CHAPTER-6

IF  $g^*$-CLOSED SETS AND IF $T_a$-SPACE

6.1 INTRODUCTION

IF generalized closed sets were studied by S. S. Thakur and Rekha Chaturvedi in [55]. IF semi generalized closed (IF sg-closed) sets were studied by S. S. Thakur and Jyoti Bajpai in [52, 53] as a generalization of the concept closed sets with the help of semi openness. IF regular closed (IFrgC) sets were introduced by S. S. Thakur and Rekha Chaturvedi in [57]. IF w-closed set was introduced by S. S. Thakur and Jyoti Prakash Bajpai in [54]. The concept of $g^*$-closed sets in fuzzy topological space was introduced by R. N. Bhumika and A. Hussin [10]. This concept in fuzzy topological space is also studied by S. S. Benchalli and G. P. Siddapur [9]. In topology the concept $g^*$-closed set is studied by N. Levine [38]. $g^*$-closed sets are also known as Strongly g-closed sets by A. Pushpalata and K. Anitha in [44].

In this chapter the concepts of IF $g^*$-closed sets and IF $T_a$ separation axioms are introduced with the help of $g^*$-closed sets in IF topological spaces. Different properties of these concepts are also studied in this chapter.
6.2 IF g*-CLOSED SETS

In this section IF g*-closed sets are defined with the help of generalized closed sets. The relationship between IF g*-closed sets with IF closed sets, IF rg-closed sets, IF sg-closed sets, IF gs-closed sets, IF w-closed sets are established and different properties of IF g*-closed sets are also studied in this section.

Definition 6.2.1 An IF set A in the IF topological space \((X, \tau)\) is said to be IF g*-closed set if \(\text{cl}(A) \subseteq O\) whenever \(A \subseteq O\) and \(O\) is an IF g-open set.

An IF set A is said to be IF g*-open if and only if its complement \(A^c\) is IF g*-closed.

Theorem 6.2.2 In an IF topological space \((X, \tau)\) every IF g*-closed set is IF g- closed set.

Proof Let A be an IF g*-closed set and O be an IF open set in \((X, \tau)\) such that \(A \subseteq O\). Since O is IF open, O is IF g-open. A being IF g*-closed set, \(\text{Cl}(A) \subseteq O\), whenever \(A \subseteq O\) and O is an IF g-open. Hence, \(\text{Cl}(A) \subseteq O\), whenever \(A \subseteq O\) and O is an IF open set. Thus A is an IF g-closed set.

But the converse is not true as shown in the next example.

Example 6.2.3 Let \(X = \{a, b\}\) be a non empty set and \(U = \langle x, (a/0.6, b/0.5), (a/0.4, b/0.5) \rangle\) be an IF set in \(X\). Then the family \(\tau = \{0_\sim, 1_\sim, U\}\) is an IF topology on \(X\). Then the IF set \(V = \langle x, (a/0.1, b/0.3), (a/0.8, b/0.6) \rangle\) is IF g-closed but not IF g*-closed.
Theorem 6.2.4  In an IF topological space \((X, \tau)\) every IF \(g^*\)-closed set is IF \(rg\)-closed set.

Proof  Let \(A\) be an IF \(g^*\)-closed set in \((X, \tau)\) such that \(A \subseteq O\) and \(O\) is an IF regular open. Since \(O\) is IF regular open, \(O\) is IF \(g\)-open. \(A\) being \(g^*\)-closed, \(\text{Cl}(A) \subseteq O\). Hence, \(\text{Cl}(A) \subseteq O\), whenever \(A \subseteq O\) and \(O\) is an IF regular open. Thus \(A\) is an IF \(rg\)-closed set.

But the converse is not true as shown in the next example.

Example 6.2.5  Let \(X = \{a, b\}\) be a non empty set and \(U = < x, (a/0.5, b/0.6), (a/0.4, b/0.4) >\) be an IF set in \(X\). Then the family \(\tau = \{0\sim, 1\sim, U\}\) is an IF topology on \(X\). Then the IF set \(A = < x, (a/0.5, b/0.4), (a/0.4, b/0.5) >\) is IF \(rg\)-closed but not IF \(g^*\)-closed.

Every IF \(g\)-closed set is IF \(rg\)-closed set [55], every IF closed set is IF \(W\)-closed set, every IF \(W\)-closed set is IF \(g\)-closed set as well as \(sg\)-closed set [54]. IF \(g^*\)-closed sets and IF \(W\)-closed sets are independent concepts. Hence the relation of IF \(g^*\)-closed sets with other types of closed sets are as follows:

\[
\begin{array}{c}
\text{IF regular closed set} \\
\downarrow \\
\text{IF closed set} \\
\downarrow \\
\text{IF } g^* \text{- closed set} \\
\downarrow \\
\text{IF } rg \text{- closed set} \\
\end{array} \quad \begin{array}{c}
\text{IF } W \text{- closed set} \\
\downarrow \\
\text{IF } g \text{- closed set} \\
\downarrow \\
\text{IF } sg \text{- closed set} \\
\end{array}
\]

However the converses are not true in general.
Theorem 6.2.6 An IF set A in an IF topological space \((X,\tau)\) is IF \(g^*\)-closed if and only if \(\text{cl}(A) \cap A^c\) does not contain any non null IF \(g\)-closed set.

Proof Let \(A\) be an IF \(g^*\)-closed and \(B\) be an IF \(g\)-closed set of \((X,\tau)\) such that \(B \subseteq (\text{cl}(A) \cap A^c)\). Then \(B^c\) is IF \(g\)-open and \(A \subseteq B^c\). So \(\text{cl}(A) \subseteq B^c\).

This implies \(B \subseteq (\text{cl}(A))^c\). So \(B \subseteq [(\text{cl}(A))^c \cap (\text{cl}(A) \cap A^c)] = 0\). Hence the condition is necessary.

Conversely, let \(A\) be an IF set in \((X,\tau)\) and \(\text{cl}(A) \cap A^c\) does not contain any non null IF \(g\)-closed set. Let \(A \subseteq O\) and \(O\) be an IF \(g\)-open. If \(\text{cl}(A)\) is not an IF subset of \(O\) then \(\text{cl}(A) \cap O \neq 0\). But \(\text{cl}(A) \cap O^c\) is a non null IF \(g\)-closed set contained in \(\text{cl}(A) \cap A^c\) a contradiction. So \(A\) is IF \(g^*\)-closed.

Theorem 6.2.7 Let \(A\) and \(B\) be IF \(g^*\)-closed sets in an IF topological space \((X,\tau)\). Then \(A \cup B\) is an IF \(g^*\)-closed set.

Proof Let \(O\) be an IF \(g\)-open set in \(X\), such that \(A \cup B \subseteq O\). Then \(A \subseteq O\) and \(B \subseteq O\). So \(\text{cl}(A) \subseteq O\) and \(\text{cl}(B) \subseteq O\). Therefore \(\text{cl}(A \cup B) \subseteq \text{cl}(A) \cup \text{cl}(B) \subseteq O\). Hence \(A \cup B\) is an IF \(g^*\)-closed set.

Remark 6.2.8 The intersection of two IF \(g^*\)-closed sets in an IF topological space \((X,\tau)\) may not be IF \(g^*\)-closed set.

Example 6.2.9 Let \(X = \{a, b\}\) be a non empty set and \(U = \langle x, (a/0.4, b/0.3), (a/0.5, b/0.4) \rangle\). Then the family \(\tau = \{0\_\sim, 1\_\sim, U\}\) is an IF
topology on X. Then IF sets \( A = < x, (a/0.5, b/0.3), (a/0.5, b/0.4) > \) and \( B = < x, (a/0.3, b/0.6), (a/0.6, b/0.4) > \) are IF g*-closed sets but \( A \cap B \) is not IF g*-closed.

**Theorem 6.2.10** Let \( A \subseteq B \subseteq \cl(A) \) and \( A \) be an IF g*-closed set in an IF topological space \((X, \tau)\). Then \( B \) is IF g*-closed.

**Proof** Let \( O \) be IF g-open set in \( X \), such that \( B \subseteq O \). Then \( A \subseteq O \) and since \( A \) is IF g*-closed, \( \cl(A) \subseteq O \). Now \( B \subseteq \cl(A) \) implies \( \cl(B) \subseteq \cl(A) \subseteq O \). Consequently, \( B \) is IF g*-closed.

**Theorem 6.2.11** An IF g*-closed and IF open set is IF closed.

**Proof** Let \( A \) be an IF g*-closed and IF open set. Then \( \cl(A) \subseteq O \), whenever \( A \subseteq O \) and \( O \) is IF g-open. Since IF open sets are IF g-open and \( A \subseteq O \), \( A \) may be considered as \( O \) such that \( \cl(A) \subseteq A \) that is \( \cl(A) = A \). Hence \( A \) is IF closed.

**Theorem 6.2.12** Let \( A \) be an IF g*-closed set and \( x(\alpha, \beta) \) be an IF point in \((X, \tau)\) such that \( x(\alpha, \beta) \notin \cl(A) \). Then \( \cl(x(\alpha, \beta)) \notin A \).

**Proof** Let \( A \) be an IF g*-closed set and let \( x(\alpha, \beta) \notin \cl(A) \). If possible let \( \cl(x(\alpha, \beta)) \notin A \), then \( A \subseteq [\cl(x(\alpha, \beta))]^c \) where \( [\cl(x(\alpha, \beta))]^c \) is IF open. Now since \( A \) is an IF g*-closed set, \( \cl(A) \subseteq [\cl(x(\alpha, \beta))]^c \subseteq [x(\alpha, \beta)]^c \). Therefore \( x(\alpha, \beta) \notin \cl(A) \), a contradiction to the hypothesis. Hence \( \cl(x(\alpha, \beta)) \notin A \).
**Theorem 6.2.13** An IF set $A$ in an IF topological space $(X, \tau)$ is IF $g^*$-open if and only if $O \subseteq \text{int}(A)$ whenever $O \subseteq A$ and $O$ is an IF $g$-closed set.

**Proof: Necessity** Let $A$ be an IF $g^*$-open and $O$ be an IF $g$-closed set in $X$ such that $O \subseteq A$. Then $O^c$ is IF $g$-open and $A^c \subseteq O^c$. By hypothesis $A^c$ is IF $g^*$-closed, so $\text{cl}(A^c) \subseteq O^c$. But $\text{cl}(A^c) = (\text{int}A)^c$. Hence $(\text{int}A)^c \subseteq O^c$ or $O \subseteq \text{int}(A)$.

**Sufficiency** Let $O$ be an IF $g$-open set in $X$ such that $A^c \subseteq O$. Then $O^c$ is IF $g$-closed and $O^c \subseteq A$. Therefore by hypothesis $O^c \subseteq \text{int}(A)$. This implies $\text{cl}(A^c) = (\text{int}A)^c \subseteq O$. Hence $A^c$ is IF $g^*$-closed and $A$ is IF $g^*$-open in $X$.

**Theorem 6.2.14** Let $A$ be an IF $g^*$-open sub set in an IF topological space $(X, \tau)$ and $\text{int}(A) \subseteq B \subseteq A$. Then $B$ is IF $g^*$-open.

**Proof** Since $A^c \subseteq B^c \subseteq \text{cl}(A^c)$ and $A^c$ is IF $g^*$-closed it follows that $B^c$ is IF $g^*$-closed. Thus $B$ is IF $g^*$-open.

**Remark 6.2.15** The union of two IF $g^*$-open sets may not be IF $g^*$-open set.

**Example 6.2.16** Let $X = \{a, b\}$ be a non empty set and $U = \langle x, (a/0.4, b/0.4), (a/0.6, b/0.5) \rangle$ be an IF set in $X$. Then the family $\tau = \{0_-, 1_-, U\}$ is an IF topology on $X$. Then IF sets $C = \langle x, (a/0.6, b/0.4), (a/0.3, b/0.5) \rangle$ and $D = \langle x, (a/0.7, b/0.5), (a/0.5, b/0.3) \rangle$ are IF $g^*$-open sets but $C \cup D$ is not IF $g^*$-open.
**Theorem 6.2.17** Let $A$ and $B$ be $q$-separated IF $g^*$-open sets in an IF topological space $(X,\tau)$. Then $A \cup B$ is an IF $g^*$-open.

**Proof** Let $O$ be IF $g^*$-closed in $X$ and $O \subseteq A \cup B$. Then $O \cap \text{cl}(A) \subseteq A$. Since $B \cap \text{cl}(A) = \emptyset$, hence $O \cap \text{cl}(A) \subseteq \text{int}(A)$ and $O \cap \text{cl}(B) \subseteq \text{int}(B)$. Now $O = O \cap (A \cup B) \subseteq (O \cap \text{cl}(A)) \cup (O \cap \text{cl}(B)) \subseteq (\text{int}(A) \cup \text{int}(B)) \subseteq \text{int}(A \cup B)$. Hence $O \subseteq \text{int}(A \cup B)$. Thus $A \cup B$ is IF $g^*$-open.

**Notation 6.2.18** In an IF topological space $(X,\tau)$, let $\text{IFGCS}(X)$ (respectively $\text{IFGOS}(X)$) be denoted by the family of all IF $g$-closed (respectively IF $g$-open) sets of $X$.

**Theorem 6.2.19** In an IF topological space $(X,\tau)$ if $\text{IFGCS}(X) = \text{IFGOS}(X)$ and every IF $g$-closed set is IF closed then every IF subset of $X$ is IF $g^*$-closed.

**Proof** Let $A$ be an IF sub set of $X$ and $A \subseteq O$ and $O$ be IF $g$-open. Since $\text{IFGCS}(X) = \text{IFGOS}(X)$, $O$ is IF $g$-closed also. According to the hypothesis $O$ is IF closed. Hence $\text{cl}(A) \subseteq \text{cl}(O) = O$ and $A$ is an IF $g^*$-closed set.

**Theorem 6.2.20** In an IF topological space $(X,\tau)$ if every IF subset of $X$ is IF $g^*$-closed then $\text{IFGCS}(X) = \text{IFGOS}(X)$.

**Proof** Assume that every subset of $(X,\tau)$ is IF $g^*$-closed and $A$ is an IF $g$-open set in $X$. Since every IF $g^*$-closed set is IF $g$-closed set, hence $A \in \text{IFGCS}(X)$. Thus $\text{IFGOS}(X) \subseteq \text{IFGCS}(X)$. Again we assume that $A$ is
an IF g-closed set in X then $A^c \in \text{IFGOS}(X) \subseteq \text{IFGCS}(X)$. Therefore $A \in \text{IFGOS}(X)$, consequently $\text{IFGCS}(X) \subseteq \text{IFGOS}(X)$. Hence $\text{IFGOS}(X) = \text{IFGCS}(X)$.

**Theorem 6.2.21** Let A and B be IF g*-closed sets in an IF topological space $(X,\tau)$ and suppose $A^c$ and $B^c$ are q-separated. Then $A \cap B$ is IF g*-closed.

**Proof** Since $A^c$ and $B^c$ are q-separated IF g*-open sets, by theorem 6.2.17, $A^c \cup B^c = (A \cap B)^c$ is IF g*-open. Hence $A \cap B$ is IF g*-closed.

**Definition 6.2.22** Two IF sets A and B in an IF topological space $(X,\tau)$ are weakly separated if and only if there exists two IF open sets C, D such that $A \subseteq C$, $B \subseteq D$ and $A \cap D = B \cap C = 0^\sim$.

**Theorem 6.2.23** In an IF topological space $(X,\tau)$, union of two weakly separated IF g*-open sets is IF g*-open.

**Proof** Let A and B be two weakly separated IF g*-open sets. Since A and B are IF weakly separated then there exists two IF g-open sets C, D such that $A \subseteq C$, $B \subseteq D$ and $A \cap D = B \cap C = 0^\sim$. Let $O_1 = C^C$ and $O_2 = D^C$. Then $O_1$ and $O_2$ are IF g-closed sets such that $A \subseteq O_1$ and $B \subseteq O_2$. Again since A and B are IF g*-open then there exists two IF g-closed sets P, Q such that $P \subseteq \text{int}(A)$ and $Q \subseteq \text{int}(B)$ whenever $P \subseteq A$ and $Q \subseteq B$. Then $P \cup Q \subseteq A \cup B$ and $P \cup Q \subseteq \text{int}(A) \cup \text{int}(B) = \text{int}(A \cup B)$. Now assume that $R \subseteq A \cup B$ is an IF g-closed set. Now $R = R \cap (A \cup B) \subseteq (R \cap O_1) \cup (R \cap O_2)$, where $R \cap O_1$ and $R \cap O_2$ are IF g-closed sets. Also
R \cap O_1 \subseteq (A \cup B) \cap O_1 \subseteq B. Hence R \cap O_1 \subseteq Q. Similarly R \cap O_2 \subseteq P. So R \subseteq P \cup Q \subseteq \text{int}(A \cup B). Hence A \cup B is IF g^* -open.

**Theorem 6.2.24** For each x \in X, the IF point \{x(\alpha,\beta)\} is IF g-closed or its complement \{x(\alpha,\beta)\}^c is IF g^* -closed sets in an IF topological space (X,\tau).

**Proof** Let \{x(\alpha,\beta)\} be not IF g-closed in X. Then X is the only IF g-open set containing \{x(\alpha,\beta)\}^c. Also \text{cl}(\{x(\alpha,\beta)\}^c) \subseteq X, hence \{x(\alpha,\beta)\}^c is IF g^* -closed.

### 6.3 IF T^*_a-SPACE

In this section T^*_a-spaces in IF topological space are defined with the help of g^* -closed sets. It is proved that both IF T_{1/2} space and IF regular T_{1/2} space is an IF T^*_a-space and different properties of T^*_a-spaces are also studied in this section.

**Definition 6.3.1** An IF topological space (X,\tau) is called an IF T^*_a-space if and only if every IF g^* -closed set in X is IF closed in X.

**Theorem 6.3.2** Every IF T_{1/2} space is an IF T^*_a-space.

**Proof** Proof of the theorem follows from the fact that in an IF topological space (X,\tau) every IF g^* -closed set is an IF g-closed set and according to the definition of IF T_{1/2} space that a space (X,\tau) is IF T_{1/2} space if and only if every IF g-closed set in X is IF closed in X.
Theorem 6.3.3 Every IF regular $T_{1/2}$ space is an IF $T_a$-space.

Proof Proof of the theorem follows from the fact that in an IF topological space $(X,\tau)$ every IF regular $T_{1/2}$ space is IF $T_{1/2}$ space and by theorem 6.3.2 which states that every IF $T_{1/2}$ space is an IF $T_a$-space.

Definition 6.3.4 An IF $g^*$-closure operator of the IF set $A$ is defined as

$$\text{Cl}^*(A) = \cap \{B : A \subseteq B \text{ and } B \text{ is IF } g^*\text{-closed set in } X\}.$$ 

Theorem 6.3.5 An IF set $A$ in an IF $T_a$-space $(X,\tau)$ is IF $g^*$-open if and only if $A$ is an IF neighbourhood of $x(\alpha,\beta)$ for each IF point $x(\alpha,\beta) \in A$.

Proof: Necessity Suppose that $x(\alpha,\beta)$ is an IF point of the IF $g^*$-open set $A$ in $X$. Since $X$ is an IF $T_a$-space, $A$ is an IF open set in $X$. Then clearly $A$ is an IF neighbourhood of $x(\alpha,\beta)$.

Sufficiency Suppose that $A$ is an IF neighbourhood of each IF point $x(\alpha,\beta)$ of $A$. Since $A$ is an IF neighbourhood of $x(\alpha,\beta)$, there exists an IF open set $B$ in $X$ such that $x(\alpha,\beta) \in B \subseteq A$. Now $A = \cup \{x(\alpha,\beta) : x(\alpha,\beta) \in A\} \subseteq \cup \{B_{x(a,b)} : x(\alpha,\beta) \in A\} \subseteq A$. This implies $A = \cup \{B_{x(a,b)} : x(\alpha,\beta) \in A \}$. Since each $B$ is an IF open set, $A$ is an IF open set.

Theorem 6.3.6 An IF set $A$ in an IF $T_a$-space $(X,\tau)$ is an IF $g^*$-open set in $X$ if and only if for each IF point $x(\alpha,\beta) \in A$, there exists an IF open set $B$ in $X$ such that $x(\alpha,\beta) \in B \subseteq A$.

Proof: Necessity Suppose that $A$ is an IF $g^*$-open set in $X$. Then we can take $B = A$ so that $x(\alpha,\beta) \in B \subseteq A$ for each IF point $x(\alpha,\beta) \in A$. 

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Sufficiency  Suppose that for each IF point $x(\alpha,\beta)$ there exists an IF $g^*$-open set $A$ in $X$ such that $x(\alpha,\beta) \in B \subseteq A$, where $B$ is an IF open set in $X$. Then $A = \bigcup_{x(\alpha,\beta) \in A} \{ x(\alpha,\beta) \} \subseteq \bigcup_{x(\alpha,\beta) \in A} \{ B_x(\alpha,\beta) \} \subseteq A$. Therefore $A = \bigcup_{x(\alpha,\beta) \in A} \{ B_x(\alpha,\beta) \}$ is an IF open set and hence $(X,\tau)$ is an IF $T_a$-space.

**Theorem 6.3.7**  An IF topological space $(X,\tau)$ is IF $T_a$-space if $K = K^*$, where $K = \{ A : \text{cl} A^c \text{ is IF closed} \}$ and $K^* = \{ A : \text{cl}^* A^c \text{ is IF } g^*\text{-closed} \}$.

**Proof**  Suppose that the IF topological space $(X,\tau)$ is IF $T_a$-space. Then $A \in K$ implies $\text{cl} A^c$ is IF closed implies $\text{cl} A^c$ is IF $g^*$-closed. Now, $\text{cl}^* A^c = \text{cl} A^c$ is also IF $g^*$-closed (since $X$ is $T_a$-space). This implies that $A \in K^*$. Similarly we can show that $A \in K^*$ implies $A \in K$. Hence $K = K^*$. 