CHAPTER-7

IF SEMI $g^*$-CLOSED SETS AND
IF SEMI $T_a$-SPACE

7.1 INTRODUCTION

In 1995, Dogan Coker [13, 16] introduced the IF topological space, IF open sets, IF continuity. IF semi open sets are defined by Hasmet Gurcay, Dogan Coker and A. Hayder Es in [26] as a generalization of IF open sets. IF generalized closed sets were studied by S. S. Thakur and Rekha Chaturvedi in [57]. IF semi generalized closed (IF sg-closed) sets were studied by S. S. Thakur and Jyoti Bajpai in [52, 53] as a generalization of the concept of closed sets with the help of semi-openness. The concept of semi $g^*$-closed sets in fuzzy topological space was introduced by R. N. Bhumik and A. Hussain [11].

In this chapter the concepts of IF semi $g^*$-closed sets, IF semi $T_a$ separation axioms are introduced with the help of semi $g^*$-closed sets in IF topological spaces and their properties are studied.
7.2 IF SEMI G*-CLOSED SETS

In this section IF semi g*-closed (IF sg*-closed) sets are defined with the help of IF semi generalized closed sets. The relationship between IF semi g*-closed sets with IF closed sets, IF semi closed sets, IF generalized semi closed sets, IF semi generalized closed sets are established in this section. Different properties of IF semi g*-closed sets are also studied.

Definition 7.2.1  An IF set A in an IF topological space \( (X, \tau) \) is said to be IF semi g*-closed if \( \text{scl}(A) \subseteq O \) whenever \( A \subseteq O \) and \( O \) is an IF sg-open set.

Every IF semi closed set is IF semi g*-closed but the converse is not true as shown in the following example.

Example 7.2.2  Let \( X = \{a, b\} \) be a non empty set, \( A = \langle x, (a/0.5, b/0.3), (a/0.5, b/0.7) \rangle \) and \( B = \langle x, (a/0.6, b/0.4), (a/0.4, b/0.6) \rangle \) be two IF sets in \( X \). Then the family \( \tau = \{0, 1, A, B\} \) is an IF topology on \( X \). Then the IF set \( C \) defined by \( C = \langle x, (a/0.2, b/0.4), (a/0.8, b/0.6) \rangle \) is IF semi g*-closed but is not IF semi closed as \( C \subseteq A^c \) but \( \text{int}(A^c) = A \nsubseteq C \).

Theorem 7.2.3  Every IF semi g*-closed set is IF semi g-closed set.

Proof  Let \( A \) be an IF semi g*-closed set and \( O \) be an IF open set in an IF topological space \( (X, \tau) \) such that \( A \subseteq O \). Since \( O \) is IF open, \( O \) is IF g-open. \( A \) being g*-closed set, \( \text{Cl}(A) \subseteq O \), whenever \( A \subseteq O \) and \( O \) is an IF
g-open. Hence, $\text{Cl}(A) \subseteq O$, whenever $A \subseteq O$ and $O$ is an IF open set. Thus $A$ is an IF semi $g$-closed set.

But the converse is not true as shown in the next example.

**Example 7.2.4** Let $X = \{a, b\}$ be a non empty set and $A = < x, (a/0.4, b/0.5), (a/0.6, b/0.5) >$. Then the family $\tau = \{0-, 1-, A\}$ is an IF topological on $X$. Then the IF set $B = < x, (a/0.7, b/0.5), (a/0.3, b/0.5) >$ is IF semi $g$-closed. But $B$ is not IF semi $g^*$-closed.

**Remark 7.2.5** Every IF sg-closed set is IF gs-closed set. Hence relationship between IF semi $g^*$-closed set and other types of IF closed sets are shown as follows:

\[
\begin{array}{c}
\text{IF closed set} \\
\text{IF g-closed set} \\
\text{IF semi closed set} \\
\text{IF gs-closed set} \\
\text{IF sg-closed set} \\
\text{IF sg}*\text{-closed set}
\end{array}
\]

However the converses are not true in general.

**Remark 7.2.6** Union of two IF semi $g^*$-closed sets may not be IF semi $g^*$-closed which is shown in the following example.

**Example 7.2.7** In example 7.2.2 let the IF set $D = < x, (a/0.4, b/0.2), (a/0.5, b/0.8) >$. Then $D$ is IF semi $g^*$-closed. Now $C \cup D = < x, (a/0.4, b/0.4), (a/0.5, b/0.6) >$ is not IF semi $g^*$-closed. Since $B$ is an IF sg-open set and $C \cup D \subseteq B$ but $\text{sc}(C \cup D) \not\subseteq B$. 
Theorem 7.2.8  Union of two IF semi g* -closed set is IF semi g* -closed if and only if union of two IF semi closed sets is IF semi g* -closed.

Proof  Let union of two IF semi g* -closed set is IF semi g* -closed, A and B be two IF semi closed sets. Then A, B are IF semi g* -closed sets and so $A \cup B$ is IF semi g* -closed set.

Conversely, let union of two IF semi closed sets be IF semi g* -closed set. Let A and B be two IF semi g* -closed sets such that $\text{scl}(A) = C$ and $\text{scl}(B) = D$, where C and D are IF semi closed sets. Then $\text{scl}(A) = C \subseteq O$ and $\text{scl}(B) = D \subseteq O$, whenever and O is an IF sg-open set such that $A \subseteq O$ and $B \subseteq O$. Now $C \cup D \subseteq O$. Since $C \cup D$ is IF semi g* -closed, therefore $\text{scl}(C \cup D) \subseteq O$ and this is true for all IF sg-open sets O containing $C \cup D$. Hence the theorem is proved.

Theorem 7.2.9  Let A be IF semi g* -closed and $A \subseteq B \subseteq \text{scl}(A)$. Then B is IF semi g* -closed.

Proof  Let $B \subseteq O$ and O be IF sg-open. Since $A \subseteq B$ so $A \subseteq O$, now since A is IF semi g* -closed $\text{scl}(A) \subseteq O$. By hypothesis $B \subseteq \text{scl}(A)$, so $\text{scl}(B) \subseteq \text{scl}(A)$. Hence $\text{scl}(B) \subseteq O$. Thus B is IF semi g* -closed.

Definition 7.2.10  An IF set A of an IF topological space $(X, \tau)$ is said to be an IF semi g* -open set if and only if $A^c$ is IF semi g* -closed.

Theorem 7.2.11  An IF set A is IF semi g* -open if and only if $F \subseteq \text{sint}(A)$, whenever F is IF semi g-closed and $F \subseteq A$. 
Proof Suppose that $A$ is an IF semi $g^*$-open set and $F$ is an IF semi $g$-closed set such that $F \subseteq A$. Then $A^c$ is IF semi $g^*$-closed and contained in the IF semi $g$-open set $F^c$. Since $A^c$ is IF semi $g^*$-closed therefore $\text{scl}(A^c) \subseteq F^c$. Now $\text{scl}(A^c) = (\text{sint}(A))^c$, where $\text{sint}(A)$ is IF semi-open and $(\text{sint}(A))^c$ is IF semi-closed. Hence $(\text{sint}(A))^c \subseteq F^c$, i.e. $F \subseteq \text{sint}(A)$.

Conversely, if $F$ is IF semi $g$-closed and $F \subseteq \text{sint}(A)$, whenever $F \subseteq A$. It follows that $A^c \subseteq F^c$ and $(\text{sint}(A))^c = \text{scl}(A^c) \subseteq F^c$, where $F^c$ is IF semi $g$-open. Hence $A^c$ is IF semi $g^*$-closed and $A$ is IF semi $g^*$-open.

Definition 7.2.12 Two IF sets $A$ and $B$ in an IF topological space $(X, \tau)$ are called semi separated if $\text{scl}(A) \cap B = 0_\sim = A \cap \text{scl}(B)$.

Theorem 7.2.13 Union of two semi separated IF semi $g^*$-open sets is IF semi $g^*$-open.

Proof Suppose that $A$, $B$ are semi separated IF semi $g^*$-open sets. Then we have $\text{scl}(A) \cap B = 0_\sim = A \cap \text{scl}(B)$. Let $F \subseteq A \cup B$ and $F$ be an IF semi closed set. Now $F \cap \text{scl}(A) = (A \cup B) \cap \text{scl}(A) = (A \cap \text{scl}(A)) \cup (B \cap \text{scl}(A)) = A \cup 0_\sim = A$. Similarly $F \cap (B) = B$. Now by theorem 7.2.12 $F \cap \text{scl}(A) = \text{sint}(A)$ and $F \cap \text{scl}(B) = \text{sint}(B)$. Hence $F = F \cap (A \cup B) = (F \cap A) \cup (F \cap B) \subseteq (F \cap \text{scl}(A)) \cup (F \cap \text{scl}(B)) \subseteq \text{sint}(A) \cup \text{sint}(B) \subseteq \text{sint}(A \cup B)$ and $A \cup B$ is IF semi $g^*$-open.

Theorem 7.2.14 An IF set $A$ is an IF semi $g^*$-closed set if and only if $\text{scl}(A) \cap A^c$ does not contain any non-null IF $sg$-closed set.
Proof Let $A$ be IF semi $g^*$-closed and $F$ be IF semi $g$-closed set such that $F \subseteq (\text{scl}(A) \cap A^c)$. Then $F^c$ is IF semi $g$-open and $A \subseteq F^c$. Hence by definition 7.2.1, $\text{scl}(A) \subseteq F^c$ or $F \subseteq (\text{scl}(A))^c$. So $F \subseteq (((\text{scl}(A))^c \cap (\text{scl}(A) \cap A^c)) = 0$. 

Conversely, let the given condition be satisfied. Let $A \subseteq O$, where $O$ be IF sg-open set. If $\text{scl}(A)$ is not an IF sub set of $O$, then $\text{scl}(A) \cap O = 0$. But $\text{scl}(A) \cap O^c$ is a non-null IF sg-closed set contained in $\text{scl}(A) \cap A^c$, a contradiction. So $A$ is IF semi $g^*$-closed.

Lemma 7.2.15 For any IF set $A$ in an IF topological space $(X, \tau)$, \[\text{sint}(\text{scl}(A) \cap A^c) = 0.\]

Theorem 7.2.16 An IF set $A$ is IF semi $g^*$-closed if and only if $\text{scl}(A) \cap A^c$ is IF semi $g^*$-open.

Proof Let $A$ be IF semi $g^*$-closed and $F$ be IF semi $g$-closed such that $F \subseteq (\text{scl}(A) \cap A^c)$, then by theorem 7.2.15 $F = 0$. Hence $F \subseteq \text{sint}(\text{scl}(A) \cap A^c)$ and by theorem 7.2.12 $\text{scl}(A) \cap A^c$ is IF semi $g^*$-open.

Conversely, let $\text{scl}(A) \cap A^c$ be IF semi $g^*$-open and $A \subseteq O$, where $O$ be an IF sg-open set. Then $O^c \subseteq A^c$ and $\text{scl}(A) \cap O^c \subseteq \text{scl}(A) \cap A^c$. So $\text{scl}(A) \cap O^c$ is an IF sg-closed sub set of $\text{scl}(A) \cap A^c$, since $\text{scl}(A) \cap A^c$ is IF semi $g^*$-open, hence by theorem 7.2.12 and lemma 7.2.16 $\text{scl}(A) \cap O^c \subseteq \text{sint}(\text{scl}(A) \cap A^c) = 0$. Hence $\text{scl}(A) \subseteq O$. Thus $A$ is IF semi $g^*$-closed.
7.3 IF SEMI Tₐ- SPACE

In this section semi Tₐ-spaces in IF topological space are defined with the help of semi g*-closed sets. Different properties of IF semi Tₐ-spaces are also studied.

**Definition 7.3.1** An IF topological space \((X, \tau)\) is said to be an IF semi Tₐ-space if and only if every IF semi g*-closed set is IF semi closed.

**Theorem 7.3.2** Every IF point \(c(\alpha, \beta)\) in an IF topological space \((X, \tau)\) is either IF semi g-closed or its complement \(\{c(\alpha, \beta)\}^c\) is IF semi g*-closed.

**Proof** Let \(c(\alpha, \beta)\) be not IF semi g-closed in \((X, \tau)\). Then \(\{c(\alpha, \beta)\}^c\) is not IF semi g-open and \(l^-\) is the only IF semi g-open set containing \(\{c(\alpha, \beta)\}^c\). Therefore \(\text{scl}(\{c(\alpha, \beta)\}^c) \subseteq l^-\) holds and so \(\{c(\alpha, \beta)\}^c\) is IF semi g*-closed.

**Definition 7.3.3** An IF semi g*-closure operator of an IF set \(A\) in an IF topological space \((X, \tau)\) is defined as \(\text{scl}^*(A) = \cap \{ F: A \subseteq F, F \text{ is IF semi g*-closed in } X \}\). If \(A\) is IF semi g*-closed set then \(\text{scl}^*(A) = A\).

**Theorem 7.3.4** In an IF topological space \((X, \tau)\), if \(c(\alpha, \beta) \neq d(\gamma, \delta)\) then \(\text{scl}^*\{c(\alpha, \beta)\} \neq \text{scl}^*\{d(\gamma, \delta)\}\).

**Proof** By theorem 7.3.2 in the IF topological space \((X, \tau)\) IF point \(\{c(\alpha, \beta)\}\) is either IF semi g-closed or its complement \(\{c(\alpha, \beta)\}^c\) is IF semi
g*-closed. Hence the proof will be complete if we consider the following two cases.

(i) If the IF point \( \{c(\alpha, \beta)\} \) is IF semi g-closed then \( \{c(\alpha, \beta)\} \) is also IF semi g*-closed, since an IF semi g-closed set is an IF semi g*-closed set. Hence \( \text{scl}^*\{c(\alpha, \beta)\} = \{c(\alpha, \beta)\} \). By the given condition \( d(\gamma, \delta) \not\in \{c(\alpha, \beta)\} \), so \( \text{scl}^*\{c(\alpha, \beta)\} \neq \text{scl}^*\{d(\gamma, \delta)\} \).

(ii) Now let \( \{c(\alpha, \beta)\}^c \) be IF semi g*-closed set and \( d(\gamma, \delta) \subseteq \{c(\alpha, \beta)\}^c \). Then \( d(\gamma, \delta) \in \text{scl}^*\{d(\gamma, \delta)\} \subseteq \{c(\alpha, \beta)\}^c \) and hence \( \text{scl}^*\{c(\alpha, \beta)\} \neq \text{scl}^*\{d(\gamma, \delta)\} \).

**Notation** By \( \text{IFSOS}(\tau) \) we mean the collection of all IF semi open sets in the space \((X, \tau)\) and similarly, we define \( \text{IFSOS}^*(\tau) = \{A : \text{scl}(A^c) = A^c \} \).

**Theorem 7.3.5** In an IF topological space \((X, \tau)\), \( \text{IFSOS}(\tau) \subseteq \text{IFSOS}^*(\tau) \).

**Proof** Let \( A \in \text{IFSOS}(\tau) \). Then \( A^c \) is IF semi closed and \( A^c = \text{scl}(A^c) \). Since \( A^c \) is IF semi closed, \( A^c \) is also IF semi g*-closed, so \( \text{scl}^*(A^c) = A^c \). Hence \( A \in \text{IFSOS}^*(\tau) \).

**Theorem 7.3.6** An IF topological space \((X, \tau)\) is IF semi T\(_a\)-space if and only if \( \text{IFSOS}(\tau) = \text{IFSOS}^*(\tau) \).

**Proof** Let \((X, \tau)\) be an IF semi T\(_a\)-space. Then \( \text{scl}(A) = \text{scl}^*(A) \) holds for every IF subset \( A \), since IF semi closed sets and IF semi g*-closed sets coincide in IF semi T\(_a\)-space. Therefore we have \( \text{IFSOS}(\tau) = \text{IFSOS}^*(\tau) \).
Conversely, let $A$ be IF semi $g^*$-closed set of $(X, \tau)$. Then $A = \text{scl}^*(A)$ and hence $A^c \in \text{IFSOS}(\tau)$. Thus $A$ is IF semi closed. Therefore $(X, \tau)$ is an IF semi $T_a$-space.

**Theorem 7.3.7** An IF topological space $(X, \tau)$ is IF semi $T_a$-space if and only if for each IF point $c(\alpha, \beta) \in X$, $\{c(\alpha, \beta)\}$ is IF semi open or IF semi $g$-closed.

**Proof** Let the IF point $c(\alpha, \beta) \in X$ and $\{c(\alpha, \beta)\}$ is not IF semi $g$-closed. Then $\{c(\alpha, \beta)\}^c$ is not IF semi $g$-open. This implies that $1_\sim$ is the only IF semi $g$-open set containing $\{c(\alpha, \beta)\}^c$. $\text{scl}\{c(\alpha, \beta)\}^c \subseteq 1_\sim$. So $\{c(\alpha, \beta)\}^c$ is IF semi $g^*$-closed set of $(X, \tau)$. Since $(X, \tau)$ is IF semi $T_a$-space, $\{c(\alpha, \beta)\}^c$ is IF semi closed. Therefore $\{c(\alpha, \beta)\}$ is IF semi open.

For the converse part it is enough to prove $\text{IFSOS}^*(\tau) \subset \text{IFSOS}(\tau)$. Let $A^c \in \text{IFSOS}^*(\tau)$ and $A \notin \text{IFSOS}(\tau)$. Then $\text{scl}^*(A^c) = A^c$ and $\text{scl}^*(A^c) \neq A^c$. Then there exist an IF point $c(\alpha, \beta)$ of $X$ such that $x(\alpha, \beta) \in \text{scl}^*(A^c)$ and $c(\alpha, \beta) \notin \text{scl}^*(A^c) = A^c$. Since $c(\alpha, \beta) \notin \text{scl}^*(A^c)$ there exists an IF semi $g^*$-closed set $F$ such that $c(\alpha, \beta) \notin F$ and $A^c \subset F$. By hypothesis, $\{c(\alpha, \beta)\}$ is IF semi open or IF semi $g$-closed.

Case I. Let $\{c(\alpha, \beta)\}$ is an IF semi open set in $X$. Since $\{c(\alpha, \beta)\}^c$ is IF semi closed and $A^c \subset \{c(\alpha, \beta)\}^c$, we have $\text{scl}^*(A^c) \subset \{c(\alpha, \beta)\}^c$, i.e. $c(\alpha, \beta) \notin \text{scl}^*(A^c)$. This contradicts the fact that $c(\alpha, \beta) \in \text{scl}^*(A^c)$. Therefore $A \notin \text{IFSOS}(\tau)$. 
Case II. Let \( \{c(\alpha, \beta)\} \) be an IF semi g-closed set in \( X \). Since \( \{c(\alpha, \beta)\}^c \) is IF semi g-open set containing the IF semi g*\(^{-}\)-closed set \( F \) and \( F \supset A^c \). Therefore we have \( \{c(\alpha, \beta)\}^c \supset \text{scl}(F) \supset \text{scl}(A^c) \). So the IF set \( \{c(\alpha, \beta)\} \notin \text{scl}(A^c) \). This is a contradiction. So \( A \in \text{IFSOS}(\tau) \). Hence in both cases \( A \in \text{IFSOS}(\tau) \). Thus \( \text{IFSOS}^*(\tau) \subset \text{IFSOS}(\tau) \).

**Definition 7.3.8** An IF topological space \((X, \tau)\) is said to be IF semi T\(_0\)-space if and only if for any pair of distinct IF points \( c(\alpha, \beta) \) and \( d(\gamma, \delta) \) in \( X \), either \( c(\alpha, \beta) \notin \text{scl}\{d(\gamma, \delta)\} \) or \( d(\gamma, \delta) \notin \text{scl}\{c(\alpha, \beta)\} \), that is \( \text{scl}\{c(\alpha, \beta)\} \neq \text{scl}\{d(\gamma, \delta)\} \).

**Theorem 7.3.9** Every IF semi T\(_a\)-space is IF semi T\(_0\)-space.

**Proof** Let \((X, \tau)\) be an IF semi T\(_a\)-space but not IF semi T\(_0\)-space. Then there exists two distinct IF points \( c(\alpha, \beta) \) and \( d(\gamma, \delta) \) in \( X \), such that \( \text{scl}\{c(\alpha, \beta)\} = \text{scl}\{d(\gamma, \delta)\} \). Let \( A = \{c(\alpha, \beta)\}^c \). Clearly \( \{c(\alpha, \beta)\} \) is not IF semi closed, otherwise \( \text{scl}\{c(\alpha, \beta)\} = \{c(\alpha, \beta)\} \neq \text{scl}\{d(\gamma, \delta)\} \). By theorem 7.3.2, \( A \) is IF semi g\(^*\)-closed but not IF semi closed, otherwise \( d(\gamma, \delta) \in \{c(\alpha, \beta)\}^c = A \), which implies \( \text{scl}\{d(\gamma, \delta)\} \subseteq \{c(\alpha, \beta)\}^c \) and \( \text{scl}\{c(\alpha, \beta)\} \neq \text{scl}\{d(\gamma, \delta)\} \), contradicting our hypothesis. Hence \((X, \tau)\) is IF semi T\(_0\)-space.