Appendix A

POSITIVE MASS THEOREM, PENROSE INEQUALITY AND INVERSE MEAN CURVATURE FLOW

INTRODUCTION

In this appendix we describe previous work on the Penrose inequality to bring out the connection with our efforts. This material is mostly review, but there are certain intriguing similarities, which suggest directions for future work. Using a pure diffeomorphism (instead of pure Ricci flow) we can see the connection with early work by Geroch, Jang and Wald [1, 2].

We recall the discussions about Positive Mass Theorem (PMT) and Penrose Inequality (PI) in chapter (1) where we consider an asymptotically flat, globally hyperbolic spacetime $(M, g_{\nu\mu})$ ($\nu = 0, 1, 2, 3$) (with signature $(-, +, +, +)$). Let $(\Sigma, h_{ab}, K_{ab})$ $(a, b = 1, 2, 3)$ be an initial data set with mass $M$ and $A$ be the area of the outermost apparent horizon. Then the statements of PMT and PI are as follows:

- **Positive Mass Theorem**: The PMT states that for a nonsingular asymptotically flat initial data set, the ADM mass $M$ is nonnegative and vanishes if and only if the initial data set is that of Minkowski space.

- **Penrose Inequality**: The Penrose inequality states that if the energy condition holds
then,

\[ A \leq 16\pi M^2 \]  

(A.1)

The PI clearly implies PMT, which only states that \( M \geq 0 \).

**GEROCH’S ARGUMENT TO PROVE POSITIVE MASS THEOREM**

In 1973 Geroch gave a proof of the PMT [1] for the restricted case of maximal slicing \((K := h^{ab}K_{ab} = 0)\) by defining a flow of 2-surfaces in \( \Sigma \) in which the surfaces flow in the outward normal direction at a rate equal to the inverse of their mean curvatures at each point. The flow is known as the Inverse Mean Curvature Flow (IMCF). Later in 1977 Jang and Wald extended the ideas of Geroch to give a proof of the Riemannian Penrose inequality [2]. Geroch’s argument [1] establishes the validity of the PMT for an initial data set \((\Sigma, h_{ab}, K_{ab})\) having the topology \( \mathbb{R}^3 \) and with the trace of its extrinsic curvature, \( K = 0 \). The constraint equations in GR and the local energy condition (3.11-3.13) imply that the scalar curvature \( R \) of \( \Sigma \) cannot be negative i.e.,

\[ R \geq 0 \]  

(A.2)

We introduce a function \( \tau \) on \( \Sigma \) such that the two dimensional surfaces \( \tau = \text{constant} \) in \( \Sigma \) are nested topological 2-spheres with the innermost surface reducing to a point. For each value of \( \tau \) let us assume that \( S \subset \Sigma \) is one such surface and \( \eta_a = D_a \tau \) defines the normal to \( S \). The unit normal is then given by \( n_a = (\eta, \eta)^{-1/2} \eta_a \). Let \( \xi^a := urn^a \) has the property that \( \xi^a D_a \tau = 1 \).

Let \( \nu^2 := \eta^a \eta_a \). So we have \( \xi^a D_a \tau = uv = 1 \). So \( u = 1/v = (\eta^a \eta_a)^{-1/2} \). Note that \( \eta_a = \xi_a/u^2 \) is an exact differential form. Next we consider the function \( C(\tau) \), which has been introduced earlier in this thesis by the name “compactness” and which, for each value of \( \tau \), is defined as

\[ C(\tau) := \int_{S \in \Sigma} (2R - k^2) dA \]  

(A.3)

where the integration extends over the surface \( S \) and \( R \) and \( k \) denote the scalar curvature and the trace of the extrinsic curvature of the surface \( S \) as a submanifold of \( \Sigma \), respectively.

Also the Hawking mass (as introduced earlier in the thesis) is defined as
\[ M_H(\tau) = \frac{A^{1/2}}{64\pi^{3/2}} \int_{S \subset \Sigma} (2R - k^2) dA = \frac{A^{1/2}}{64\pi^{3/2}} C(\tau) \quad \text{(A.4)} \]

We note that the Gauss-Bonnet theorem implies that
\[ \int_{S \subset \Sigma} RdA = 8\pi \quad \text{(A.5)} \]

The trace of the extrinsic curvature \( k \) of \( S \subset \Sigma \) is defined as
\[ k = D_a n^a \quad \text{(A.6)} \]

The rate of change of any quantity with respect to \( \tau \) is its Lie derivative by \( \xi^a \). The rate of change of \( k \) with respect to \( \tau \) is then given by
\[ \frac{dk}{d\tau} = \xi^b D_b(D_a n^a) = \xi^b D_b D_a n^a - \xi^b R^a_{\quad mb} n^b = \xi^b D_a D_b n^a - uR_{ab} n^b n^m \quad \text{(A.7)} \]

Next after a calculation using the Gauss-Codazzi equation and standard projection techniques in GR [3] we arrive at the result that
\[ \frac{d}{d\tau} C(\tau) = \int_{S \subset \Sigma} [2kD^a \tilde{D}_a u + ukk^{ab} k_{ab} - ukR + ukR] dA \quad \text{(A.8)} \]

where \( \tilde{D}_a \) is the covariant derivative operator on the 2-surface \( S \) with respect to the induced metric. Next we suppose that we can choose the 2-surfaces, \( S \), such that
\[ uk = 1 \quad \text{(A.9)} \]

This defines the flow of the 2-surfaces considered. This is the IMCF. Under this condition the equation (A.8) becomes
\[ \frac{d}{d\tau} C(\tau) = -\frac{1}{2} C(\tau) + \int_{S \subset \Sigma} [R + (k^{ab} k_{ab} - \frac{1}{2} k^2) + 2u^{-2} \tilde{D}_a u \tilde{D}^a u] dA \]
\[ \geq -\frac{1}{2} C(\tau) \quad \text{(A.10)} \]

But since \( C(\tau) \to 0 \) as the surface reduces to a point, equation (A.10) implies \( C(\tau) \geq 0 \) for all \( \tau \). As \( \tau \to \infty \) the surface \( S \) expands to a round sphere at infinity and the Hawking mass \( M_H \) becomes the ADM mass \( M \). By equation (A.4) we have \( M \geq 0 \). The equality holds (i.e., \( M = 0 \)) only if \( K^{ab} = 0 \) ans \( h_{ab} \) is flat, i.e., initial data for flat space.
ARGUMENT BY JANG AND WALD TO PROVE PENROSE INEQUALITY

Jang and Wald extended the ideas of Geroch to give a proof of the Riemannian Penrose inequality [2]. Their argument runs as follows: We consider a time symmetric initial data set whose apparent horizon $\mathcal{H}$ has only one component. It is known that $\mathcal{H}$ must have the topology of a sphere. Consider a nested family of 2-sphere analogous to those used in the earlier section on Geroch's argument, but now this family has the additional property that the surface defined by $\tau = 0$ is $\mathcal{H}$. The family of surfaces for $\tau > 0$ is again defined by the property of the IMCF that

$$uk = 1$$  \hspace{1cm} (A.11)

Now by considering the function

$$C(\tau) := \int_{S \in \Sigma} (2R - k^2)dA$$  \hspace{1cm} (A.12)

we again find that

$$\frac{d}{d\tau} C(\tau) \geq -\frac{1}{2} C(\tau)$$  \hspace{1cm} (A.13)

i.e., we have

$$\frac{d}{d\tau} [\exp(\tau/2)C] \geq 0$$  \hspace{1cm} (A.14)

On the other hand, the rate of change of area $A(\tau)$ of the surface $\tau = \text{const.}$ is given by

$$\frac{d}{d\tau} A(\tau) = \int k u dA = \int dA = A(\tau)$$  \hspace{1cm} (A.15)

Since, the surface $\tau = 0$ is just $\mathcal{H}$, we have

$$A(\tau) = A \exp(\tau)$$  \hspace{1cm} (A.16)

where $A$ is the area of $\mathcal{H}$. Equation (A.14) then implies

$$\lim_{\tau \to \infty} \frac{A(\tau)^{1/2}}{A^{1/2}} C(\tau) \geq C(0)$$  \hspace{1cm} (A.17)

Using equation (A.4), the LHS of the equation (A.17) is

$$\lim_{\tau \to \infty} \frac{A(\tau)^{1/2}}{A^{1/2}} C(\tau) = \frac{64\pi^{3/2}}{A^{1/2}} M$$  \hspace{1cm} (A.18)
On the other hand, since $\mathcal{H}$ is extremal ($k = 0$), by the Gauss-Bonnet theorem we have

$$C(0) = 16\pi \quad (A.19)$$

So from equations (A.17, A.18, and A.19) we have

$$A \leq 16\pi M^2 \quad (A.20)$$

which is PI (equation A.1).

**CONCLUSION**

We note that the effect of varying $S$ in $\Sigma$ can be achieved by using a diffeomorphism to change the metric and keeping $S$ fixed. In earlier work (chapter (6)), we took the point of view that $S$ is fixed and only the tensor field $h_{ab}$ is changing. So the equation responsible for the change in the metric is

$$\frac{dh_{ab}}{d\tau} = D_a \xi_b + D_b \xi_a \quad (A.21)$$

which is the same equation for the rate of change of the metric under IMCF (where we vary $S$ in $\Sigma$) given by the Lie derivative of the metric by $\xi$. In chapter (6) we have studied the evolution of geometric quantities of interest under this flow.

The proof of the PMT using Geroch's argument was only for the restricted case of time symmetric initial data set ($K = 0$). Although there is some evidence [4] for believing that all all nonsingular, asymptotically flat spacetimes must contain at least one asymptotically flat slice with $K = 0$, one would still want to prove the most general statement of PMT that an initially regular spacetime must have $M \geq 0$ even if it develops singularities later.

In both the proofs of PMT and PI described above, the existence of the flow of surfaces as required in the above arguments was assumed. It can happen that as the flow evolves there appear points where $k = 0$ and the flow $uk = 1$ becomes ill defined and difficulties with the surface evolution can occur.

The work of Huisken and Ilmanen in 1997 [5] reformulated the IMCF in such a way that the new generalized IMCF always exists. In the new formulation the surface sometimes
jumps outward. However when the flow is smooth it equals the original IMCF, and the Hawking mass still remains monotone.

Geometrically, the idea of Huisken and Ilmanen can be described as follows: Let $S(\tau)$ be the surface resulting from IMCF for "time" $\tau$ beginning with the minimal surface $S_0$. Define $\bar{S}(\tau)$ to be the outermost minimal area enclosure of $S(\tau)$. Typically, $S(\tau) = \bar{S}(\tau)$ in the flow, but in the case that the two surfaces do not coincide, we immediately replace $S(\tau)$ with $\bar{S}(\tau)$ and then continue the IMCF. This is similar in essence to the geometric surgeries done in the Ricci flow when it hits a singularity in order to continue the flow further.

The proof of PI by the Inverse mean curvature flow applies only for a single black hole and also it does not work in higher dimensions. Bray considered a flow on the space of three metrics and presented a technique that proves the inequality for any number of black holes and which can likely be generalized to higher dimensions [6]. We hope that a combination of Ricci flow and diffeomorphism can be exploited to give a general approach to the Penrose inequality.
Bibliography


Appendix B

EVOLUTION OF THE SCALAR CURVATURE

Notation:

Our notation will be as follows:

1. For Scalar and Tensor quantities we have the following rules:
   - **In dimension 4:** the index of dimension is explicitly written e.g \( ^4R, ^4R_\mu \) etc. except for the metric tensor while we use the symbol \((g)\) for the 4 metric, e.g \((g_{ab})\) etc.
   - **In dimension 3:** the index of dimension is not written explicitly e.g \((R, R_{ab})\) etc. while we use the symbol \((h)\) for the 3 metric, e.g \((h_{ab})\) etc.

2. For operators:
   - **In dimension 4:** \(\nabla\) is for 'Grad', \(\nabla^2\) is for 'Laplacian'
   - **In dimension 3:** \(D\) is for 'Grad', \(D^2\) is for 'Laplacian'

3. Our metric conventions are those of Poisson [Eric Poisson 2004 A Relativist’s Toolkit
   *Cambridge Univ. Press*]
General expression for the evolution of the 3 scalar curvature

Here we calculate the general expression for the evolution of the 3-scalar curvature when $h_{ab}$ is varied according to

$$h_{ab} = H_{ab} \tag{B.1}$$

$$\dot{R} = \frac{d}{d\tau}(h^{bd} R_{bd}) = -R_{bd} H^{bd} + h^{bd} \dot{R}_{bd} \tag{B.2}$$

$$H^{bd} = h^{kk} h^{ld} H_{kl} \tag{B.3}$$

the term,

$$h^{bd} \dot{R}_{bd} = h^{bd} \left[ D_c \Gamma^c_{bd} - D_d \Gamma^c_{bc} \right]$$

$$= h^{bd} \left[ D_c \frac{1}{2} h^{ce} \left( D_b H_{cd} + D_d H_{bc} - D_e H_{bd} \right) - D_d \frac{1}{2} h^{ce} \left( D_b H_{ce} + D_c H_{eb} - D_e H_{bc} \right) \right]$$

$$= \delta^2 H - D^2 (tr H) \tag{B.4}$$

where,

$$\delta^2 H := D^d D^b H_{ab} = h^{cd} h^{bd} (D_c D_d H_{ab}) \tag{B.5}$$

$$tr H := h^{ab} H_{ab} \tag{B.6}$$

and

$$D^2 := h^{ab} D_a D_b \tag{B.7}$$

so we have,

$$\dot{R} = \frac{d}{d\tau}(h^{bd} R_{bd}) = -R_{bd} H^{bd} + \delta^2 H - D^2 (tr H) \tag{B.8}$$
Appendix C

PERELMAN’S GRADIENT FORMULATION

Perelman flow using the variational technique

Here we give a derivation of the Perelman flow using a variational technique. This is based on Topping’s exposition of Perelman’s work [Peter Topping 2006 London Mathematical Society Lecture Notes Series Cambridge University Press]. Let us consider the following functional :

\[ \mathcal{F}_p(h, f) := \int_{M} (R + |Df|^2) e^{-f} dV \]  \hspace{1cm} (C.1)

where,

\[ \bar{k} := \frac{\partial f}{\partial \tau} \]  \hspace{1cm} (C.2)

now,

\[ \frac{\partial}{\partial \tau} |Df|^2 = -H^{ab}D_a f D_b f + 2k^{ab} D_a \bar{k} D_b f \]  \hspace{1cm} (C.3)

and

\[ \frac{\partial}{\partial \tau} dV = \frac{1}{2} (trH) dV \]  \hspace{1cm} (C.4)

so,

\[ \frac{d}{d\tau} \mathcal{F}_p(h, f) = \int_{M} \left[ -H^{ab}D_a f D_b f + 2k^{ab} D_a \bar{k} D_b f - R_{ab} H^{ab} + \delta^2 H - D^2 (trH) \right] e^{-f} dV \]

\[ + \int_{M} (R + |Df|^2) \left[ - \bar{k} + \frac{1}{2} (trH) \right] e^{-f} dV \]  \hspace{1cm} (C.5)
we notice the following things:

\[
\int_M (2\kappa^{ab} D^b \kappa D_a f) e^{-f} dV = \int_M -2\bar{k}(D^2 f - |Df|^2) e^{-f} dV \tag{C.6}
\]

\[
\int_M (\delta^2 H) e^{-f} dV = \int_M (F^{ab} D_a f D_b f - H^{ab} D_a D_b f) e^{-f} dV \tag{C.7}
\]

and

\[
\int_M -D^2 (tr H) e^{-f} dV = \int_M (D^2 f - |Df|^2)(tr H) e^{-f} dV \tag{C.8}
\]

so,

\[
\frac{d}{dt} \mathcal{F}_P(h, f) = \int_M \left[ (-R^{ab} - D^a D^b f) H_{ab} + (2D^2 f - |Df|^2 + R)(-\bar{k} + \frac{1}{2} tr H) \right] e^{-f} dV \tag{C.9}
\]

we demand that the scaled volume remains constant under the flow:

\[
0 = \frac{\partial}{\partial \tau} (e^{-f} dV) = (-\bar{k} + \frac{1}{2} tr H) e^{-f} dV \tag{C.10}
\]

so,

\[
\bar{k} = \frac{\partial f}{\partial \tau} = \frac{1}{2} tr H \tag{C.11}
\]

and we have,

\[
\frac{d}{dt} \mathcal{F}_P(h, f) = \int_M \left[ (-R^{ab} - D^a D^b f) H_{ab} \right] e^{-f} dV \geq 0 \tag{C.12}
\]

i.e., \( \mathcal{F}_P \) is nondecreasing along the flow if we choose the metric \( h_{ab} \) to be varying according to the following equation

\[
H_{ab} = \frac{\partial}{\partial \tau} h_{ab} = -2(R_{ab} + D_a D_b f) \tag{C.13}
\]

and hence

\[
\bar{k} = \frac{\partial f}{\partial \tau} = \frac{1}{2} tr H = -R - D^2 f \tag{C.14}
\]
Appendix D

EVOLUTION OF GEOMETRIC QUANTITIES UNDER PERELMAN’S RICCI FLOW

The rate of change of 3 scalar curvature under the Perelman flow

Here we calculate the rate of change of 3 scalar curvature under the Perelman flow. (Here we follow Perelman’s convention. The sign of $f$ agrees with Perelman and is reversed between this and the bulk of the thesis).

Method: 1

we have for the Perelman flow:

$$\frac{\partial}{\partial \tau} h_{ab} = -2(R_{ab} + D_aD_bf)$$  \hspace{1cm} (D.1)

using,

$$h^{ab}h_{bc} = \delta_c^a$$ \hspace{1cm} (D.2)

we have,

$$\frac{d h^{ab}}{dt} := h^{am}h^{bn}\frac{dh^{mn}}{dt} = -\frac{d}{d\tau}(h^{ab})$$ \hspace{1cm} (D.3)

we move to a local flat coordinate system to calculate the following things:

$$\Gamma^a_{bc} = \frac{1}{2} h^{ad}(D_bh_{dc} + D_c h_{db} - D_dh_{bc})$$ \hspace{1cm} (D.4)
We introduce a standard notation for covariant derivative: $T^a_{bc} := D_c T^a_b$ for any tensor $T^a_b$.

now,

\[ \mathring{R} = h^{bd} R_{bd} + h^{bd} \mathring{R}_{bd} \]

\[ = 2|R_{bd}|^2 + (2D^b D^d f) R_{bd} + \text{term.} \tag{D.7} \]

where,

\[ \text{term} = h^{bd} R_{bd} = h^{bd} (D_c \Gamma^c_{bd} - D_d \Gamma^e_{bd}) \]

\[ = h^{bd} \left[ D_c \left( \frac{1}{2} \delta^{ce} (h_{edcb} + h_{edbc} - h_{cdeb}) \right) - D_d \left( \frac{1}{2} \delta^{ce} (h_{edcb} + h_{edbc} - h_{cdeb}) \right) \right] \]

\[ = h^{bd} \left[ -R_{edcb} - (D_c D_d f)_{bc} - R_{edcb} - (D_c D_d f)_{cd} + R_{bde} + (D_b D_d f)_{ec} \right] \]

\[ - \left[ -R_{edcb} - (D_c D_d f)_{bd} - R_{edbc} - (D_c D_d f)_{cd} + R_{bdec} + (D_b D_d f)_{ed} \right] \]

\[ = \text{term1} + \text{term2} + \text{term3} + \text{term4} \tag{D.8} \]

now,

\[ \text{term1} = h^{bd} h^{ce} \left[ R_{edcb} - R_{edbc} \right] \]

\[ = h^{bd} \left[ R^e_{bdc} - R^e_{bde} \right] \]

\[ = h^{bd} \left[ -R^e_{mcd} R^m_b + R^m_{bcd} R^e_m \right] \]

\[ = h^{bd} \left[ -R_{md} R^m_b + R^m_{bcd} R^e_m \right] = -R_{md} R^md + R_{mc} R^mc = 0 \tag{D.9} \]

and

\[ \text{term2} = -2h^{bd} h^{ce} R_{edbc} \]

\[ = -h^{ce} D_c 2D_b R^b_e \]

\[ = -h^{ce} D_c D_e R = -D^2 R \tag{D.10} \]
where we have used the Bianchi Identity:

$$2D_a R^a = D_c R$$  \hspace{1cm} (D.11)

also,

$$term 3 = 2h^{bd} h^{ce} R_{bdec} = 2D^2 R$$  \hspace{1cm} (D.12)

and

$$term 4 = -h^{bd} h^{ce} \left[ D_c D_b D_d D_f + D_c D_d D_e D_b f - D_c D_e D_b D_d f 
- D_d D_b D_c D_f - D_d D_c D_e D_b f + D_d D_e D_b D_c f \right]$$

$$= term(a) + term(b) + term(c)$$  \hspace{1cm} (D.13)

where,

$$term(a) = -h^{bd} h^{ce} \left[ (D_e D_b f)_{l(d-c-d0)} \right]$$

$$= -h^{ce} \left[ (D_e \xi^d)_{l(d-c-d0)} \right]$$

$$= -h^{ce} \left[ - R^d \epsilon_{d} D_{e} \xi^l + R^l \epsilon_{d} D_{l} \xi^d \right]$$

$$= -h^{ce} \left[ - R_{i} \epsilon_{i} D_{e} \xi^l - R^l \epsilon_{d} D_{l} \xi^d \right]$$

$$= h^{ce} R_{i} \epsilon_{i} D_{e} \xi^l - h^{m} R^m \epsilon_{l} D_{d} \xi^d$$

$$= term(b)$$

$$= 0$$  \hspace{1cm} (D.14)

where,

$$\xi_m := D_m f$$  \hspace{1cm} (D.15)

we notice that:

$$term(b) = -h^{bd} h^{ce} \left[ (D_c D_b D_d D_f - D_c D_b D_d) f \right]$$

$$= -h^{bd} h^{ce} \left[ (D_d D_e D_b D_c - D_d D_b D_e D_c) f \right]$$

$$= term(c)$$  \hspace{1cm} (D.16)
so,

$$
term(b) + term(c) = -2h^{cd} \varepsilon^e \left( (D_e D_d D_e D_d - D_e D_d D_e D_d) f \right)$$

$$= -2h^{cd} \varepsilon^e \left( \varepsilon_{de(b)c} \right)$$

$$= -2h^{cd} \left( \varepsilon_{de(b)c} \right)$$

$$= -2h^{cd} \left( -R^{\alpha}_{\varepsilon de\beta} \right)$$

$$= -2h^{cd} \left( R^{\alpha}_{\varepsilon de\beta} R^{\beta}_{\varepsilon de\alpha} \right)$$

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$$= -2h^{cd} \left( R^{\alpha}_{\varepsilon de\beta} R^{\beta}_{\varepsilon de\alpha} \right)$$

therefore,

$$\dot{R} = h^{bd} R_{bd} + h^{bd} \dot{R}_{bd}$$

$$= 2|R_{bd}|^2 + D^2 R - \xi R$$

(D.17)

where,

$$\mathbb{L}_\xi R = \xi^m D_m R$$

(D.19)

**Method: 2**

This method is a shorter one where we will use the general evolution equation for the three scalar curvature $R$:

we have here:

$$H_{ab} = h_{ab} = -2(R_{ab} + D_a D_b f)$$

(D.20)

and so,

$$trH = -2(R + D^2 f)$$

(D.21)
and we also have the general equation for the evolution of $R$:

$$\dot{R} = -R_{ab} H^{ab} + D_a D_b H^{ab} - D^2 \text{tr} H$$  \hspace{1cm} (D.22)

We calculate term by term the above expression of $\dot{R}$:

$$-R_{ab} H^{ab} = 2|R_{ab}|^2 + 2R_{ab} D^a D^b f$$  \hspace{1cm} (D.23)

Next we have:

$$D_a D_b H^{ab} = -2D_a D_b R^{ab} - 2D_a D_b D^a D^b f$$  \hspace{1cm} (D.24)

and,

$$-D^2 \text{tr} H = 2D^2 R + 2D^2 D^2 f$$  \hspace{1cm} (D.25)

We now note the following identities:

$$-2D_a D_b D^a D^b f + 2D^2 D^2 f = 2D_a \left[ (D^a D_b D^b - D_b D^a D^b) f \right]$$

$$= 2D_a \left[ R_{dab} D^d f \right]$$

$$= 2D_a \left[ R_{ad} D^d f \right]$$

$$= -(D_a R)(D^a f) - 2R^{ad} D_a D_d f$$  \hspace{1cm} (D.26)

where to get the last step we had used the Bianchi identity. Collecting all the terms we have:

$$\dot{R} = 2|R_{ab}|^2 + D^2 R - \mathcal{L}_\zeta R$$  \hspace{1cm} (D.27)

where,

$$\mathcal{L}_\zeta R = \zeta^a D_a R = (D^a f)(D_a R)$$  \hspace{1cm} (D.28)