Pure mathematics must remain a source of aesthetic joy, no more, no less. To that extent, modern mathematics is modern art, whereas applied mathematics is a modern language.

- J.M. Hemmerseley.
CHAPTER VI.

(a) Some Generating Functions of Extended Jacobi Polynomials.

(b) Extended Jacobi Function of Second kind.
6a.1 The search for generating function of polynomial sets involved many techniques Recently some generating functions have been found by the use of differential operators [3]. We employ the technique to record a few generating functions of extended Jacobi polynomials.

6a.2 We shall use the following properties of the differential operator $D_x = \frac{d}{dx}$

\[
D_x^n \{ u \cdot v \} = \sum_{j=0}^{n} \binom{n}{j} D_x^{n-j} \{ u \} \cdot D_x^j \{ v \}
\]

\[ F( D_x ) \{ x^\alpha f(x) \} = x^\alpha F( x^{\alpha-1} + D_x ) f(x)
\]

\[ e^t D_x \{ f(x) \} = f(x+t)
\]

and

\[\frac{f(y)}{1 - t \phi(y)} = \sum_{n=0}^{\infty} D_x^n \{ \phi(x) \}^n f(x) \cdot \frac{t^n}{n!}
\]

where $y = x + t \phi(x)$

6a.3 We have seen in Chapter II that the extended Jacobi polynomial is defined by
From definition (6a.3.1) we have

\[
\sum_{n=0}^{\infty} F_n (\alpha-n, \beta-n; x) \, t^n = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!} \, t^n \, (x-a)^{n-\alpha} (b-x)^{n-\beta} \, D_x^n \left[ (x-a)^{n+\alpha} (b-x)^{n+\beta} \right]
\]

\[
= (x-a)^{-\alpha} (b-x)^{-\beta} \, \exp \left[ -c \, t \, (x-a) \, (b-x) \, D_x \right] \times \left[ (x-a)^{\alpha} (b-x)^{\beta} \right]
\]

Using (6a.2.3) we obtain following generating function.

\[
\sum_{n=0}^{\infty} F_n (\alpha-n, \beta-n; x) \, t^n = \left[ 1 - c \, t \, (b-x) \right] \left[ 1 + c \, t \, (x-a) \right]^{\beta}
\]

In (6a.3.2) put \(-a = b = 1, c = \frac{1}{2}\) and interchange \(\alpha\) and \(\beta\) then we get the generating function for Jacobi polynomial \([2]\),

\[
\sum_{n=0}^{\infty} \left( \begin{array}{c} m+n \\ n \end{array} \right) \, F_{m+n} (\alpha-m, \beta-m; x) \, t^n
\]

Again from definition (6a.3.1) we may write
\[
\sum_{n=0}^{\infty} \frac{(-c)^{m+n} t^n}{n!} (x-a)^{m+n-\alpha} (b-x)^{m+n-\beta} \\
\times D_x^{m+n} \left[ (x-a)^{\alpha} (b-x)^{\beta} \right] \\
= \sum_{n=0}^{\infty} \frac{(-c)^n t^n}{n!} (x-a)^{m+n-\alpha} (b-x)^{m+n-\beta} \\
\times D_x^n \left[ (x-a)^{-m+\alpha} (b-x)^{-m+\beta} F_m(\alpha-m, \beta-m; x) \right] \\
= (x-a)^{m-\alpha} (b-x)^{m-\beta} \exp \left[ -c t (x-a) (b-x) D_x \right] \\
\times \left[ (x-a)^{-m} (b-x)^{-m} F_{m-n}(\alpha-m, \beta-m; x) \right]
\]

Now using the relation (6a.2.3) we get the generating function

\[(6a.3.3) \sum_{n=0}^{\infty} \binom{m+n}{n} F_{m+n}(\alpha-m-n, \beta-m-n; x) t^n \]

\[= \left[ 1 - c t (b-x) \right]^{\alpha-m} \left[ 1 + c t (x-a) \right]^{\beta-m} x \\
\times F_m(\alpha-m, \beta-m; x-c t (x-a) (b-x)) \]

In (6a.3.3) put \(-a = b = 1, c = \frac{1}{2}\) and interchanging \(\alpha\) and \(\beta\) then we get the generating function of Jacobi polynomial [7]:

\[\sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}(\alpha-m-n, \beta-m-n)(x) t^n \]

\[= \left[ 1 + \frac{x}{2} t \right]^{\alpha-m} \left[ 1 + \frac{x-1}{2} t \right]^{\beta-m} x \]
Now using (6a.2.1) we see that
\[ D_x^n \left[ (x-a)^{n+\alpha} (b-x)^{n+\beta} f_1(x) \right] \]
\[ = \sum_{j=0}^{n} \binom{n}{j} D_x^{n-j} \left\{ (x-a)^{n+\alpha} (b-x)^{n+\beta} \right\} x \]
\[ \times D_x^j \left\{ f_1(x) \right\} \]

[where \( f(x) = x^\alpha f(x) \)]

\[ (6a.3.4) \]
\[ = \sum_{j=0}^{n} \binom{n}{j} \frac{(n-j)!}{(-c)^{n-j}} (x-a)^\alpha (b-x)^\beta x \]
\[ \times F_{n-j} (\alpha, \beta; x) \ D_x^j \left\{ x^\alpha f(x) \right\} \]

Also using the relation (6a.2.2) we see that
\[ (6a.3.5) \]
\[ D_x^n \left[ (x-a)^{n+\alpha} (b-x)^{n+\beta} f_1(x) \right] \]
\[ = x^\alpha (x-a)^{n+\alpha} (b-x)^{n+\beta} \left[ \frac{\alpha}{x} + \frac{n+\alpha}{x-a} - \frac{n+\beta}{b-x} + D_x \right]^n f_1(x) \]

Comparing (6a.3.4) and (6a.3.5) we see that
\[ (6a.3.6) \]
\[ x^\alpha (x-a)^{n+\alpha} (b-x)^{n+\beta} \left[ \frac{\alpha}{x} + \frac{n+\alpha}{x-a} - \frac{n+\beta}{b-x} + D_x \right]^n f_1(x) = \]
\[
\sum_{j=0}^{n} \frac{n!}{j! (c)^{n-j} \left\{ x^j \right\}} = \sum_{j=0}^{n} \frac{n!}{j! (-c)^{n-j} \left\{ x^j \right\}} F_{n-j}(\alpha, \beta; x) \partial_x^j \left\{ x^\alpha f(x) \right\}
\]

Now let
\[
\phi(x) = (-c)(x-a)(b-x)
\]
and
\[
f(x) = (x-a)^\alpha (b-x)^\beta
\]

Using (6a.2.4) in (6a.3.1) we get
\[
(6a.3.7) \sum_{n=0}^{\infty} F_n(\alpha, \beta; x) x^n = \frac{(x-a)^\alpha (b-x)^\beta \left\{ 1 - ct(b-x) \right\}^\alpha \left\{ 1 + ct(x-a) \right\}^\beta}{1 - 2 ct \left\{ x - ct(x-a)(b-x) - \frac{1}{2} (a+b) \right\}^2}
\]

(b) **EXTENDED JACOBI FUNCTION OF SECOND KIND:**

6b.1 Thakare \[8\] has stated that the extended Jacobi differential equation is given by
\[
(x-a)(b-x) y'' + \left\{ c(b+a) \beta + a + b \right\} - (\alpha + \beta + 2)x \right\} y' + \alpha (\alpha + \beta + 1) y = 0.
\]

This equation can be reduced to the hypergeometric differential equation:
\[
(6b.1.2) x(1-x) u'' + \left[ c - (a+b+1) z \right] u' - abu = 0
\]

where a, b, c, are the parameters of the equation; they are arbitrary complex numbers.

Extended Jacobi polynomial \( P_n(\alpha, \beta; x) \) is that solution of (6b.1.1) which is regular at \( x = b \), has the value.
Hypergeometric forms of \( F_n(\alpha, \beta; b) \) have been obtained by Thakare (1972) [Chapter II, (2b.1.14) - (2b.1.19)]

From [4, p.105, 2.9(14)] we can write another solution of the equation (6b.1.1) as

\[
(6b.1.4) \quad F_n(\alpha, \beta; x) = \lambda^n (b-a)^{n+\alpha+\beta} x \\
\times \frac{\binom{n+\alpha+1}{n+\beta+1}}{(x-b)^{n+\beta+1}} \frac{\binom{n+\beta+1}{n+\alpha+1}}{(x-a)^{n+\alpha+\beta+2}} \\
\times 2 \binom{n+1; n+\beta+1; b-a}{2n+\alpha+\beta+2; b-x}
\]

We shall call it as the extended Jacobi function (not a polynomial) of second kind on the analogy of Jacobi function of second kind. It satisfies the same recurrence and differential formula (given below [8]) as the extended Jacobi polynomial [except that \( n = 0 \) is not admissible with the \( \mathcal{G} \)], it vanishes at infinity when \( \text{Re} (\alpha + \beta) > n - 1 \).

(6b.1.5) \[
\frac{n(\alpha+\beta+n)(\alpha+\beta+n-2)}{-a\alpha-b\beta-n(a+b)} \left[ a(\alpha+\beta+2n) \right] F_n(\alpha, \beta; x) \\
= \lambda(\beta+n) [ (b-a)(\alpha+n)(\alpha+\beta+n)(\alpha+\beta+2n-2) \\
+ (n-1)^2 (a+b)(\alpha+\beta+2n)+ (a+\beta)(a+b)(\alpha+\beta+2n-1) \\
+ \lambda(\alpha+\beta+2n)(\alpha+\beta+n)(\alpha+\beta+2n-2) +
\]
Extended Jacobi polynomials and extended Jacobi functions of second kind are connected by several relations. Erdelyi [4] has given twenty linear relations with constant coefficients between various solutions of the hypergeometric equation (6b.1.2). Using the relation [4, p. 107 (39)].

\[
\begin{aligned}
(6b.2.1) \quad u_4 &= \frac{\Gamma(1-c) \Gamma(1+b-a)}{\Gamma(1-a) \Gamma(1+b-c)} u_1 - \frac{\Gamma(c) \Gamma(1-a) \Gamma(b+1-a)}{\Gamma(a-c) \Gamma(c-a) \Gamma(b)} \times \\
&\quad \times e^{i \pi (c-1)} u_5
\end{aligned}
\]

where \( u_1, u_4, u_5 \) are given as [4, p. 105(1), (14), (18)].
\[(6b.2.3) \quad U_4 = (-z)^{a-c} (1-z)^{c-a-b} \times \]
\[\times F(1-a, c-a; b+1-a; z^{-1}) \]

and

\[(6b.2.4) \quad U_5 = z^{1-c} (1-z)^{c-a-b} \times \]
\[\times F(1-a, 1-b, c-c; z) \]

with the substitutions

\[a = -n, \quad b = n + \alpha + \beta + 1, \quad c = \beta + 1, \quad z = \frac{b-a}{b-a} \]

we get

\[(6b.2.5) \quad \mathcal{G}_n(\alpha, \beta; x) = - \frac{\lambda^n \pi}{b-a} \cos \sec (\pi \beta) F_n(\alpha, \beta; x) \]
\[+ \frac{(b-a)^{\alpha+\beta-1}}{\Gamma(n+\alpha+\beta+1)} \frac{\Gamma(\alpha) \Gamma(n+\alpha+\beta+1)}{(x-a) (x-b)} \]
\[\times \mathcal{F}_1 \left[ n+1, -n-\alpha-1; \frac{b-a}{1-\beta}; \frac{b-a}{b-a} \right] \]

The integral relation:

For the integral (2b.3.7) we have

\[(6b.2.6) \quad \mathcal{Z}_n(\alpha, \beta; \alpha) = (b-a)^{-1} (x-a)^{-\alpha} (x-b)^{-\beta} \times \]
\[\times \int_a^b (x-t)^{-1} A_n(\alpha, \beta; t) \, dt \]

where

\[A_n(\alpha, \beta; t) = (t-a)^\alpha (b-t)^\beta F_n(\alpha, \beta; t) \]
This integral is valid for all points $x$ in the complex plane cut along the segment $(a, b)$.

6b.3 According to [4, p. 162(6)] associated polynomial is defined by

$$ q_n = \frac{b_n(x) - b_n(t)}{x - t} \omega(t) \, dt $$

Thus we have a polynomial $g_n(\alpha, \beta, x)$ associated with extended Jacobi polynomials and may be written as

$$ (6b.3.1) \quad g_n(\alpha, \beta; x) = \int_a^b (t-x)^{-1} (t-a)^\alpha (b-t)^\beta \, dt $$

Therefore $(6b.2.6)$ may be written as

$$ (6b.3.2) \quad f_n(\alpha, \beta; x) = - (b-a)^{-1} (x-a)^{-\alpha} (b-x)^{-\beta} $$

$$ x \cdot g_n(\alpha, \beta; x) $$

$$ + f_0(\alpha, \beta; x) \cdot F_n(\alpha, \beta; x) $$

6b.4 Integral Representation:

Using the Rodrigues' formula of extended Jacobi polynomial in $(4b.2.6)$, we have

$$ f_n(\alpha, \beta; x) = (b-a)^{-1} (x-a)^{-\alpha} (x-b)^{-\beta} \frac{(-c)^n}{n!} $$

$$ x \cdot \frac{d^2}{dt^2} \left[ (t-a)^{n+\alpha} (b-t)^{n+\beta} \right] dt $$

$$ = \frac{c^n}{n!} (b-a)^{-1} (x-a)^{-\alpha} (x-b)^{-\beta} $$

$$ x \cdot \int_a^b (t-a)^{n+\alpha} (b-t)^{n+\beta} \frac{1}{(x-t)^{n-1}} dt $$
Hence

$$G_n(\alpha, \beta; x) = c^n (b-a)^{-1} (x-a)^{\alpha} (b-x)^{\beta}$$

$$\times \int_a^b (t-a)^{n+\alpha} (b-t)^{n+\beta} x (x-t)^{-n-1} dt,$$

**Particular Case:**

Put $a = b = \lambda = 1$ in (6b.1.4) and interchange $\alpha$ and $\beta$ then we get the Jacobi function of second kind $Q_n^{(\alpha, \beta)}(x)$ [5, p.170(18)]:

$$Q_n^{(\alpha, \beta)}(x) = 2^n \Gamma(n+\alpha+1) \Gamma(n+\beta+1)$$

$$\times \frac{\Gamma(\alpha+\beta+2)(x-1)^n+\alpha+1}{\Gamma(\alpha+\beta+2)(x-1)(x+1)^{\beta}} x$$

$$\times \sqrt[2]{F_1} \left[ \frac{n+1, n+\alpha+1; \frac{a}{1-x}}{a n+\alpha+\beta+2, 1-x} \right].$$

With the same substitution (6b.2.5) (6b.2.6) reduce to the following result [5, p.171 (19), (20)]:

$$Q_n^{(\alpha, \beta)}(x) = -\frac{\pi}{\alpha} \csc\pi(\alpha \pi) \frac{P_n^{(\alpha, \beta)}(x)}{\alpha}$$

$$+ 2^{\alpha+\beta-1} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)} (x-1)^{n+\beta}$$

$$\times \sqrt[2]{F_1} \left[ \frac{n+1, -n-\alpha-\beta; 1-x}{1-x} \right]$$

and

$$Q_n^{(\alpha, \beta)}(x) = \frac{1}{\alpha} (x-1)^{-\alpha} (x+1)^{-\beta} x$$

$$\times \int_{-1}^1 (x-t)^{-1} (1-t)^{\alpha} (1+t)^{\beta} \frac{P_n^{(\alpha, \beta)}}{P_n(t)} dt.$$
Taking the same substitution we have from (6b.3.1), (6b.3.2) and (6b.4.1) the following relations for extended Jacobi function of second kind [4, p. 171-172]:

\[
q_n^{(\alpha, \beta)}(x) = \int_{-1}^{1} (t-x)^{-1} (1-t)^{\alpha} (1+t)^{\beta} x \\
\times \left[ P_n^{(\alpha, \beta)}(t) - P_n^{(\alpha, \beta)}(x) \right] dt
\]

\(q_n^{(\alpha, \beta)}(x)\) is the polynomial associated with Jacobi polynomial.

\[
Q_n^{(\alpha, \beta)}(x) = -\frac{1}{2} (x-1) \cdot (x+1) \cdot q_n^{(\alpha, \beta)}(x) \\
+ Q_0^{(\alpha, \beta)}(x) \cdot P_n^{(\alpha, \beta)}(x)
\]

and

\[
Q_n^{(\alpha, \beta)}(x) = 2^{-n-1} (x-1)^{-\alpha} (x+1)^{-\beta} x \\
\times \int_{-1}^{1} (x-t)^{-n-1} (1-t)^{n+\alpha} (1+t)^{n+\beta} dt,
\]

when \(x\) is in the plane cut along the segment \((-1, 1)\).
**REFERENCES:**

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