\]

\[
= (1 - \lambda t - \lambda T) \quad F \left[
\begin{array}{ccc}
A+1 & a_1, a_2, \ldots, a_A & -\lambda t \\
B & b_1, b_2, \ldots, b_B, b_B^\prime & -\lambda T \\
C & c_1, \ldots, c_C & -y \lambda T \\
D & d_1, d_2, \ldots, d_D, d_D^\prime & (1 - \lambda t - \lambda T)
\end{array}
\right]
\]

**Particular Cases:**

(i) In (11a.4.1) set \( A = D = 1, B = C = 0, \)
\( a_1 = a_1 = b + b' + a, \)
\( d_1 = b, d_1^\prime = b', x = 1, \)
then we get the generating function for the polynomial

\[
\text{L}^{(a-m-n)}_{m,n} (b, b', x, y) [1, p.6]
\]

(11a.4.2) \[
\sum_{m,n=0}^{\infty} \frac{(b+b'+a)_m}{b)_m} \frac{(a-m-n)}{b')_n} \text{L}^{(a-m-n)}_{m,n} (b, b', x, y) t^m T^n
\]

\[
= (1 - t - T) \quad F_4 \left[
\begin{array}{cccc}
b - b' & a_1, b + b' + a & -\frac{\lambda t}{1 - \lambda t - \lambda T} & -\frac{\lambda T}{1 - \lambda t - \lambda T}
\end{array}
\right]
\]

where \( F_4 \) is Appell's function and a special case
of Kampe de Feriet function given by [3, p.30]:

\[
\text{p}_{2,0}^{2,0} = F_4.
\]
In (11a.4.1) set $A = C = 0$, $B = D = 1$,
$$a_1 = 1 + y + \delta, \quad b_1 = n + y + \delta + 1, \quad d_1 = \delta + 1,$$
$$m = 0, \quad y = \frac{b - z}{b - a},$$
then we get the generating function for extended Jacobi polynomial

\[
\sum_{n=0}^{\infty} \frac{(1+y+\delta)^{n}}{(\delta+1)^{n}} F_n(y, \delta; z) T^n
\]
\[
= (1 - \lambda T)^{-1-y-\delta} \binom{1+y+\delta}{\delta+1} \frac{\lambda(z-b) T}{(b-a)(1-\lambda T)}
\]

In (11a.4.1) set $A = C = 0$, $B = D = 1$,
$$a_1 = 1 + \alpha + \beta, \quad b_1 = \alpha + \beta + 1, \quad d_1 = \alpha + 1, \quad m = 0,$$
$$\lambda = \frac{1}{2} \quad \text{and} \quad y = \frac{1 - z}{2},$$
then we get the generating function for Jacobi polynomial [1, p.7] :

\[
\sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)^{n}}{(1+\alpha)^{n}} \binom{\alpha}{\beta-n} (z) T^n
\]
\[
= (1 - T)^{-1-\alpha-\beta} \binom{1+\alpha+\beta}{1+\alpha} \frac{(z-1) T}{z(1-T)}
\]

Another form of extended Jacobi polynomial in two variables and some generating functions :

Malave and Bhonsle have investigated certain properties of Jacobi polynomials of class II defined by Koornwinder [5]. Likewise we define
two variable analogue of extended Jacobi polynomials as follows:

For \( z > 1 \), the polynomials

\[
F_{n,k}^{(z; x, y)} = F_{n-k}^{(z+k+\frac{1}{2}, z+k+\frac{1}{2}; x)}.
\]

\[
\cdot \left[ (x-a)(b-x) \right]^{k/2} F_k^{(z, z; \frac{y}{(x-a)(b-x)})};
\]

are orthogonal w.r.t. the weight function

\[
(b - x^2 - y^2)^z \text{ on } b \leq x \leq a.
\]

The purpose of this section is to obtain certain generating functions for the classical orthogonal polynomial set \( F_{n,k}(z; x, y) \) of two variables defined by (11b.1.1)

If we put \(-a = b = 1\) in the relation (11b.1.1) so that it reduces to the two variable analogue of the Jacobi polynomials of class II [5].

\[
x P_{n,k}^{(z)}(x, y) = P_{n-k}^{(z+k+\frac{1}{2}, z+k+\frac{1}{2})}(x) . \]

\[
\cdot (1-x^2)^{k/2} P_k^{(z, z; \frac{y}{(c^{1-x^2})})};
\]

for \( z > 1, n \geq k \geq 0 \).
Formulas Required:

We have the following generating functions for the extended Jacobi polynomials [7],[8],[11]:

(11b.2.1) \[ \sum_{n=0}^{\infty} F_n(\alpha-n, \beta-n; \chi) t^n = \left[1 + ct(x-a)\right]^\alpha \left[1 + ct(x-b)\right]^\beta \]

(11b.2.2) \[ \sum_{n=0}^{\infty} \frac{F_n(\alpha; \chi) b^n t^n}{(1+\alpha)_n (1+\beta)_n} = {}_1F_1\left[1+\beta; bct(x-b)\right] \cdot {}_1F_1\left[1+\alpha; bct(x-a)\right] \]

(11b.2.3) \[ \sum_{n=0}^{\infty} \frac{\gamma^n}{\gamma_n} F_n(\alpha-n, \beta-n; \chi) t^n = {}_1F_1\left[\gamma; -\alpha, -\beta, \gamma; -ct(x-a), ct(b-x)\right] \]

(11b.2.4) \[ \sum_{n=0}^{\infty} F_n(\alpha; \chi) t^n = 2^{\alpha+\beta} R^{-1}(1+\lambda t+R)^{-\alpha} \\
\quad \cdot (1-\lambda t+R)^{-\beta} \]

where

\[ R = \left[1 + \frac{\sqrt{\lambda t}}{b-a} (a+b-2x) + \lambda^2 t^2\right]^{1/2} \]
\[
\sum_{n=0}^{\infty} \frac{(\delta)_n}{(1+\alpha)_n(1+\beta)_n} F_n(\alpha, \beta; x) \frac{t^n}{(n+\gamma)^2}
\]

\[
= \psi_2(\delta; 1+\beta, 1+\alpha; \frac{\lambda t(x-a)}{b-a}, \frac{\lambda t(x-b)}{b-a})
\]

where \( \psi_2(a, \gamma, \gamma', x, y) \) is Horn's function[2, p.225].

\[
\sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{(1+\beta)_n} F_n(\alpha, \beta; x) \frac{t^n}{(n+\gamma)^2}
\]

\[
= (1-\lambda t)^{-\alpha-\beta-1} 2 F_1 \left[ \frac{1+\alpha+\beta}{2}, \frac{2+\alpha+\beta}{2}; 1+\beta; \frac{4\lambda t(x-b)}{(b-a)(1-\lambda t)^2} \right]
\]

\[
\sum_{n=0}^{\infty} \frac{(\gamma)_n (\delta)_n}{(1+\gamma)_n(1+\beta)_n} F_n(\alpha, \beta; x) \frac{t^n}{(n+\gamma)^2}
\]

\[
= F_4 \left[ \psi(\delta; 1+\beta, 1+\alpha; \frac{\lambda t(x-a)}{b-a}, \frac{\lambda t(x-b)}{b-a} \right]
\]

and

\[
\sum_{n=0}^{\infty} F_n(\alpha, \beta; x) \frac{t^n}{(n+\gamma)^2}
\]

\[
= F_4 \left[ 1+\alpha, 1+\beta; 1+\beta, 1+\gamma; \frac{\Delta_\alpha(x-b)}{b-a}, \frac{\Delta(t(x-a))}{b-a} \right]
\]
First Generating Function for the Polynomial $F_{n,k}(a; x, y)$:

From definition (11b.1.1) we have

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{\alpha} \frac{(Y)_k}{(\alpha + \frac{3}{2})^n} \frac{(b \cdot t)^{n+k}}{(\alpha - k - \frac{1}{2})^n} F_{n+k,k}(\alpha - k; x, y)
$$

Using (11b.2.2) and (11b.2.3) we get the generating function

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{\alpha} \frac{(Y)_k}{(\alpha + \frac{3}{2})^n} \frac{(b \cdot t)^{n+k}}{(\alpha - k - \frac{1}{2})^n} F_{n+k,k}(\alpha - k; x, y)
$$

Particular Case:

In (11b.3.1) put $-a = b = 1, c = \frac{1}{2}$, then we get the generating function for $\mathfrak{F}_{n,k}(x, y)$ due to Malave and Bhonsale [5, p.33].
llb.4 Second Generating Function for the Polynomial

\( F_{n,k}(x; x, y) : \)

From definition (llb.1.1) we have

(11.4.1) \[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a^\alpha x)^n (y)_k}{(a+\frac{3}{2})_k (y)_k} F_{n+k}(\alpha-k; x, y) t^{n+k} \]

\[ = \sum_{n=0}^{\infty} \frac{(n+2)_n}{(n+\frac{3}{2})_n} F_n(\alpha+\frac{1}{2}, \alpha+\frac{1}{2}; x) t^n \cdot \]

\[ \cdot \sum_{k=0}^{\infty} \frac{(y)_k}{(y)_k} F_k \left[ \gamma - k, x - k; \frac{y}{(x-a)(b-x)} \right] \]

\[ \cdot \left[ t \left( (x-a)(b-x) \right)^{\gamma_2} \right]^k \]

Using the results (llb.2.6) and (llb.2.3) we get

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a^\alpha x)^n (y)_k}{(a+\frac{3}{2})_n (y)_k} F_{n+k}(\alpha-k; x, y) t^{n+k} = \]
\[
= (1 - \lambda t)^{-2(\alpha+1)} \Gamma_0 \left[\alpha+1; \frac{4\lambda t (x-b)}{(b-a)(1-\lambda t)^2}\right]
\]

\[
\cdot \Gamma_1 \left[y; -\alpha, -\alpha; \frac{c t}{e} \left(y - a \{x - a \{(b-x)\}^{\frac{1}{2}}\},
\right.ight.
\]
\[
\left.\left(c t \left(b \{x - a \{(b-x)\}^{\frac{1}{2}}\}^{\frac{1}{2}} - y\right)\right]\right]
\]

\[
= \left[1 - \frac{4\lambda t (x-b)}{b-a}\right]^{-\alpha+1}
\]

\[
= \Gamma_1 \left[y; -\alpha, -\alpha; \frac{c t}{e} \left(y - a \{x - a \{(b-x)\}^{\frac{1}{2}}\},
\right.ight.
\]
\[
\left.\left(c t \left(b \{x - a \{(b-x)\}^{\frac{1}{2}}\}^{\frac{1}{2}} - y\right)\right]\right]
\]

Hence the generating function is given by

\[
(1.1.2.4) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2\alpha+x)_n}{(\alpha+3/2)n} \frac{\Gamma_{n+k}(\alpha-k; x, y)}{\Gamma_{n+k}(\alpha-k; x, y)} t^{n+k}
\]

\[
\left[1 - \frac{4\lambda t (x-b)}{b-a}\right]^{-\alpha+1}
\]

\[
= \Gamma_1 \left[y; -\alpha, -\alpha; \frac{c t}{e} \left(y - a \{x - a \{(b-x)\}^{\frac{1}{2}}\},
\right.ight.
\]
\[
\left.\left(c t \left(b \{x - a \{(b-x)\}^{\frac{1}{2}}\}^{\frac{1}{2}} - y\right)\right]\right]
\]

**Particular Case:**

In (1.1.4.2) put \(-a = b = \lambda = 1\) to get the generating function for the polynomial

\[
\frac{q^n x (x, y)}{\Gamma_n^x (x, y)} \text{ due to Malave and Bhonsle [6, p.34]}
\]
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(2\alpha+2)n(y)_{n+k}}{(\alpha+3/2)_n(y)} t^{n+k} \frac{\Gamma(\alpha-k)}{2^{n+k} k!} (x, y)
\]

\[
= \left[ t^2 - 2tx + 1 \right]^{-\alpha - 1}.
\]

\[
\cdot \frac{F_1 \left( y; -\alpha; \frac{1}{2}; \frac{y}{2} \right) - \frac{t}{2} \left\{ y + \sqrt{1-x^2} \right\} \frac{t}{2} \left\{ \sqrt{1-x^2} - y \right\}}{t^2 - 2tx + 1}
\]

\textbf{11b.5 Third Generating Function for the polynomial}

\( F_{n,k}(\alpha; x, y) : \)

Again from definition (11b.1.1) we have

\[
(11b.5.1) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_n (\delta)_n (\epsilon)_n}{[(\alpha+3/2)_n]^2 (\gamma)_k} F_{n+k,k} (\alpha-k; x, y) t^{n+k}
\]

\[
= \sum_{n=0}^{\infty} \frac{(\gamma)_n (\delta)_n}{[(\alpha+3/2)_n]^2} F_n (\alpha+\frac{1}{2}, \frac{1}{2}; x, y) k^n.
\]

\[
\cdot \sum_{k=0}^{\infty} \frac{(\epsilon)_k}{(\gamma)_k} F_k \left[ \alpha-k, \alpha-k; \frac{y}{(x-a)(b-x)} \right] \frac{t}{2} \left\{ (x-a) (b-x) \right\}^{1/2} \right]^k
\]

Using the results (11b.2.7) and (11b.2.3) we get

\[
(11b.5.2) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_n (\delta)_n (\epsilon)_n}{[(\alpha+3/2)_n]^2 (\gamma)_k} F_{n+k,k} (\alpha-k; x, y) t^{n+k}
\]
Particular Case:

In (llb.5.2) put \(-a = b = \lambda = 1\) then we get the generating function for the polynomial

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_n \, (\delta)_n \, (\omega)_k}{(\alpha + \frac{3}{2})_n \, (\beta)_k} \, F_{n+k}^{\alpha-k} (\alpha, \beta; x, y) \, t^{n+k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_n \, (\delta)_n \, (\omega)_k}{(\alpha + \frac{3}{2})_n \, (\beta)_k} \, F_{n+k}^{\alpha-k} (\alpha, \beta; x, y) \, t^{n+k}
\]

11b.6 Fourth generating function for the Polynomial

\[
F_{n,k}^{\alpha} (\alpha; x, y) :
\]

From definition (llb.1.1) we have

\[
(11b.6.1) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_k \, (\delta)_k}{(\beta)_k} \, F_{n+k}^{\alpha-k} (\alpha-k; x, y) \, t^{n+k}
\]
we get the generating function as:

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma_k} \operatorname{F}_{n+k, k} \left(\alpha-k, \alpha-k, \frac{y}{(x-a)(b-z)} \right) \cdot \frac{y^{n+k}}{n+k}.
\]

With the help of the results (11b.2.4) and (11b.2.3) we get the generating function as:

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma_k} \operatorname{F}_{n+k, k} \left(\alpha-k, \alpha-k, \frac{y}{(x-a)(b-z)} \right) \cdot \frac{y^{n+k}}{n+k}.
\]

In (11b.6.2) put \(-a = b = \lambda = 1, c = \frac{1}{2}\) then we get the generating function for the polynomial \(P_{n, k} (-a, y)\) due to Malave and Bhonsle [6, p.35]

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma_k} \operatorname{F}_{n+k, k} \left(\alpha-k, \alpha-k, \frac{y}{(x-a)(b-z)} \right) \cdot \frac{y^{n+k}}{n+k}.
\]

\[
= z^{2\gamma+1} R^{-1} \left[ (1 + \lambda t + R)(1 - \lambda t + R) \right]^{-e+\frac{1}{2}} \cdot \frac{F_1 \left[ \lambda, -\gamma, -\gamma; \frac{1}{x}(y + \sqrt{y^2}), \frac{e}{x}(\sqrt{y^2} - y) \right]}{R}.\]
Fifth Generating Function for the polynomial $F_{n,k}(x; z, y)$:

From definition (11b.1.1) we have

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_{n+k, k} (\alpha-k; x, y) t^{n+k} = \sum_{n=0}^{\infty} F_{n} (\alpha+\frac{1}{2}, \alpha+\frac{1}{2}; x) t^{n} \cdot \sum_{k=0}^{\infty} \frac{y}{(x-a)(b-x)^{\frac{1}{2}}} \cdot \left[ t \left( \frac{x-a}{(b-x)} \right)^{\frac{1}{2}} \right]^{k}
$$

Using the results (11b.2.4) and (11b.2.1) we get the generating function as follows:

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_{n+k, k} (\alpha-k; x, y) t^{n+k} = 2^{2\alpha+1} R^{-1} \left( 1 + \lambda t + R \right) \left( 1 - \lambda t + R \right) \cdot \left[ 1 + ct \left( \frac{y-a}{(x-a)(b-x)^{\frac{1}{2}}} \right) \right] \cdot \left[ 1 + ct \left( \frac{y-b}{(x-a)(b-x)^{\frac{1}{2}}} \right) \right]
$$

**Particular Case:**

In (11b.7.2) put $-a = b = \lambda = 1, c = \frac{1}{2}$ then we get

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_{n+k, k} (x; y) t^{n+k} = \ldots
$$
\[ 2^{xR} R^{-1} \left[ (1+t+R)(1-t+R) \right]^{-(\alpha+\frac{1}{2})} \]
\[ \cdot \left[ 1+\frac{t}{2} \left( y+(1-x^2)^{\frac{3}{2}} \right) \right] \left[ 1+\frac{t}{2} \left( y-(1-x^2)^{\frac{3}{2}} \right) \right] \]
\[ = 2^{xR} R^{-1} \left[ (1+t)^2 - R^2 \right]^{\alpha+\frac{1}{2}} \left[ (1+\frac{ty}{2})^2 - \frac{t^2(1-x^2)}{4} \right] \]

Where \( R = (1 - 2xt + t^2)^{\frac{1}{2}} \)

Which is obtained by Malave and Bhonsle [6, p.36].

**11b.8** Sixth generating function for the polynomial \( F_{n,k}(a; x, y) \):

From definition (11b.1.1) we have

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\delta)n(Y)k}{(\alpha+\frac{3}{2})n} \frac{t^{n+k}}{2!(s)_k} F_{n+k,k}(\alpha-k; x, y) \]

\[ = \sum_{n=0}^{\infty} \frac{(\delta)n t^n}{(\alpha+\frac{3}{2})n} F_n (\alpha+\frac{1}{2}, \alpha+\frac{1}{2}; x) \cdot \sum_{k=0}^{\infty} \frac{(Y)k}{(s)_k} \frac{t^k}{k!} F_k (\alpha-k, \alpha-k; \frac{y}{((x-a)(b-x))^\frac{1}{2}}) \]

Using (11b.2.5) and (11b.2.3) we get

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\delta)n(Y)k}{(\alpha+\frac{3}{2})n} \frac{t^{n+k}}{2!(s)_k} F_{n+k,k}(\alpha-k; x, y) = \]
Particular Case:

In (11b.8.2) put \(-a = b = \lambda = 1, c = \frac{1}{2}\) then we get a generating function for the polynomial \(P_{n,k}^\alpha(x, y)\)

\[
(11b.8.3) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\delta)_n \Gamma_k}{[\left(\alpha + \frac{3}{2}\right)_n]^2} \frac{\Gamma_k}{\Gamma(\delta)_k} P_{n+k, k}^{\alpha-k}(x, y) t^{n+k} = \Psi_2 \left[ \delta, \alpha + \frac{3}{2}, \alpha + \frac{3}{2}; \frac{1}{2} (\alpha+1), \frac{1}{2} (\alpha-1) \right] \cdot \cdot \cdot F_1 \left[ \gamma; -\alpha_3 - \alpha; \frac{1}{2} (y+\sqrt{-x^2}), \frac{1}{2} (d-\xi^2) \right],
\]
REFERENCES: