Chapter 4

Fixed Point Theorems Using Implicit Relations

4.1 Introduction

In this chapter we prove some fixed point theorems using implicit functions in fuzzy metric space. These implicit functions are found to be viable, productive and powerful tool in finding the existence of common fixed point for non surjective mappings satisfying certain expansion conditions. For this we recall some required definitions. After using semicompatibility Singh and Jain[96] defined weakly compatible mappings in fuzzy metric space as follows:

Definition 4.1.1. [95] Two self mappings $A$ and $B$ in fuzzy metric spaces are said to be weakly compatible if they commute at their coincidence points i.e.,

$$M(ABu, BAu, t) = 1, \text{ whenever } \ M(Au, Bu, t) = 1 \ \forall \ t > 0 \ \text{forsome } \ u \in X.$$ 

After the introduction of the notion of occasionally weakly compatible by M.A.Al-Thagafi and N.Shahzad[2] in metric space we fuzzify as follows:

Definition 4.1.2. Two self mappings $A$ and $B$ of fuzzy metric space $(X, M, *)$ are said to be occasionally weakly compatible if they commute at one of their coincidence
points i.e., there exists a point \( u \in X \) such that

\[
M(Au, Bu, t) = 1 \text{ then } M(ABu, BAu, t) = 1 \quad \forall \ t > 0.
\]

**Remark 4.1.3.** It is seen that occasionally weakly compatible mappings are weakly compatible mappings but converse is not true.

**Example 4.1.4.** Let \( X = \mathbb{R} \) be a fuzzy metric space with fuzzy metric \( M(x, y, t) \forall x, y \in X, \ t : [0, \infty) \to [0, \infty) \). Define

\[
S, T : \mathbb{R} \to \mathbb{R} \text{ by } Sx = 2x \text{ and } Tx = x^2, \ \forall \ x \in \mathbb{R}.
\]

then

\[
M(Sx, Tx, t) = 1 \text{ for } x = 0, 2
\]

gives \( M(ST(0), TS(0), t) = 1 \) and \( M(ST(2), TS(2), t) \neq 1 \).

Hence \( S \) and \( T \) are occasionally weakly compatible self maps but not weakly compatible.

**Definition 4.1.5.** [1] The pair of two self mappings \((A, B)\) satisfies the property of E.A. if there exists a sequence \( \{x_n\} \in X \) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \in X
\]

**Remark 4.1.6.** [90] is called \( z \) as tangent point to \((A, B)\) and \((A, B)\) is called tangential.

**Definition 4.1.7.** [1] Let \( A, B, S \) and \( T \) be self mappings then the pairs \((A, S)\) and \((B, T)\) satisfy a common property of E.A. if there exists two sequences \( \{x_n\}, \{y_n\} \in X \) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z \in X
\]

**Remark 4.1.8.** If \( B = A \) and \( T = S \) we obtain the definition of property (E.A.)
Definition 4.1.9. [45] Let $S$ and $T$ be self mappings a point $z \in X$ is said to be a weak tangent point to $(S, T)$ if there exists two sequences $\{x_n\}, \{y_n\} \in X$ such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n = z \in X$$

Definition 4.1.10. [45] Let $A, B, S$ and $T$ be self mappings then the pairs $(A, B)$ is called tangential with respect to the pair $(S, T)$ whenever the weak tangent point of $(S, T)$ is also weak tangent point of $(A, B)$ i.e. for sequences $\{x_n\}, \{y_n\} \in X$, a point $z \in X$ exists such that

$$\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n = z \in X$$

then

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} By_n = z$$

Remark 4.1.11. [45] If $B = A$ and $T = S$ we say that $A$ is tangential with respect to $S$.

Remark 4.1.12. [45] If $S = A$ and $T = B$ we say that $(A, B)$ is tangential with respect to itself.

By examples[45] 2.2, 2.3, 2.5 they show that:

(1) every pair of mappings $(S, T)$ which satisfies the property E.A. also has a weak tangent point to $(S, T)$ but the converse is not necessarily true.

(2) if $(A, B)$ is tangential w.r.t. $(S, T)$ it doesn’t mean that $(S, T)$ is tangential w.r.t. $(A, B)$.

(3) $A, B, S$ and $T$ satisfy common E.A. property if they satisfy the tangential property but the converse is not necessarily true.

### 4.2 Implicit relations

There are two types are functions, explicit and implicit. In explicit functions (in two variables ‘$x, y$’) ‘$y$’ can be written in form of ‘$x$’ or ‘$x$’ can be written in form
of 'y' but in implicit functions we cannot always separate the variables so every explicit functions are implicit functions but converse is not true. We can say that implicit functions are more general form of functions.

To unify all contraction conditions V. Popa[82] introduced the notion of implicit functions in metric spaces. In this section we fuzzify and define some types of implicit relations and furnish various examples to verify them also with these examples we define a variety of contraction conditions as corollaries.

**F-1**

Let \( \mathbb{R} \) and \( \mathbb{R}_+ \) denote the set of real and non-negative real numbers respectively. Let \( \mathcal{F} \) be the set of all real-valued continuous functions \( F \) defined by

\[
F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R}
\]
satisfying the following conditions:

for \( u, v \geq 0 \) with

\[
(F_1) : F(u, v, v, u, u) \leq 0 \\
(F_2) : F(u, v, u, v, u) \leq 0
\]

then \( u \geq r(v) \), where \( r : [0, 1] \to [0, 1] \) defined as \( r(t) > t \) for \( t < 1 \) and \( r(0) = 0, r(1) = 1 \) (for example \( r(t) = \sqrt{t} \)).

**Example 4.2.1.** Define \( F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^5 \to \mathbb{R} \) as:-

\[
F(t_1, t_2, t_3, t_4, t_5) = \phi(t_2, t_3, t_4, t_5) - t_1 = kt_2.t_3.t_4.t_5 - t_1
\]

where \( k \leq 1, u \geq r(v) > v \) then

\[
(F_1) : F(u, v, v, u, u) = ku^4 - u = u(ku^3 - 1) \leq 0 \\
(F_2) : F(u, v, u, v, u) = ku^2.v^2p - u = u(ku^3 - 1) \leq 0
\]

**F-2**

Let \( \mathbb{R} \) and \( \mathbb{R}_+ \) denote the set of real and non-negative real numbers respectively. Let \( \mathcal{F} \) the set of all real-valued continuous functions \( F(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}_+^6 \to \mathbb{R} \) satisfying the following conditions:

for \( u, v \geq 0 \) with

\[
(F_a) : F(u, v, v, u, u * v, 1) \leq 0 \\
(F_b) : F(u, u, v, v, u, u) \leq 0 \\
(F_c) : F(u, v, u, v, 1, u * v) \leq 0
\]
then \( u \geq r(v) \), where \( r : [0, 1] \rightarrow [0, 1] \) is some continuous function defined as \( r(t) > t \) for \( t < 1 \). (for example \( r(t) = \sqrt{t} \)).

**Example 4.2.2.** Define \( F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as:

\[
F(t_1, t_2, t_3, t_4, t_5, t_6) = kt_2.t_3.t_4.t_5.t_6 - t_1
\]

then \( u \geq r(v) > v \)

\( (F'_a) : F(u, v, u, v, v, u) = ku^2.v^3 - u = u(ku^4 - 1) \)

if we choose \( k \) such that \( k \leq 1 \) then

\( (F'_b) : F(u, v, u, v, v, u) = ku^2.v^3 - u < 0 \) or

\( (F'_c) : F(u, v, u, u, v, u) = ku^2.v^3 - u = u(ku^4 - 1) < 0 \) or

\( (F'_d) : F(u, u, v, v, u, u) = u^3.v^2 - u = u(ku^4 - 1) < 0 \)

**Example 4.2.3.** Define \( F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as:

\[
F(t_1, t_2, t_3, t_4, t_5, t_6) = \phi(\min\{t_2, t_3, t_4, t_5, t_6\}) - t_1
\]

and \( \phi : [0, 1] \rightarrow [0, 1] \) is an increasing function defined as \( \phi(t) > t \) and \( \phi(0) = 0 \), \( \phi(1) = 1 \) also \( u \geq r(v) > v \)

then

\( (F'_a) : F(u, v, u, v, v, u) = \phi(v) - v = v - u < 0 \) or

\( (F'_b) : F(u, v, u, v, u, v) = \phi(u) - u = v - u < 0 \) or

\( (F'_c) : F(u, u, v, v, u, u) = \phi(u) - u = v - u < 0 \)

**Example 4.2.4.** Define \( F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R} \) as

\[
F(t_1, t_2, t_3, t_4, t_5, t_6) = at_2 + bt_3 + ct_4 + \max\{t_5, t_6\} - qt_i
\]

and \( a + b + c + 1 > q, \ a, b, c \geq 0, \ q > 0 \)

then

\( (F'_a) : F(u, v, u, v, v, u) = av + bu + cv + \max\{v, u\} - qu \)
\[(a + c + 1)v - (q - b)u = (q - b)(\frac{a + c + 1}{q - b}v - u) < 0 \text{ or} \]

\[(F'_l): F(u, v, v, u, u, v) = av + bv + cu + \max\{u, v\} - qu\]

\[(a + b + 1)v - (q - c)u = (q - c)(\frac{a + b + 1}{q - c}v - u) < 0 \text{ or} \]

\[(F'_c): F(u, u, v, v, u, v) = au + bv + cv + \max\{u, u\} - qu\]

\[(b + c)v - (q - a - 1)u = (q - a - 1)(\frac{b + c}{q - a - 1}v - u) < 0 \]

**Example 4.2.5.** Define

\[F(t_1, t_2, t_3, t_4, t_5, t_6) = \phi(\min\{t_2, t_3, t_4, t_5, t_6\}) - t_1\]

where \(\phi(t) = r(t) > t \ \forall \ t \in (0, 1) \text{ and } \phi(0) = r(0) = 0, \ \phi(1) = r(1) = 1\)

then

\[(F_1): F(u, u, v, v, u, u) = \phi(\min\{u, v\}) - u \leq 0\]

if and only if

\[u \geq \phi(\min\{u, v\}).\]

It is possible only when \(\min\{u, v\} = v \text{ i.e. } u \geq \phi(v) = r(v) > v.\)

Also if \(v = 1\) then \(u \geq \phi(1) = 1 \text{ i.e. } u = 1.\)

**Example 4.2.6.** Define

\[F(t_1, t_2, t_3, t_4, t_5, t_6) = at_2 + bt_3 + ct_4 + \max\{t_5, t_6\} - qt_1\]

\(\forall \ a, b, c \geq 0, \ q > 0, \ a + b + c + 1 > q\)

then

\[(F_1): F(u, u, v, v, u, u) = au + bv + cv + \max\{u, u\} - qu\]

\[= (b + c)v - (q - a - 1)u \leq 0\]

iff

\[u \geq \frac{b + c}{q - a - 1} > v\]

i.e.

\[u \geq r(v) = \frac{b + c}{q - a - 1} > v.\]
Example 4.2.7. Define

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = at_2 + bt_3 + \frac{c \max\{t_4, t_5\}}{t_6} - qt_1, \quad t_6 \neq 0$$

\[\forall \quad a + b + c > q, \quad a, b, c \geq 0, \quad q > 0\]

then

$$(F_1): F(u, u, v, v, u, u) = au + bv + \frac{c \max\{v, u\}}{u} - qu,$$

If $v \geq u$ then $\max\{v, u\} = v$ then

$$(q - a)u \geq bv + \frac{cv}{u}$$

$$(q - a)u^2 \geq buv + cv > bu^2 + cu$$

$$u > \frac{c}{q - a} > 1$$

which is absurd. Hence $\max\{v, u\} = u$ then

$$(F_1) \text{ gives } (q - a)u \geq (q - a)u^2 \geq bv + c$$

i.e. $u \geq \frac{bv + c}{q - a} > v$

$$u \geq r(v) = \frac{bv + c}{q - a} > v.$$

(4.2.1)

4.3 Common fixed point for weakly compatible mappings

In this section we fuzzify a theorem of Bouhadjera[15] and improve it by taking the setting of Khan[57]. The results of Bouhadjera[15] and Khan et. al[57] are as follows:

Theorem 4.3.1. [15]. Let $S, T, I, J$ be mappings from a complete metric space $(X, d)$ into itself satisfying the conditions

(a) $S(X) \subset J(X)$ and $T(X) \subset I(X)$,

(b) one of $S, T, I, J$ is continuous,
(c) $S$ and $I$ as well as $T$ and $J$ are compatible of type-(C),
(d) the inequality

$$F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)) \leq 0$$

holds for all $x, y \in X$, where $F \in \mathcal{F}$. Then $S, T, I$ and $J$ have an unique common fixed point.

**Theorem 4.3.2.** [57] Let $(X, M, \ast)$ be a complete FM-space with $t \ast t > t, t \in [0, 1]$ and $P, Q, S$ and $T$ are self mappings of $X$ satisfying

(4.1)\(P(X) \subseteq T(X) \text{ and } Q(X) \subseteq S(X),\)

(4.2)\([1 + pM(Sx, Ty, kt)] \ast M(Px, Qy, kt) \geq p[M(Px, Sx, kt) \ast M(Qy, Ty, kt) + M(Px, Ty, kt) \ast M(Qy, Sx, kt)] + M(Sx, Ty, t) \ast M(Px, Sx, t) \ast M(Qy, Ty, t) \ast M(Px, Ty, \alpha t) \ast M(Qy, Sx, (2 - \alpha)t)\)

\(\forall x, y \in X, p \geq 0, t > 0, \alpha \in (0, 2) \text{ and } k \in (0, 1).\)

(4.4)the pairs $(P, S)$ and $(Q, T)$ are compatible of type $(A - 1)$ or $(A - 2)$.

(4.5)one of $S$ and $T$ is continuous.

Then $P, Q, S$ and $T$ have a unique fixed point.

**Theorem 4.3.3.** [57] Let $(X, M, \ast)$ be a complete FM-space with $t \ast t > t, t \in [0, 1]$ and $P$ and $Q$ be two maps of product space $X \times X$ with values in $X$. If there exists a constant $k \in (0, 1)$ such that

$$[1 + pM(x, u, kt)] \ast M(P(x, y), Q(u, v), kt) \geq p[M(P(x, y), x, kt) \ast M(Q(u, v), u, kt) + M(P(x, y), u, kt) \ast M(Q(u, v), x, kt)] + M(P(x, y), x, t) \ast M(Q(u, y), u, t) \ast M(x, u, t) \ast M((x, y), u, \alpha t) \ast M(Q(u, y), x, (2 - \alpha)t)$$

\(\forall x, y, u \in X, p \geq 0, t > 0 \text{ and } \alpha \in (0, 2)\) then there exists one point $w \in X$ such that $P(w, w) = w = Q(w, w)$. 

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In the theorem[15] we take
(1) the space as fuzzy metric space,
(2) change the properties of function from compatible of type \((A − 1)\) and \((A − 2)\) to weakly compatible,
(3) improve the inequality condition taking \(\alpha\) inspired by theorem 5.1[57].
Also we construct example taking idea from theorem 5.2[57] to verify our theorem.

**Theorem 4.3.4.** 1 Let \((X, M, \ast)\) be a complete \(FM\) – space and \(A, B, S\) and \(T\) be self maps of \((X, M, \ast)\) such that
(1) \(A(X) \subset T(X)\) and \(B(X) \subset S(X)\),
(2) \(F[M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M(Sx, By, \alpha t), M(Ty, Ax, (2 - \alpha) t)] \leq 0\),
for all \(x, y \in X\) and \(t > 0\) and \(\alpha \in (0, 2)\) and
(3) \((A, S)\) and \((B, T)\) are weakly compatible.
(4)\(M(x, y, t)\) is continuous in \(x, y\) for each fixed \(t\), and
(5) one of the subspaces \(A(X), B(X), S(X)\) or \(T(X)\) is closed. Then \(A, B, S\) and \(T\) have a unique fixed point.

**Proof.** Suppose \(x_0\) is an arbitrary point in \(X\). Since \(A(X) \subset T(X)\) and \(B(X) \subset S(X)\), there must exists a point \(x_1 \in X\) such that \(Ax_0 = Tx_1\) and for this \(x_1\) there exists \(x_2 \in X\) such that \(Bx_1 = Sx_2\). Continuing this process we define the sequence \(\{y_n\}\) in \(X\) such that
\[
y_{2n} = Ax_{2n} = Tx_{2n+1}
\]
and
\[
y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}
\]1The Journal of Fuzzy Mathematics 17 (3)(2009)699-710
for $n = 0, 1, 2, \cdots$.

Using inequality (2), we get

$$F[M(Ax_{2n}, Bx_{2n+1}, t), M(Sx_{2n}, Tx_{2n+1}, t), M(Sx_{2n}, Ax_{2n}, t),
M(Tx_{2n+1}, Bx_{2n+1}, t), M(Sx_{2n}, Bx_{2n+1}, \alpha t), M(Tx_{2n+1}, Ax_{2n}, (2 - \alpha)t)] \leq 0.$$ 

gives

$$F[M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t),
M(y_{2n-1}, y_{2n+1}, \alpha t), M(y_{2n}, y_{2n}, (2 - \alpha)t)] \leq 0.$$ 

Denoting by $M_m = M(y_m, y_{m+1}, t)$, we get

$$F[M_{2n}, M_{2n-1}, M_{2n-1}, M_{2n}, M(y_{2n-1}, y_{2n+1}, \alpha t), 1] \leq 0.$$ 

Using condition of triangle inequality and nondecreasing character of $F$ in the fifth variable coordinate, we get

$$F[M_{2n}, M_{2n-1}, M_{2n-1}, M_{2n}, M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, (\alpha - 1)t), 1] \leq 0.$$ 

Considering the limit as $\alpha \to 2$ and using the continuity of $F$, we get

$$F[M_{2n}, M_{2n-1}, M_{2n-1}, M_{2n}, M_{2n} * M_{2n-1}, 1] \leq 0.$$ 

By $F_a$, we obtain

$$M_{2n} = M(y_{2n}, y_{2n+1}, t) \geq r(M(y_{2n-1}, y_{2n}, t)) = r(M_{2n-1}) > M_{2n-1}. \quad (4.3.1)$$ 

Using inequality (2) again with $x = x_{2n+2}$, $y = x_{2n+1}$, we get for $t > 0$,

$$F[M(Ax_{2n+2}, Bx_{2n+1}, t), M(Sx_{2n+2}, Tx_{2n+1}, t), M(Sx_{2n+2}, Ax_{2n+2}, t),
M(Tx_{2n+1}, Bx_{2n+1}, t), M(Sx_{2n+2}, Bx_{2n+1}, \alpha t), M(Tx_{2n+1}, Ax_{2n+2}, (2 - \alpha)t)] \leq 0.$$ 

i.e. $F[M(y_{2n+2}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+1}, t),
M(y_{2n+1}, y_{2n+1}, \alpha t), M(y_{2n}, y_{2n+2}, (2 - \alpha)t)] \leq 0$

i.e. $F[M(y_{2n+2}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t),
M(y_{2n}, y_{2n+1}, t), 1, M(y_{2n}, y_{2n+2}, (2 - \alpha)t)] \leq 0$

Denoting by $M_m = M(y_m, y_{m+1}, t)$, we get

$$F[M_{2n+1}, M_{2n}, M_{2n+1}, M_{2n}, 1, M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, (1 - \alpha)t)] \leq 0.$$ 

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\[ F[M_{2n+1}, M_{2n}, M_{2n+1}, M_{2n}, 1, M_{2n} \ast M_{2n+1}] \leq 0, \]

which by \((F_c)\) implies

\[ M_{2n+1} \geq r(M_{2n}) \geq M_{2n}. \tag{4.3.2} \]

From (4.3.1) and (4.3.2), we deduce that \( \{M_{2n} = M(y_n, y_{n+1}, t)\}_{n \geq 0} \) is a non-decreasing sequence of positive real numbers in \([0,1]\) and, therefore, tends to a limit \( L \leq 1 \). We claim \( L = 1 \). If \( L < 1 \), then letting \( n \to \infty \) in (4.3.1) and using the definition of \( r \), we get

\[ L \leq r(L) < L, \]

which is a contradiction. Therefore, we must have \( L = 1 \). Now for any positive integer \( p \), we have

\[ M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) \ast M(y_{n+2}, y_{n+1}, t/p) \ast \ldots \]

\[ \ast M(y_{n+p-1}, y_{n+p}, t/p) \]

implying

\[ \lim_{n \to \infty} M(y_n, y_{n+p}, t) \geq 1 \ast 1 \ast \ldots \ast 1 = 1. \]

Thus \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there is a point \( z \in X \) such that \( y_n \to z \), and so the sequences \( \{Ax_{2n}\} = \{Tx_{2n+1}\} \) and \( \{Bx_{2n+1}\} = \{Sx_{2n+2}\} \) also converges to \( z \) in \( X \). Suppose \( A(X) \) is continuous and closed. Since \( A(X) \subset T(X) \), there exists a point \( u \in X \) such that \( z = Tu \). Now using inequality (2), we obtain

\[ F[M(Ax_{2n}, Bu, t), M(Sx_{2n}, Tu, t), M(Sx_{2n}, Ax_{2n}, t), \]

\[ M(Tu, Bu, t), M(Sx_{2n}, Bu, \alpha t), M(Tu, Ax_{2n}, (2 - \alpha)t)] \leq 0. \]

Taking limit as \( n \to \infty \), we obtain

\[ F[M(z, Bu, t), M(z, Tu, t), M(z, z, t), M(Tu, Bu, t), M(z, Bu, \alpha t), \]

\[ M(Tu, z, (2 - \alpha)t)] \leq 0 \]

i.e., \( F[M(z, Bu, t), 1, 1, M(z, Bu, t), M(z, Bu, \alpha t), 1] \leq 0 \).
If we take $\alpha = 1$ in the above inequality, then

$$F[M(z, Bu, t), 1, 1, M(z, Bu, t), M(z, Bu, t), 1] \leq 0,$$

so, by $F_a$, $M(z, Bu, t) \geq r(1) = 1$. Hence, $Bu = z$. Therefore, $Tu = z = Bu$. Since $B$ and $T$ are weakly compatible, $BTu = TBu$ and so $Tz = TBu = BTu = Bz$.

Again by inequality (2), we have

$$F[M(Ax_{2n}, BTu, t), M(Sx_{2n}, TBu, t), M(Sx_{2n}, Ax_{2n}, t), M(TBu, BTu, t),
M(Sx_{2n}, BTu, \alpha t), M(TBu, Ax_{2n}, (2 - \alpha)t)] \leq 0.$$

Taking limit as $n \to \infty$, we obtain

$$F[M(z, Bz, t), M(z, Tz, t), M(z, z, t), M(Tz, Bz, t), M(z, Bz, t),
M(Tz, z, (2 - \alpha)t)] \leq 0$$

i.e., $F[M(z, Bz, t), M(z, Bz, t), 1, 1, M(z, Bz, t), M(z, Bz, t) \leq 0.$

If $\alpha = 1$ then by the above inequality, we get

$$F[M(z, Bz, t), M(z, Bz, t), 1, 1, M(z, Bz, t), M(z, Bz, t)] \leq 0.$$

Then, by $F_b$ we obtain $Bz = z = Tz$. Since $B(X) \subset S(X)$, there must exists an element $v \in X$ such that $Tz = z = Sv$. Again using condition (2), we obtain

$$F[M(Av, Bu, t), M(Sv, Tu, t), M(Sv, Av, t), M(Tu, Bu, t), M(Sv, Bu, t),
M(Tu, Av, (2 - \alpha)t)] \leq 0$$

i.e., $F[M(Av, z, t), M(z, z, t), M(z, Av, t), M(z, z, t), M(z, z, t),
M(z, Av, (2 - \alpha)t)] \leq 0$

i.e., $F[M(Av, z, t), 1, M(z, Av, t), 1, 1, M(z, Av, (2 - \alpha)t)] \leq 0.$

For $\alpha = 1$, we get

$$F[M(Av, z, t), 1, M(z, Av, t), 1, 1, M(z, Av, t)] \leq 0,$$
which, by $F_c$, gives $Av = z = Sv$. Since $(A, S)$ is weakly compatible and $Av = Sv = z$ we have $ASv = SAv = Sz = Az$. Thus the condition (2) gives

$$F[M(Az, Bz, t), M(Sz, Tz, t), M(Sz, Az, t), M(Tz, Bz, t),$$

$$M(Sz, Bz, \alpha t), M(Tz, Az, (2 - \alpha) t)] \leq 0$$

i.e.,

$$F[M(Az, z, t), M(Sz, z, t), M(Sz, Sz, t), M(z, z, t),$$

$$M(Sz, z, \alpha t), M(z, Az, (2 - \alpha) t)] \leq 0$$

i.e.,

$$F[M(Az, z, t), M(Az, z, t), 1, 1, M(Az, z, \alpha t), M(z, Az, (2 - \alpha) t)] \leq 0.$$

For $\alpha = 1$, we get

$$F[M(Az, z, t), M(Az, z, t), 1, 1, M(Az, z, t), M(z, Az, t)] \leq 0,$$

which, by $F_b$, yields $M(Az, z, t) = 1$. This gives $Az = z = Sz$. Hence $z$ is a common fixed point of $A, B, S$ and $T$. If $T(X)$ is closed the proof is similar.

On the other hand, if $B(X)$ or $S(X)$ is closed, then there exists a point $u \in X$ such that $z = Su$. From inequality (2) with $x = u$ and $y = x_{2n+1}$, we get for a fixed $t > 0$,

$$F[M(Au, Bx_{2n+1}, t), M(Su, Tx_{2n+1}, t), M(Su, Au, t),$$

$$M(Tx_{2n+1}, Bx_{2n+1}, t), M(Su, Bx_{2n+1}, \alpha t), M(Tx_{2n+1}, Au, (2 - \alpha) t)] \leq 0.$$

Taking limit as $n \to \infty$, we get

$$F[M(Au, z, t), M(Su, z, t), M(Su, Au, t), M(z, z, t), M(Su, z, \alpha t), M(z, Au, (2 - \alpha) t)] \leq 0$$

i.e.,

$$F[M(Au, z, t), 1, M(z, Au, t), 1, 1, M(z, Au, (2 - \alpha) t)] \leq 0$$

If $\alpha = 1$ in the above inequality, we get

$$F[M(Au, z, t), 1, M(Au, z, t), 1, 1, M(z, Au, t)] \leq 0$$

Then, by $F_b$ we obtain $M(Au, z, t) \geq r(1) = 1$, implies $Au = z$. Since $Au = zSu$ and $A$ and $S$ are weakly compatible, $Az = Sz$. To prove that $Az = z$, we use condition (2) with $x = z$ and $y = x_{2n+1}$, then

$$F[M(Az, Bx_{2n+1}, t), M(Sz, Tx_{2n+1}, t), M(Sz, Az, t),$$

$$M(Tx_{2n+1}, Bx_{2n+1}, t), M(Su, Bx_{2n+1}, \alpha t), M(Tx_{2n+1}, Au, (2 - \alpha) t)] \leq 0.$$
\[ M(Tx_{2n+1}, Bx_{2n+1}, t), M(Sz, Bx_{2n+1}, \alpha t), M(Tx_{2n+1}, Az, (2-\alpha)t) \leq 0. \]

Taking limit as \( n \to \infty \), we get

\[ F[M(Az, z, t), M(Sz, z, t), M(Sz, Az, t), M(z, z, t), M(Sz, Az, \alpha t), M(z, Az, (2-\alpha)t)] \leq 0 \]

i.e., \( F[M(Az, z, t), M(z, Az, t), 1, M(z, Az, \alpha t), M(z, Az, (2-\alpha)t)] \leq 0 \)

If \( \alpha = 1 \) in the above inequality, we get

\[ F[M(Az, z, t), M(Az, z, t), 1, M(Az, z, t), M(Az, z, t)] \leq 0. \]

Applying \( F_b \) we obtain \( M(Az, z, t) \geq r(1) = 1 \), implies \( Az = z \). Since \( A(X) \subseteq T(X) \), there exists \( v \in X \) such that \( Az = zTv \). From condition (2) with \( x = u \), \( y = v \) we have

\[ F[M(Au, Bv, t), M(Su, Tt, t), M(Su, Au, t), M(Tv, Bv, t), M(Su, Bv, \alpha t), M(Tv, Au, (2-\alpha)t)] \leq 0, \]

i.e.

\[ F[M(z, Bv, t), M(z, z, t), M(z, Bv, t), M(z, Bv, \alpha t), M(z, z, (2-\alpha)t)] \leq 0, \]

i.e.

\[ F[M(z, Bv, t), 1, 1, M(z, Bv, t), M(z, Bv, \alpha t), 1] \leq 0. \]

If \( \alpha = 1 \) then

\[ F[M(z, Bv, t), 1, 1, M(z, Bv, t), M(z, Bv, t), 1] \leq 0, \]

which by \( F_a \) implies \( z = Bv = Tv \). Since \( (B, T) \) are weakly compatible so \( Tz = TBv = BTv = Bz \). To check that \( Tz = Bz = z \), in condition (2), take \( x = y = z \),

\[ F[M(z, Bz, t), M(z, Bz, t), M(z, z, t), M(Bz, Bz, t), M(z, Bz, \alpha t), M(Bz, z, (2-\alpha)t)] \leq 0. \]

For \( \alpha = 1 \), we deduce

\[ F[M(z, Bz, t), M(z, Bz, t), 1, 1, M(z, Bz, t), M(Bz, z, t)] \leq 0, \]

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which by $F_b$ implies $M(z, Bz, t) \geq 1$ and $z = Bz = Tz$.

Now for the uniqueness of common fixed point, suppose $A, S, B$ and $T$ have another common fixed point $z' \neq z$, then by condition (2) we have

$$F[M(Az, Bz', t), M(Sz, Tz', t), M(Sz, Az, t), M(Tz', Bz', t), M(Sz, Bz', \alpha t),$$

$$M(Tz', Az, (2 - \alpha)t)] \leq 0$$

i.e., $F[M(z, z', t), M(z, z', t), M(z, z, t), M(z', z', t), M(z, z', \alpha t),$ $M(z', z, (2 - \alpha)t)] \leq 0$

If $\alpha = 1$, then by $F_b$ we obtain

$$M(z, z', t) \geq r(1) = 1$$

gives $z' = z$. Hence $z$ is the unique fixed point of $A, B, S$ and $T$.

In theorem 4.3.4 if we take $S = T = I_X$ (the identity map on $X$) then we obtain the following.

**Corollary 4.3.5.** Let $(X, M, \ast)$ be a complete FM-space and $A$ and $B$ be self maps of $(X, M, \ast)$ such that

$$F[M(Ax, By, t), M(x, y, t), M(Ax, x, t), M(By, y, t), M(By, x, \alpha t),$$

$$M(Ax, y, (2 - \alpha)t)] \leq 0$$

for all $x, y \in X$ and $t > 0$ and $\alpha \in (0, 2)$.

Then $A$ and $B$ have a unique fixed point.

### 4.3.1 An application

Now we give an application in product spaces.
Theorem 4.3.6. Let \((X, M, \ast)\) be a complete \(FM-\) space with \(t \ast t \geq t\) and let \(A\) and \(B\) be self maps on the product \(X \times X \rightarrow X\) such that

\[
F[M(A(x, y), B(u, v), t), M(x, u, t), M(A(x, y), x, t), M(B(u, v), u, t),
M(B(u, v), x, \alpha t), M(A(x, y), u, (2 - \alpha)t)] \leq 0 \quad (4.3.3)
\]

for all \(x, y, u, v \in X\) and \(t > 0\) and \(\alpha \in (0, 2)\). Then there exists exactly one point \(w \in X\) such that \(A(w, w) = w = B(w, w)\).

**Proof.** By the implication relation (4.3.3), we have

\[
F[M(A(x, y), B(u, y), t), M(x, u, t), M(A(x, y), x, t), M(B(u, y), u, t),
M(B(u, y), x, \alpha t), M(A(x, y), u, (2 - \alpha)t)] \leq 0,
\]

for all \(x, y, u \in X\). Therefore, by corollary (4.3.5), for each \(y \in X\), there exists one and only one \(z(y)\) in \(X\) such that

\[
A(z(y), y) = z(y) = B(z(y), y).
\]  \quad (4.3.4)

Now, for any \(y, y' \in X\) and \(\alpha = 1\), (5.1) gives

\[
F[M(z(y), z(y'), t), M(z(y), z(y'), t), M(A(z(y), y), z(y), t),
M(B(z(y'), y'), z(y'), t), M(B(z(y'), y'), z(y'), t), M(A(z(y), y), z(y'), t)] \leq 0
\]

\[
F[M(z(y), z(y'), t), M(z(y), z(y'), t), 1, 1, M(z(y'), z(y'), t), M(z(y), z(y'), t)] \leq 0.
\]

By \(F_b\), we obtain

\[
M(z(y), z(y'), t) \geq r(1) = 1.
\]

Hence \(z(y) = z(y')\). Therefore, the map \(z(.)\) of \(X\) into itself has exactly one fixed point \(w\) in \(X\), i.e. \(z(w) = w\). Hence by (??) \(w = z(w) = A(w, w) = B(w, w)\). This completes the proof.
4.4 For occasionally weakly compatible mappings

In this section we fuzzify the definition of occasionally weakly compatible (owc) which is introduced by M.A.Al-Thagafi and N.Shahzad[2] in metric space as $f(g(x)) = g(f(x))$ for some $x \in C(f, g) = \{x \in (X, d) | f(x) = g(x)\}$.

**Definition 4.4.1.** Two self mappings $A$ and $B$ of fuzzy metric space $(X, M, *)$ are said to be occasionally weakly compatible if they commute at one of their coincidence points i.e., there exists a point $u \in X$ such that

$$M(Au, Bu, t) = 1 \text{ then } M(ABu, BAu, t) = 1 \text{ } \forall \text{ } t > 0.$$  

**Remark 4.4.2.** In occasionally weakly compatible mappings the mappings are not necessarily commute at all points but they need to commute at least one coincidence point of mappings. The advantage of this is that we left the condition of completeness and subset-hood of any image space. No need of Cauchy sequence, which helps us to minimize the calculation and requirement of its limit.

In 2008 Imdad and Javid[51] used implicit relations to find fixed point in fuzzy metric space with weakly compatible condition and in four coordinates.

**Theorem 4.4.3.** Let $A, B, S$ and $T$ be four self-mappings of a fuzzy metric space $(X, M, *)$ satisfying the condition:

$F(M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)) \leq 0$

for all distinct $x, y \in X$ and $t > 0$, where $F \in \Psi$. If $A(X) \subset T(X), B(X) \subset S(X)$ and one of $A(X), B(X), S(X)$ or $T(X)$ is a complete subspace of $X$. Then (a) the pair $(A, S)$ has a point of coincidence, and (b) the pair $(B, T)$ has a point of coincidence. Moreover, if the pairs $(A, S)$ and $(B, T)$ are weakly compatible, then $A, B, S$ and $T$ have a unique common fixed point.

Here we are taking improved condition of occasionally weakly compatible and six coordinates.
Theorem 4.4.4. \(^2\)Let \((X, M, \ast)\) be a fuzzy metric space and \(A, B, S, T\) be self-mappings of \(X\) satisfying the conditions:

(1) mappings \((A, S)\) and \((B, T)\) are occasionally weakly compatible (owc).

(2) the inequality

\[
F(M(Ax, By, t), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\
M(By, Sx, t), M(Ax, Ty, t)) \leq 0.
\]

holds for all \(x, y \in X, \forall \ t > 0\) and \(F \in \mathcal{F}_6\) then we have a unique common fixed point for the mappings \(A, B, S\) and \(T\).

**Proof** Since \((A, S)\) and \((B, T)\) are owc, there exists points \(u, v \in X\) such that

\[
M(Au, Su, t) = 1 \text{ then } M(ASu, SAu, t) = 1 \text{ for } t > 0
\]

and

\[
M(Bv, Tv, t) = 1 \text{ then } M(TBv, BTv, t) = 1 \text{ for } t > 0.
\]

First we take that \(Au \neq Bv\) i.e. \(\forall t > 0, \ M(Au, Bv, t) \neq 1\) then by inequality (2)

\[
F(M(Au, Bv, t), M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t), \\
M(Bv, Su, t), M(Au, Ty, t)) \leq 0
\]

\[
F(M(Au, Bv, t), M(Au, Bv, t), M(Au, Au, t), M(Bv, Bv, t), \\
M(Bv, Au, t), M(Au, Bv, t)) \leq 0
\]

\[
F(M(Au, Bv, t), M(Au, Bv, t), 1, 1, M(Au, Bv, t), M(Au, Bv, t)) \leq 0
\]

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so $F_b$ gives $M(Au, Bv, t) \geq r(1) = 1$. Hence $Au = su = Bv = Tv$. Now suppose $A^2u \neq Au$ i.e. $\forall t > 0, M(A^2u, Au, t) \neq 1$ then by inequality (2)

$$F(M(A^2u, Bv, t), M(SAu, Tv, t), M(A^2u, SAu, t), M(Bv, Tv, t),$$

$$M(Bv, SAu, t), M(A^2u, Tv, t)) \leq 0$$

$$F(M(A^2u, Au, t), M(MSu, Au, t), M(A^2u, ASu, t), M(Au, Au, t),$$

$$M(Au, ASu, t), M(A^2u, Au, t)) \leq 0$$

$$F(M(A^2u, Au, t), M(A^2u, Au, t), 1, 1, M(Au, A^2u, t), M(A^2u, Au, t)) \leq 0,$$

so $F_b$ gives $M(Au, A^2u, t) \geq r(1) = 1$. Hence $A^2u = Au = su = Bv = Tv$. If we take $Au = z$ then $Az = z$. Similarly we can show that $Bz = Sz = Tz = z$.

**Uniqueness** Now for uniqueness of $z$ let there is another point $w \in X$ such that $Aw = Bw = Tw = Sw = w$ and $w \neq z$ i.e. $\forall t > 0, M(z, w, t) \neq 1$ then by inequality (2)

$$F(M(Az, Bw, t), M(Sz, Tw, t), M(Az, Sz, t), M(Bw, Tw, t),$$

$$M(Bw, Sz, t), M(Az, Tw, t)) \leq 0$$

$$F(M(z, w, t), M(z, w, t), M(z, z, t), M(w, w, t), M(w, z, t),$$

$$M(z, w, t)) \leq 0$$

$$F(M(z, w, t), M(z, w, t), 1, 1, M(w, z, t), M(z, w, t)) \leq 0$$

so $F_b$ gives $M(z, w, t) \geq r(1) = 1$. Hence $z = w$.

Hence the fixed point of all the functions $A$, $B$, $S$ and $T$ is unique.
Corollary 4.4.5. Let $(X, M, *)$ be a fuzzy metric space and $A, S$ be self-mappings of $X$ such that $(A, S)$ is owc and satisfying the condition

$$F(M(Ax, Ay, t), M(Sx, Sy, t), M(Ax, Sx, t), M(Ay, Sy, t), M(Ay, Sx, t), M(Ax, Sy, t)) \leq 0$$

holds for all $x, y \in X$, $\forall \ t > 0$ and $F \in \mathcal{F}_6$ Then then we have a unique common fixed point for the mappings $A$ and $S$.

Corollary 4.4.6. Let $(X, M, *)$ be a fuzzy metric space and $A, B, S$ be self-mappings of $X$ such that $(A, S)$ and $(B, S$ are owc and satisfying the condition

$$F(M(Ax, By, t), M(Sx, Sy, t), M(Ax, Sx, t), M(By, Sy, t), M(By, Sx, t), M(Ax, Sy, t)) \leq 0$$

holds for all $x, y \in X$, $\forall \ t > 0$ and $F \in \mathcal{F}_6$ Then then we have a unique common fixed point for the mappings $A$, $B$ and $S$.

Corollary 4.4.7. Let $(X, M, *)$ be a fuzzy metric space and $A, B, S, T$ be self-mappings of $X$ such that $(A, S)$ and $(B, T$ are owc and satisfying the inequality

$$M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(By, Sx, t), M(Ax, Ty, t)\})$$

where $\phi(t) = r(t) > t$ $\forall \ t \in (0, 1)$ and $\phi(0) = r(0) = 0$, $\phi(1) = r(1) = 1$

holds for all $x, y \in X$, $\forall \ t > 0$ then $A, B, S$ and $T$ have common fixed point.

Corollary 4.4.8. Let $(X, M, *)$ be a fuzzy metric space and $A, B, S, T$ be self-mappings of $X$ such that $(A, S)$ and $(B, T$ are owc and satisfying the inequality

$$q M(Ax, By, t) \geq a M(Sx, Ty, t) + b M(Ax, Sx, t) + c M(By, Ty, t),$$

$$+ \max\{M(By, Sx, t), M(Ax, Ty, t)\})$$

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holds for all \( x, y \in X, \quad \forall \ t > 0 \quad \forall \quad a, b, c \geq 0, \quad q > 0, \quad a + b + c + 1 > q \) then \( A, B, S \) and \( T \) have common fixed point.

**Corollary 4.4.9.** Let \((X, M, \ast)\) be a fuzzy metric space and \( A, B, S, T \) be self-mappings of \( X \) such that \((A, S)\) and \((B, T)\) are owc and satisfying the inequality

\[
q M(Ax, By, t) \geq a M(Sx, Ty, t) + \frac{b M(Ax, Sx, t) + c \max\{M(Ax, Ty, t), M(By, Sx, t)\}}{M(By, Ty, t)}
\]

holds for all \( x, y \in X, \quad \forall \ t > 0 \quad \forall \quad a, b, c \geq 0, \quad q > 0, \quad a + b + c > q \) then \( A, B, S \) and \( T \) have common fixed point.

**Corollary 4.4.10.** Let \((X, M, \ast)\) be a fuzzy metric space and \( A, B, S, T \) be self-mappings of \( X \) such that \((A, S)\) and \((B, T)\) are owc and satisfying the inequality

\[
q M(Ax, By, t) \geq a M(Sx, Ty, t) + \frac{b M(By, Ty, t) + c \max\{M(Ax, Ty, t), M(By, Sx, t)\}}{M(Ax, Sx, t)}
\]

holds for all \( x, y \in X, \quad \forall \ t > 0 \quad \forall \quad a, b, c \geq 0, \quad q > 0, \quad a + b + c > q \) then \( A, B, S \) and \( T \) have common fixed point.

**Corollary 4.4.11.** Let \((X, M, \ast)\) be a fuzzy metric space and \( A, B, S, T \) be self-mappings of \( X \) such that \((A, S)\) and \((B, T)\) are owc and satisfying the inequality

\[
q M(Ax, By, t) \geq b M(Ax, Sx, t) + c M(By, Ty, t) + \frac{a \max\{M(Ax, Ty, t), M(By, Sx, t)\}}{M(Sx, Ty, t)}
\]

holds for all \( x, y \in X, \quad \forall \ t > 0 \quad \forall \quad a, b, c \geq 0, \quad q > 0, \quad a + b + c > q \) then \( A, B, S \) and \( T \) have common fixed point.

### 4.5 For mappings satisfying tangential condition

In this part, we fuzzify the tangential condition and prove some coincidence and fixed point theorems for fuzzy metric spaces in this setting. We also fuzzify the concept of *weak tangent point* and *pair-wise tangential* property for a dual pair of mappings introduced by Pathak and Shahzad[79].
Definition 4.5.1. Let $A$, $B$, $S$ and $T$ be self mappings of a fuzzy metric space $(X, M, \ast)$ then the pairs $(A, B)$ is called tangential with respect to the pair $(S, T)$ whenever the weak tangent point of $(S, T)$ is also weak tangent point of $(A, B)$ i.e. for sequences $\{x_n\}, \{y_n\} \in X$, a point $z \in X$ exists such that
\[
\lim_{n \to \infty} M(Sx_n, Ty_n, t) = \lim_{n \to \infty} M(Sx_n, z, t) = \lim_{n \to \infty} M(Ty_n, z, t) = 1
\]
then
\[
\lim_{n \to \infty} M(Ax_n, By_n, t) = \lim_{n \to \infty} M(Ax_n, z, t) = \lim_{n \to \infty} M(By_n, z, t) = 1.
\]

Theorem 4.5.2. Let $(X, M, \ast)$ be a fuzzy metric space and $A, B, S, T$ be self-mappings of $X$ satisfying the conditions:

(1) $(A, B)$ is tangential with respect to $(S, T)$.

(2) the inequality
\[
F(M(Ax, By, t), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t),
\]
\[
\min\{M(By, Sx, t), M(Ax, Ty, t)\}) \leq 0
\]
holds for all $x, y \in X$ and $F \in \mathcal{F}_5$ then the pairs $(A, S)$ and $(B, T)$ have coincidence points.

Proof Since $(A, B)$ is tangential with respect to $(S, T)$ it means the weak tangent point of $(S, T)$ is also weak tangent point of $(A, B)$ i.e. there exist sequences $\{x_n\}, \{y_n\} \in X$ and a point $z \in X$ such that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Ty_n = z \in X
\]
then
\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} By_n = z
\]
also $z \in S(X) \cap T(X)$ hence there must exists points $u, v \in X$ such that $z = Su = Tv$.  

\(^3\)the Journal of Fuzzy Mathematics 19(2), 2011,263-274.
If \( Bv \neq z \) we get by inequality

\[
F(M(Ax_n, Bv, t), M(Sx_n, Tv, t), M(Ax_n, Sx_n, t), M(Bv, Tv, t),
\min\{M(Bv, Sx_n, t), M(Ax_n, Tv, t)\}) \leq 0.
\]

Taking limit as \( n \to \infty \) we obtain

\[
F(M(z, Bv, t), M(z, z, t), M(z, z, t), M(Bv, z, t),
\min\{M(Bv, z, t), M(z, z, t)\}) \leq 0.
\]

\[
F(M(z, Bv, t), 1, 1, M(Bv, z, t), \min\{M(Bv, z, t), 1\}) \leq 0.
\]

\[
F(M(z, Bv, t), 1, 1, M(Bv, z, t), M(Bv, z, t)) \leq 0.
\]

By \( F_1 \) we get \( M(z, Bv, t) \geq r(1) > 1 \)

which gives \( Bv = z \).

Further if \( Au \neq z \) using inequality we get

\[
F(M(Au, By_n, t), M(Su, Ty_n, t), M(Au, Su, t), M(By_n, Ty_n, t),
\min\{M(By_n, Su, t), M(Au, Ty_n, t)\}) \leq 0
\]

Taking limit as \( n \to \infty \) we obtain

\[
F(M(Au, z, t), M(z, z, t), M(Au, z, t), M(z, z, t),
\min\{M(z, z, t), M(Au, z, t)\}) \leq 0
\]

\[
F(M(Au, z, t), 1, M(Au, z, t), 1, \min\{1, M(Au, z, t)\}) \leq 0
\]

\[
F(M(Au, z, t), 1, M(Au, z, t), 1, M(Au, z, t)) \leq 0,
\]

By \( F_2 \) we get \( M(Au, z, t) \geq r(1) > 1 \)

which gives \( Au = z \).

Thus \( Au = Su = Bv = Tv = z \) showing that the pairs \( (A, S) \) and \( (B, T) \) have coincidence points \( u \) and \( v \) respectively.

**Corollary 4.5.3.** In the setting of Theorem 4.5.2 if the pairs of mappings \( (A, S) \) and \( (B, T) \) are occasionally weakly compatible then we have a unique common fixed
point for the mappings $A$, $B$, $S$ and $T$.

**Corollary 4.5.4.** Let $(X, M, *)$ be a fuzzy metric space and $A$, $B$, $S$ and $T$ be self-mappings of $X$ satisfying the conditions:

1. $(A, B)$ is tangential with respect to $(S, T)$.
2. the inequality
   
   $$F(M(Ax, By, t), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t),$$
   
   $$M(By, Sx, t), M(Ax, Ty, t)) \leq 0$$

   holds for all $x, y \in X$ and $F \in \mathcal{F}_5$. Then the pairs $(A, S)$ and $(B, T)$ have coincidence points.

**Remark 4.5.5.** We can change the condition (1) of the theorem 4.5.2 by

(a) $(S, B)$ is tangential with respect to $(A, T)$.

(b) $(A, S)$ and $(B, T)$ have a common E.A. property.

(c) $(A, S)$ and $(B, T)$ are each occasionally weakly compatible.

**Remark 4.5.6.** We can change the condition (2) of the theorem 4.5.2 by

$$(a_1) \quad M(Ax, By, t) \geq k\{M(Sx, Ty, t)M(Ax, Sx, t)$$

$$M(By, Ty, t), M(By, Sx, t), M(Ax, Ty, t)\}$$

$$(a_2) \quad M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t)M(Ax, Sx, t)$$

$$M(By, Ty, t), M(By, Sx, t), M(Ax, Ty, t)\})$$

and $\phi : [0, 1] \to [0, 1]$ is an increasing function defined as $\phi(t) > t$ also $\phi(0) = 0$, $\phi(1) = 1$

$$(a_3) \quad qM(Ax, By, t) \geq aM(Sx, Ty, t) + bM(Ax, Sx, t)$$

$$+ cM(By, Ty, t) + \max\{M(By, Sx, t), M(Ax, Ty, t)\}).$$

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