Chapter 11

Two layers supply chain imperfect production inventory model with fuzzy credit period, time and production rate dependent imperfectness

11.1 Introduction

In today’s highly competitive business world, the supply chain management (SCM) is a vital issue for manufacturers, retailers and customers. It is a methodology to improve the business performance. As a result, Supply Chain Management (SCM) is in the form to enhance the revenue and to reduce operational costs, to improve flow of supplies, to reduction delays of production and increase customer satisfaction. Researchers as well as practitioners in manufacturing industries have given importance to develop inventory control problems in supply chain management. Adoption of supply chain management practices in industries has steadily increased since the 1980s. However, Manna et al. [147], Munson and Rosenblatt [157], Yang and Wee [236], Khouja [117], Yao et al. [234], Chaharsooghi et al. [16], Wang et al. [216] and others provided excellent review on supply chain management literature. These articles define the concept, principals, nature and development of SCM and indicate that there is an intense research being conducted around the world in this field.

In present business culture, usually a supplier offers a permissible delay in payments to a manufacturer, a manufacturer offers a permissible delay in payments to a retailer and a retailer offers a permissible delay in payments to customers, known as trade credit period, in paying for
purchasing cost, which is a very common business practice. Suppliers often offer trade credit as a marketing strategy to increase sales and reduce on hand stock level. Once a trade credit has been offered, the amount of period that the retailer’s capital tied up in stock is reduced, and that leads to a reduction in the retailer’s holding cost of finance. In addition, during trade credit period, the retailer can accumulate revenues by selling items and by earning interests. As a matter of fact, retailers, especially small businesses which tend to have a limited number of financing opportunities, rely on trade credit as a source of short-term funds. In this research field, Goyal [78] was the first who established an EOQ model with a constant demand rate under the condition of permissible delay in payments. Chung and Liao [46] studied a lot-sizing problem under a supplier’s trade credit depending on the retailer’s order quantity. Abad et al. [2] developed a seller-buyer model under permissible delay in payments by game theory to determine the optimal unit price with trade credit period, considering that the demand rate is a function of the retail price. Recently Das et al. [51] developed an integrated model under trade credit policy. Summary of related literature for multi-retailer EOQ/EPQ models with credit period is shown in Table 11.1.

Table 11.1: Summary of related literature for EPQ/EOQ models with Credit period

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>EOQ/EPQ</th>
<th>Defective Rate</th>
<th>Demand rate dependency</th>
<th>Environment</th>
<th>Credit Period</th>
<th>Retailer/Agent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chavan et al. [16]</td>
<td>EPQ and EOQ</td>
<td>-</td>
<td>Stochastic lead-time</td>
<td>Crisp (fixed)</td>
<td>Multiple</td>
<td></td>
</tr>
<tr>
<td>Chung &amp; Liao [46]</td>
<td>EOQ</td>
<td>Constant</td>
<td>Fuzzy random-lead time</td>
<td>Fuzzy-stochastic</td>
<td>-</td>
<td>Single</td>
</tr>
<tr>
<td>Das et al. [51]</td>
<td>EPQ</td>
<td>Constant</td>
<td>Constant</td>
<td>Crisp</td>
<td>Crisp (fixed)</td>
<td>Single</td>
</tr>
<tr>
<td>Datta &amp; Pal [52]</td>
<td>EOQ</td>
<td>Stock level</td>
<td>Crisp</td>
<td>-</td>
<td>Multiple</td>
<td></td>
</tr>
<tr>
<td>Dey et al. [56]</td>
<td>EOQ</td>
<td>Dynamic</td>
<td>Fuzzy</td>
<td>-</td>
<td>Single</td>
<td></td>
</tr>
<tr>
<td>Jabar et al. [103]</td>
<td>EOQ</td>
<td>Random</td>
<td>Constant</td>
<td>Stochastic</td>
<td>-</td>
<td>Single</td>
</tr>
<tr>
<td>Khanna [117]</td>
<td>EPQ</td>
<td>Constant</td>
<td>Crisp</td>
<td>-</td>
<td>Multiple</td>
<td></td>
</tr>
<tr>
<td>Moin et al. [147]</td>
<td>EPQ</td>
<td>Constant</td>
<td>Stock level</td>
<td>Fuzzy</td>
<td>-</td>
<td>Single</td>
</tr>
<tr>
<td>Panda &amp; Malik [163]</td>
<td>EPQ</td>
<td>Price dependent</td>
<td>Fuzzy</td>
<td>-</td>
<td>Single</td>
<td></td>
</tr>
<tr>
<td>Yang &amp; Wei [230]</td>
<td>EPQ</td>
<td>Constant</td>
<td>Stochastic</td>
<td>-</td>
<td>Multiple</td>
<td></td>
</tr>
<tr>
<td>Present model</td>
<td>EPQ</td>
<td>Production rate</td>
<td>Stock and time dependent</td>
<td>Crisp linked</td>
<td>Fuzzy</td>
<td>Multiple</td>
</tr>
</tbody>
</table>

In recent years, the green house effect and global warming have gained much attention due to strong and more frequent extreme weather events. In every developing countries, there is a scope of measuring and maintaining such carbon-emission. Benjaafar et al. [9] first presented a series of model formulations that illustrate how carbon emission considerations can be incorporated into a decision-making problem. Dye and Yang [70] study a deteriorating
11.2. NOTATIONS AND ASSUMPTIONS

Inventory system under various carbon emissions policies.

Uncertainty of the parameters in a decision making is a well established phenomenon in recent years. Estimation of such parameters in the objective functions using traditional econometric methods is not always possible due to the insufficient historical data, especially for newly launched products. Generally, nature of uncertainties can be classified into three major groups such as random (stochastic), fuzzy (imprecise) and rough (approximation). Several research works on fuzzy inventory problem [26, 56] have been done in the existing literature. Panda et al. [163] extended the single period inventory problem in a multi-product manufacturing system under chance and imprecise constraints. Chang [25] developed an EOQ model with fuzzy defective rate and demand. Recently Maroja et al. [147] and Das et al. [51] considered different imprecise parameters in their supply chain models.

This chapter considers a manufacturer-retailer-customer supply chain model involving bi-level trade credit and random carbon emission. In this chapter, a two-echelon supply chain system with several markets, by taking the imperfect production process is considered. Here, we establish a bi-level trade credit model to enhance the demand of the customers, which actually is a Stackelberg model with the customer's satisfaction and whole system being the leader in the management. Here, we introduce an integrated production-inventory model with rework policy. Here, the manufacturer offers trade credit period to retailer and as well as retailer offers trade credit to the customers. Here, both the credit periods are fuzzy in nature and the model is defuzzified using the expression of expectation. The demand of the customers is considered as stock dependent.

11.2 Notations and Assumptions

The following notations and assumptions have been used to develop the proposed model:

11.2.1 Notations

The following notations have been used to develop the model.

- \( q_{m}(t) \) : Inventory level of the manufacturer at any time \( t \) of perfect quality items.
- \( q^r_i(t) \) : Inventory level of the \( i \)th retailer at any time \( t \) of perfect quality items.
- \( n \) : Number of retailers.
- \( P \) : Production rate in units, \( P > D \).
- \( \theta \) : Rate of produced defective item which depend on time and production rate.
- \( \eta \) : Rate of carbon emissions associated per unit produce item, a random variable.
- \( \eta^r \) : Rate of carbon emissions associated per unit rework item, a random variable.
- \( \eta^d \) : Rate of carbon emissions associated per unit disposal item, a random variable.
- \( f(\eta) \) : The probability density function of \( \eta, \eta \in [0, 1] \).
- \( \delta \) : Percentage of rework of defective units per unit time.
CHAPTER 11. TWO LAYERS SUPPLY CHAIN IMPERFECT PRODUCTION INVENTORY MODEL WITH FUZZY CREDIT PERIOD, TIME AND PRODUCTION RATE DEPENDENT IMPERFECTION

\[ \hat{d}_i^p \]: Demand rate of perfect quality items of the \( i \)th retailer.

\[ d_r^p \]: Demand rate of perfect quality items of the customers from \( i \)th retailer.

\[ \hat{D} \]: Selling rate of perfect quality items of the manufacturer, where \( \hat{D} = \sum_{i=1}^{n} d_r^p \).

\[ t_1 \]: Production run-time in one period.

\[ t_2 \]: Manufacturer business period.

\[ T^* \]: Time at which the selling season ends for \( i \)th retailer.

\[ \bar{M} \]: Imprecise credit period offered by the the manufacturer to the retailers.

\[ N^i \]: Credit period offered by the \( i \)th retailer to the customers, \( 0 < N^i < \bar{M} \).

\[ I_{cm} \]: Rate of interest per year earned manufacturer from retailer.

\[ I_{cr} \]: Rate of interest per year earned retailer from customer.

\[ s_c \]: Screening cost per unit item.

\[ A_m \]: Set up cost of manufacturer, \( A_m = A_m0 + A_m1 f^k, k > 0 \).

\[ h_m \]: Holding cost per unit for per unit time for perfect item in manufacturer.

\[ h_r^p \]: Holding cost per unit for per unit time of perfect quality items of the \( i \)th retailer.

\[ r_{cm} \]: Reworking cost per unit for manufacturer.

\[ c_p \]: Production cost per unit.

\[ c_c \]: Cost per unit carbon emission.

\[ c_d \]: Disposal cost per unit.

\[ s_m \]: Selling price per unit of perfect quality items for manufacturer.

\[ A_r^p \]: Set up cost of the \( i \)th retailer.

\[ s_r^p \]: Selling price per unit of perfect quality items of the \( i \)th retailer \( s_r^p \in [s_{min}, s_{max}] \).

### 11.2.2 Assumptions

The following assumptions have been used to develop the model.

(i) Manufacturer produces a mixture of perfect and imperfect quality items. Some portion of imperfect items are reworked and transformed into a perfect quality items.

(ii) The defective rate is not constant, it increased with time and production rate. So, the defective rate depend on time and production rate, it is defined as follows: \( \hat{\theta} = \hat{\beta} + \lambda P + \xi t \), where \( \hat{\beta}, \lambda \) and \( \xi \) are positive constants as well as taken suitable values.

(iii) The demand rate of the customers depend on displayed stock/inventory of the item and credit period offered them, i.e., \( d_r^p = d_{r0}^p + d_{r1}^p c^{\sigma N^i} + d_{r2}^p (t), d_{r0}^p > 0, d_{r1}^p > 0, d_{r2}^p > 0, \mu > 0 \).

(iv) The production-project might involve rolling out clean energy technologies or soaking up carbon emission from the production, that need to include in a production problem as a carbon emission cost.

(v) Set up cost of manufacturer has been considered as production rate dependent.
(vi) It is assumed that the fuzzy credit period \( (\hat{M}) \) offered by supplier must be within replenishment period \( (T) \), i.e., \( \hat{M} < T \).

(vii) The \( i \)th retailer provide a down-stream credit period \( (N^i) \) to his / her customers, where \( N^i < \hat{M} \).

### 11.3 Mathematical Formulation of the Proposed Model

We consider a manufacturing system which produces both perfect and imperfect item in each production run at a rate \((1 - \theta)P\) and \(\theta P\) respectively. Among the imperfect item few items are repaired at a rate \(\delta \theta P\) portion We consider a manufacturing system which produces the lot size \( Q \) in each production run, with constant production and demand rates denoted by \( p \) and \( d \), respectively. In these process of production, screening and repaired unavoidable carbons are emission at a rate \( \eta^\mu, \eta^\nu \) and \( \eta^\tau \) respectively. The fresh units are transported to several market with their individual demand along with an imprecise trade credit \( \hat{M} \). The retailers sold the units in their respective markets as per customers demand \( d^i_c(t) = d^i_c + d^i_{cN^i} + d^i_{cR^i}(t) \). Here it necessary to mention that the demand depend on the displayed stock of the retailer and the credit period \( N^i \) offered to the customers. Such supply chain inventory model is derived to formulate different cost expression.

![Figure 11.1: The flow of the produce items of the integrated model](image-url)
CHAPTER 11. TWO LAYERS SUPPLY CHAIN IMPERFECT PRODUCTION INVENTORY MODEL WITH FUZZY CREDIT PERIOD, TIME AND PRODUCTION RATE DEPENDENT IMPERFECTION

11.3.1 Formulation of the Manufacturer

The rate of change of inventory level of manufacturer for perfect quality items can be represented by the following differential equations:

\[
\frac{dq_m}{dt} = \begin{cases} P - D - (1 - \delta) (\beta + \lambda P + \xi t)P, & 0 \leq t \leq t_1 \\ -D, & t_1 \leq t \leq t_2 \end{cases}
\]

with boundary conditions \( q_m(0) = 0, \quad q_m(t_2) = 0. \)

The solution of above differential equations are given by

\[
q_m(t) = \begin{cases} \{ P - D - (1 - \delta) (\beta + \lambda P)P\} t - (1 - \delta) \frac{\xi}{2} Pt^2, & 0 \leq t \leq t_1 \\ -D(t - t_2), & t_1 \leq t \leq t_2 \end{cases}
\]

Lemma 11.1. The manufacturer’s production time length \( t_1 \) and production rate \( P \) must satisfy the condition: \( t_2 = \frac{1}{P} \{ P - (1 - \delta) (\beta + \lambda P)Pt - (1 - \delta) \frac{\xi}{2} Pt^2 \}. \)

Proof. From the continuity conditions of \( q_m(t) \) at \( t = t_1 \), the following is obtained:

\[
\{ P - D - (1 - \delta) (\beta + \lambda P)Pt_1 - (1 - \delta) \frac{\xi}{2} Pt_1^2 \} = -D(t_1 - t_2)
\]

\[
\Rightarrow Pt_1 - (1 - \delta) (\beta + \lambda P)Pt_1 - (1 - \delta) \frac{\xi}{2} Pt_1^2 = Dt_2
\]

\[
\Rightarrow t_2 = \frac{1}{P} \{ P - (1 - \delta) (\beta + \lambda P)Pt - (1 - \delta) \frac{\xi}{2} Pt^2 \}
\]

Inventory holding cost for perfect items is:

\[
HCM = h_m \left[ \int_0^{t_1} q_m(t) dt + \int_{t_1}^{t_2} q_m(t) dt \right]
\]

\[
= h_m \left[ \int_0^{t_1} \{ P - D - (1 - \delta) (\beta + \lambda P)Pt - (1 - \delta) \frac{\xi}{2} Pt^2 \} dt - \int_{t_1}^{t_2} D(t - t_2) dt \right]
\]

\[
= \frac{h_m}{2} \left[ \{ P - D - (1 - \delta) (\beta + \lambda P)Pt_1^2 - (1 - \delta) \frac{\xi}{2} Pt_1^3 - D(t_1 - t_2)^2 \} \right]
\]

Production cost for the manufacturer = \( c_p Pt_1 \).

Inspection cost = \( s_P t_1 \).

Reworking cost for manufacture = \( r_m \int_0^{t_1} \delta (\beta + \lambda P + \xi t)P dt = r_m \delta \left( \beta + \lambda P + \frac{\xi}{2} t_1 \right) Pt_1 \)

Revenue of perfect quality items for the manufacturer = \( s_m \delta t_1 \).

Disposal cost during \((0, t_2) = c_d (1 - \delta) \left( \beta + \lambda P + \frac{\xi}{2} t_1 \right) Pt_1 \)

The total amount of carbon emissions during the production run time can be calculated as follows:

\[
CE(t_1) = \hat{\eta} \int_0^{t_1} P dt + \hat{\eta} \int_0^{t_1} \delta (\beta + \lambda P + \xi t)P dt + \hat{\eta} \int_0^{t_1} (1 - \delta) (\beta + \lambda P + \xi t)P dt
\]

\[
= \hat{\eta} Pt_1 + \left\{ \hat{\eta} \delta + \hat{\eta} (1 - \delta) \right\} \left( \beta + \lambda P + \frac{\xi}{2} t_1 \right) Pt_1 
\]

264
11.3. MATHEMATICAL FORMULATION OF THE PROPOSED MODEL

The expected carbon emission cost during the production run time is given by

\[
E[CE(t_1; \hat{\eta})] = c_e \left[ E[\hat{\eta}] P t_1 + \{E[\hat{\gamma}] \delta + E[\hat{\eta}] (1 - \delta) \} \left( \beta + \lambda P + \frac{\xi}{2} t_1 \right) \right] P t_1
\]

**Expected total profit** \( E[\Pi_m(t_1)] \) of manufacturer during the period \((0, T)\) is given by

\[
E[\Pi_m(t_1; \hat{\eta})] = s_m D t_2 - (c_p + s_c + c_e E[\hat{\eta}]) P t_1 - (r_{cm} + c_e E[\hat{\eta}]) \delta \left( \beta + \lambda P + \frac{\xi}{2} t_1 \right) P t_1
\]

\[\quad - \left( c_d + c_e E[\hat{\eta}] \right) (1 - \delta) \left( \beta + \lambda P + \frac{\xi}{2} t_1 \right) P t_1 - A_{cm}
\]

\[\quad - \frac{h_m}{2} \left[ \{P - D - (1 - \delta) (\beta + \lambda P) P \} t_1^2 - (1 - \delta) \frac{\xi}{2} P t_1^2 + D(t_2 - t_1)^2 \right]
\]

11.3.2 Formulation of the \(i\)th Retailer

The \(i\)th retailer receives his/her required quantity per unit time \(d_i^0\) from the manufacturer and fulfill the customers’ demand rate \(d_i\). Those retailer start their business on or before the production run time \(t_1\), pay \(r\) portion of the price amount payable initially and the remaining \((1 - r)\) portion pay at the end of his/her business period. But those retailer’s arrive after the production run time \(t_1\), pay the total amount at their business starting time. They pay the initial amount by getting loan from a bank at the rate of interest of \(I_p\) per year. Every retailer’s earns interest at the rate of \(I_c\) by depositing sales revenue continuously. The inventory level \(q_i(t)\) for the \(i\)th retailer’s is governed by the following differential equation:

\[
\frac{dq_i(t)}{dt} = \begin{cases}
(d_i^0 - d_i), & 0 \leq t \leq t_2 \\
-d_i, & t_2 \leq t \leq T_i
\end{cases}
\]

with boundary conditions \(q_i(0) = 0\) and \(q_i(T_i) = 0\).

The customer demand is \(d_i(t) = d_i^0 + d_i e^{\delta N t} + d_i q_i(t) = d_i^0 + d_i^e q_i(t),\) where \(d_i^0 = d_i^0 + d_i e^{\delta N t}.\) Therefore the solutions of above differential equations are given by

\[
q_i(t) = \begin{cases}
\frac{d_i^0 - d_i}{\delta} (1 - e^{-\delta t}), & 0 \leq t \leq t_2 \\
\frac{d_i^0}{\delta} \left(1 - e^{-\delta(t - t_2)} \right), & t_2 \leq t \leq T_i
\end{cases}
\]

**Lemma 11.2.** The retailer time length of inventory \(T_i\) is given by

\[T_i = t_2 + \frac{1}{\delta_i} \log \left\{ 1 + \frac{d_i^0 - d_i}{\delta_i} (1 - e^{-\delta_i t_2}) \right\}\]
CHAPTER 11. TWO LAYERS SUPPLY CHAIN IMPERFECT PRODUCTION INVENTORY MODEL WITH FUZZY CREDIT PERIOD, TIME AND PRODUCTION RATE DEPENDENT IMPERFECTION

Proof. From the continuity conditions of \( q_i(t) \) at \( t = t_2 \), we have

\[
\left( \frac{d_i}{d_1} - d_0^2 \right)(1 - e^{-d_i(t_2 - t^*)}) = -d_0^2 \left( 1 - e^{-d_i(t_2 - t^*)} \right)
\]

\[
\Rightarrow (d_i - d_0^2)(1 - e^{-d_i(t_2 - t^*)}) = -d_0^2 \left( 1 - e^{-d_i(t_2 - t^*)} \right)
\]

\[
\Rightarrow e^{-d_i(t_2 - t^*)} = 1 + \frac{d_i}{d_0^2} \left( 1 - e^{-d_i(t_2 - t^*)} \right)
\]

\[
T_i = t_2 + \frac{1}{d_i} \log \left( 1 + \frac{d_i}{d_0^2} (1 - e^{-d_i(t_2 - t^*)}) \right)
\]

\[\square\]

Holding cost of the \( i \)-th retailer is given by

\[
HCR^i = h_i^* \left[ \int_0^{T_i} q_i(t) \, dt + \int_{t_2}^{T_i} q_i(t) \, dt \right]
\]

\[
= h_i^* \left[ \int_0^{T_i} \frac{d_i}{d_0^2} \left( t_2 + \frac{1}{d_i} (e^{-d_i(t_2 - t^*)} - 1) \right) - \frac{d_0^2}{d_0} \left( T_i - t_2 \right) + \frac{1}{d_i} \left( 1 - e^{-d_i(t_2 - t^*)} \right) \right]
\]

Holding cost (HCR) for all retailers’ is given by

\[
HCR = \sum_{i=1}^{n} h_i^* \left[ \frac{d_i}{d_0^2} \left( t_2 + \frac{1}{d_i} (e^{-d_i(t_2 - t^*)} - 1) \right) - \frac{d_0^2}{d_0} \left( T_i - t_2 \right) + \frac{1}{d_i} \left( 1 - e^{-d_i(t_2 - t^*)} \right) \right]
\]

Sales revenue from perfect quality items of the \( i \)-th retailer is given by

\[
SRR^i = s_i^* \left[ \int_0^{T_i} (d_1^i + d_0^i q_i(t)) \, dt + \int_{t_2}^{T_i} (d_1^i + d_0^i q_i(t)) \, dt \right]
\]

\[
= s_i^* \left[ \int_0^{T_i} d_0^i t_2 + \frac{d_0^i}{d_1} \left( t_2 + \frac{1}{d_0^i} (e^{-d_0^i(t_2 - t^*)} - 1) \right) - \frac{d_0^2}{d_0} \left( T_i - t_2 \right) + \frac{1}{d_0^i} \left( 1 - e^{-d_0^i(t_2 - t^*)} \right) \right]
\]

All retailers’ total sales revenue (SRR) is given by

\[
SRR = \sum_{i=1}^{n} s_i^* \left[ \frac{d_0^i}{d_1^i} \left( e^{-d_0^i(t_2 - t^*)} - 1 \right) - \frac{d_0^2}{d_0^i} \left( 1 - e^{-d_0^i(t_2 - t^*)} \right) \right]
\]

All retailers’ total purchase cost (PCR) is given by

\[
PCR = \sum_{i=1}^{n} s_i m_i d_0^i t_2
\]

Here it is assumed that the retailer’s trade credit period offered by the manufacturer is \( M \) and that of customer’s offered by the retailer is \( N^i \) (where \( N^i < M \)). The retailer is charged by the manufacturer, an interest at the rate of \( L_c \) per year per unit for the unpaid amount after the delay period and can earn an interest at the rate of \( L_o \) (\( L_o > L_c \)) per year per unit for the amount sold during the period \( (N^i, M) \) respectively. Depending on the cycle times of the retailer and offering as well as receiving credit periods, three different cases may arise, which have been discussed separately.
11.3. MATHEMATICAL FORMULATION OF THE PROPOSED MODEL

Case-I: when $N^i < M < t_2 < T^i$

![Figure 11.2: Total interest earned and paid representation when $N^i < M < t_2 < T^i$](image)

Interest paid by the $i$th retailer ($IP^i_i$).

\[
IP^i_i = s_mI_p \int_{M}^{T^i} q_f(t) \, dt
= s_mI_p \left[ \int_{t_2}^{T^i} q_f(t) \, dt + \int_{t_2}^{M} q_f(t) \, dt \right]
= s_mI_p \left[ \frac{(d^i_q - d^i_{f})}{d^i_l} \left\{ (t_2 - M) + \frac{1}{d^i_l} (e^{-d^i_l(t_2 - M)} - e^{-d^i_lM}) \right\} \right]
\]

Interest earned by the $i$th retailer ($IE^i_i$).

\[
IE^i_i = s_iL_c \left[ (T^i - N^i) \int_{0}^{N^i} d^i_q(t) \, dt + (T^i - M) \int_{M}^{N^i} (M - t)d^i_q(t) \, dt \right]
+ (T^i - t_2) \int_{M}^{t_2} (t_2 - t)d^i_q(t) \, dt + \int_{t_2}^{T^i} (T^i - t)d^i_q(t) \, dt
= s_iL_c \left[ (T^i - N^i) \left\{ d^i_qN^i + (d^i_q - d^i_{f}) \left\{ N + \frac{1}{d^i_l}(e^{-d^i_lN^i} - 1) \right\} \right\} \right]
+ (T^i - M) \left\{ \frac{d^i_q}{2}(M - N^i)^2 - (M - N^i)\frac{e^{-d^i_lN^i}}{d^i_l} + \frac{1}{(d^i_l)^2}(e^{-d^i_lN^i} - e^{-d^i_lM}) \right\}
+ (T^i - t_2) \left\{ \frac{d^i_q}{2}(M - t_2)^2 - (t_2 - M)\frac{e^{-d^i_lM}}{d^i_l} + \frac{1}{(d^i_l)^2}(e^{-d^i_lM} - e^{-d^i_lt_2}) \right\}
+ \frac{d^i_q}{d^i_l} (T^i - t_2)e^{-d^i_l(t_2 - t)} + \frac{d^i_q}{(d^i_l)^2} (1 - e^{-d^i_l(t_2 - t)^2}) \right\}
\]

267
All retailers’ total interest payable \((IP_1)\) is expressed as

\[
IP_1 = \sum_{j=1}^{n} s_m I_p \left\{ \left( \frac{d_i^E - d_i^E_i}{d_i^E} \right) \left( t_2 - M \right) + \frac{1}{d_i^E} \left( e^{-d_i^R (t_2 - T^i)} - e^{-d_i^R M} \right) \right\}
- \frac{d_i^E}{d_i^E} \left\{ \left( T^i - t_2 \right) + \frac{1}{d_i^E} \left( 1 - e^{-d_i^R (t_2 - T^i)} \right) \right\} \right\}
\]

All retailers’ total interest earned \((IE_1)\) is obtained as

\[
IE_1 = \sum_{j=1}^{n} s_m I_p \left\{ \left( T^i - N^i \right) \left\{ \frac{d_i^E}{d_i^E} \left( N^i + \frac{1}{d_i^E} \left( e^{-d_i^R M} - 1 \right) \right) \right\} \right\}
+ \left( T^i - M \right) \left\{ \frac{d_i^E}{d_i^E} \left( (M - N^i)^2 - (M - N^i) \right) e^{-d_i^R M} \right\}
+ \left( T^i - t_2 \right) \left\{ \frac{d_i^E}{d_i^E} \left( (M - t_2)^2 - (t_2 - M) \right) e^{-d_i^R (t_2 - T^i)} \right\}
+ \frac{d_i^E}{d_i^E} \left( T^i - t_2 \right) e^{-d_i^R (t_2 - T^i)} \right\}
\]

Therefore, all retailers’ total profit is given by

\[
\Pi_0^{1}(t_1) = SRR - PCR - HCR - IP_1 + IE_1 - \sum_{j=1}^{n} \lambda_j^i
\]

So, the total profit \((ITP)\) for this case of the integrated system is written as

\[
E[ITP_0(t_1)] = E[\Pi(t_1)] + \Pi_0^{1}(t_1)
\]

**Case-II: when \(N^i < t_2 < M < T^i\)**

Interest paid by the retailer \((IP_2)\),

\[
IP_2 = s_m I_p \int_{M}^{T^i} q_i^E(t) \, dt
= s_m I_p \int_{M}^{T^i} \frac{d_i^E}{d_i^E} \left\{ (t - T^i) e^{-d_i^R (t - T^i)} - 1 \right\} \, dt
= \frac{d_i^E}{d_i^E} s_m I_p \left\{ \frac{1}{d_i^E} \left( e^{-d_i^R (M - T^i)} - 1 \right) - (T^i - M) \right\}
\]

268
11.3. MATHEMATICAL FORMULATION OF THE PROPOSED MODEL

Figure 11.3: Total interest earned and paid representation when \( N^i < t_2 < M < T^i \)

Interest earned by the retailer \( (IE_2) \),

\[
IE_2 = \sum_{i=1}^{n} s_i L_i \left[ \left( T^i - N^i \right) \int_{0}^{N^i} \hat{d}_i(t) \, dt + \left( T^i - t_2 \right) \int_{N^i}^{t_2} \hat{d}_i(t) \, dt \right. \\
+ \left. \left( M - t_2 \right) \int_{t_2}^{M} \hat{d}_i(t) \, dt + \int_{M}^{T^i} \left( T^i - t \right) \hat{d}_i(t) \, dt \right]
\]

\[
= \sum_{i=1}^{n} s_i L_i \left[ \left( T^i - N^i \right) \left\{ \hat{d}_i N^i + (\hat{d}_i - \hat{d}_0) \left\{ N^i + \frac{1}{d_1} (e^{-\delta N^i} - 1) \right\} \right\} \\
+ \left( T^i - t_2 \right) \left\{ \frac{d_i}{2} (t_2 - N^i)^2 - (t_2 - N^i) \frac{e^{-\delta N^i}}{d_1} + \frac{1}{(d_i t_2)} (e^{-\delta N^i} - e^{-\delta t_2}) \right\} \\
+ \left( T^i - M \right) \left\{ \frac{d_i}{d_1} (M - t_2) e^{-\delta (t_2 - M)} + \frac{d_i}{(d_i t_2)} \left\{ 1 - e^{-\delta (M - t_2)} \right\} \right\} \\
+ \left( T^i - M \right) e^{-\delta (M - T^i)} + \frac{d_i}{(d_i t_2)} \left\{ 1 - e^{-\delta (M - T^i)} \right\} \right]
\]

All retailers’ total interest payable \( (IP_2) \) is expressed as

\[
IP_2 = \sum_{i=1}^{n} \frac{d_0 \hat{d}_i}{d_i} L_i \left[ \frac{1}{d_1} \left\{ e^{-\delta (M - T^i)} - 1 \right\} - (T^i - M) \right]
\]

All retailers’ total interest earned \( (IE_2) \) is obtained as

\[
IE_2 = \sum_{i=1}^{n} s_i L_i \left[ \left( T^i - N^i \right) \left\{ \hat{d}_i N^i + \frac{(\hat{d}_i - \hat{d}_0)}{d_1} (e^{-\delta N^i} - 1) \right\} + \left( T^i - t_2 \right) \left\{ \frac{d_i}{2} (t_2 - N^i)^2 - (t_2 - N^i) \frac{e^{-\delta N^i}}{d_1} + \frac{1}{(d_i t_2)} (e^{-\delta N^i} - e^{-\delta t_2}) \right\} \\
+ \left( T^i - M \right) \left\{ \frac{d_i}{d_1} (M - t_2) e^{-\delta (t_2 - M)} \right\} + \left( T^i - M \right) e^{-\delta (M - T^i)} + \frac{d_i}{(d_i t_2)} \left\{ 1 - e^{-\delta (M - T^i)} \right\} \right]
\]

269
Therefore, all retailers’ total profit is given by

$$
\Pi_{1,2}^{(t_1)} = SRR - PCR - HCR - IP_2 + IF_2 - \sum_{s=1}^{n} N_s
$$

So, the total profit (ITP) for this case of the integrated system is written as

$$
E[\Pi_1^{(t_1; \tilde{y})}] = E[\Pi_1^{(t_1; \tilde{y})}] + \Pi_1^{(t_1)}
$$

**Case-III: when** $t_2 < N^i < M < T^i$

Interest paid by the retailer ($IP_2^i$),

$$
IP_2^i = s_m I_p \int_{M}^{N^i} d_1^i(t) \, dt
$$

$$
= s_m I_p \int_{M}^{T^i} \frac{d_0}{d_1^i} \left[ 1 - e^{-d_1^i(t-T^i)} \right] \, dt
$$

$$
= \frac{d_0}{d_1^i} s_m I_p \left[ \frac{1}{d_1^i} \left( e^{-d_1^i(M-T^i)} - 1 \right) - (T^i - M) \right]
$$

![Graph](image)

**Figure 11.4:** Total interest earned and paid representation when $t_2 < N^i < M < T^i$

Interest earned by the retailer ($IE_2^i$),

$$
IE_2^i = s_f L \left[ (T^i - N^i) \left\{ \int_{0}^{t_2} d_2^i(t) \, dt + \int_{t_2}^{N^i} d_1^i(t) \, dt \right\} + \int_{N^i}^{T^i} (T^i - t) d_1^i(t) \, dt \right]
$$

$$
= s_f L \left[ (T^i - N^i) \left\{ d_0 N^i + \left[ d_1^i - d_0 \right] t_2 + \frac{1}{d_1^i} (e^{-d_1^i t_2} - 1) \right\} - d_0 \left[ N^i - t_2 \right]
$$

$$
+ \frac{1}{d_1^i} \left\{ 1 - e^{-d_1^i (T^i - N^i)} \right\} \right\} + \frac{d_0}{d_1^i} (T^i - N^i) e^{-d_1^i (N^i - T^i)} + \frac{d_0}{(d_1^i)^2} \left[ 1 - e^{-d_1^i (N^i - T^i)} \right]\right]
$$

270
11.3. MATHEMATICAL FORMULATION OF THE PROPOSED MODEL

All retailers’ total interest payable \( IP_3 \) is expressed as

\[
IP_3 = \sum_{i=1}^{n} \frac{d\bar{s}_m}{d\bar{t}_i} \left[ \frac{1}{d\bar{t}_i} \{e^{-\bar{d}(\bar{M}-\bar{T}_2)} - 1\} - (\bar{T}_2 - \bar{M}) \right]
\]

All retailers’ total interest earned \( IE_3 \) is obtained as

\[
IE_3 = \sum_{i=1}^{n} s_i \bar{c}_i \left[ (T^i - N^i) \left\{ \frac{d^i}{d\bar{t}_i} N^i + (d^i - d^m) \{T_2 + \frac{1}{d\bar{t}_i} (e^{-\bar{d}(\bar{T}_2 - 1)} - 1) \} - d^m \left\{ (N^i - T_2) \right. \right. \\
+ \left. \left. \frac{1}{d\bar{t}_i} (1 - e^{-\bar{d}(\bar{T}_2 - \bar{N}_2)}) \right\} \right\} \right] + \frac{d^m}{d\bar{t}_i} (T^i - N^i) e^{-\bar{d}(N^i - T^i)} + \frac{d^m}{d\bar{t}_i} \left\{ 1 - e^{-\bar{d}(N^i - T^i)} \right\}
\]

Therefore, all retailers’ total profit is given by

\[
\Pi^3(t_i) = SSR - PCR - HCR - IP_3 + IE_3 - \sum_{i=1}^{n} A_i
\]

So, the total profit (ITP) for this case of the integrated system is written as

\[
E[ITP_3(t_1; \bar{t}_i)] = E[\Pi(t_1; \bar{t}_i)] + \Pi^3(t_i)
\]

11.3.3 Objective of the Proposed Model

The integrated total profit (ITP) for this case of the integrated system is written as

\[
E[ITP_3(t_1; \bar{t}_i)] = E[\Pi(t_1; \bar{t}_i)] + \Pi^3(t_i)
\]

When manufacturer and retailers’ have decided to share resources to undertake mutually beneficial cooperation, the joint total profit which is a function of \( t_1 \) can be obtained by maximizing \( E[ITP(t_1; \bar{t}_i)] \) and is given by

\[
\text{Maximize } E[ITP(t_1; \bar{t}_i)] = \begin{cases} 
E[\Pi(t_1; \bar{t}_i)] = E[\Pi_m(t_1; \bar{t}_i)] + \Pi^3(t_i), & \text{if } N^i < M < t_2 < T^i \\
E[\Pi(t_1; \bar{t}_i)] = E[\Pi_m(t_1; \bar{t}_i)] + \Pi^3(t_i), & \text{if } N^i < t_2 < M < T^i \\
E[\Pi(t_1; \bar{t}_i)] = E[\Pi_m(t_1; \bar{t}_i)] + \Pi^3(t_i), & \text{if } t_2 < N^i < M < T^i
\end{cases}
\]

11.3.4 Model with Fuzzy Credit Period

If we assume that the manufacturer gives an opportunity to all the retailers’ by offering a fuzzy credit period \( (\bar{M}) \). Here, the credit period \( \bar{M} \) is represented in form of triangular fuzzy number. So due to fuzzy credit period \( (\bar{M}) \), the optimum value of integrated profit function \( ITP(t_1) \) will be different for various values of \( \bar{M} \) with some degree of belonging ness. Therefore in
such situation, the profit function will be fuzzy in nature and is denoted by \( E[\tilde{\Pi}(t; \tilde{M}, \tilde{\eta})] \), where

\[
\begin{align*}
E[\tilde{\Pi}(t; \tilde{M}, \tilde{\eta})] = \left\{ \begin{array}{ll}
E[\tilde{\Pi}^{(1)}(t; \tilde{M}, \tilde{\eta})] = E[\tilde{\Pi}^{(1)}(t; \tilde{M})] + \tilde{\Pi}^{(1)}(t; \tilde{M}), & \text{at } N^i < t < T^i \\
E[\tilde{\Pi}^{(2)}(t; \tilde{M}, \tilde{\eta})] = E[\tilde{\Pi}^{(2)}(t; \tilde{M})] + \tilde{\Pi}^{(2)}(t; \tilde{M}), & \text{at } N^i < t < \tilde{M} < T^i \\
E[\tilde{\Pi}^{(3)}(t; \tilde{M}, \tilde{\eta})] = E[\tilde{\Pi}^{(3)}(t; \tilde{M})] + \tilde{\Pi}^{(3)}(t; \tilde{M}), & \text{at } t_2 < N^i < \tilde{M} < T^i
\end{array} \right.
\end{align*}
\]

The imprecise expression of \( E[\tilde{\Pi}^{(1)}(t_1; \tilde{M}, \tilde{\eta})] \), \( E[\tilde{\Pi}^{(2)}(t_1; \tilde{M}, \tilde{\eta})] \) and \( E[\tilde{\Pi}^{(3)}(t_1; \tilde{M}, \tilde{\eta})] \) are given below:

\[
E[\tilde{\Pi}^{(1)}(t_1; \tilde{M}, \tilde{\eta})] = s_m \sum_{i=1}^{n} \left[ \frac{d_i}{d_1} t_2 - \left( r_c + c_r E[\tilde{\Pi}^{(1)}] \right) \delta \left( 1 + \frac{\zeta}{\lambda_P} + \frac{z}{t_1} \right) t_1 \left( \beta + 1 - \frac{\zeta}{\lambda_P} \right) \right] P t_i
\]

\[
- \left( c_p + s_c + c_r E[\tilde{\Pi}^{(1)}] \right) P t_i - \left( c_d + c_r E[\tilde{\Pi}^{(1)}] \right) \left( 1 - \delta \right) \left( 1 + \frac{\zeta}{\lambda_P} + \frac{z}{t_1} \right) P t_i
\]

\[
- \frac{h_m}{2} \left[ \frac{P - \sum_{i=1}^{n} d_i P t_i}{\sum_{i=1}^{n} d_i P t_i} - \frac{(1 - \delta) P}{\lambda_P} \left( 1 + \frac{\zeta}{\lambda_P} \right) \frac{z}{t_1} + \sum_{i=1}^{n} d_i (t_1 - t_i)^2 \right] - A_m
\]

\[
+ \sum_{i=1}^{n} s_i \left( \frac{d_i}{d_1} t_2 - \frac{d_i}{d_1} \left( 1 - e^{-\delta_1 t_2} \right) \left( 1 - e^{-\delta_i (t_2 - T^i)} \right) \right) - \sum_{i=1}^{n} s_i \left( 1 - e^{-\delta_1 (t_2 - T^i)} \right)
\]

\[
- \frac{h_m}{2} \left[ \left( \frac{d_i}{d_1} \right) \left( t_i - t_i \right) \left( t_i - t_i \right) - \frac{d_i}{d_1} \left( 1 - e^{-\delta_i (t_2 - T^i)} \right) \right] - \sum_{i=1}^{n} s_i \left( 1 - e^{-\delta_i (t_2 - T^i)} \right)
\]

\[
- \frac{h_m}{2} \left[ \left( \frac{d_i}{d_1} \right) \left( t_i - t_i \right) \left( t_i - t_i \right) - \frac{d_i}{d_1} \left( 1 - e^{-\delta_i (t_2 - T^i)} \right) \right] - \sum_{i=1}^{n} s_i \left( 1 - e^{-\delta_i (t_2 - T^i)} \right)
\]
11.3. MATHEMATICAL FORMULATION OF THE PROPOSED MODEL

\[ E[\overline{P}(t; \tilde{M}, \tilde{n})] = sm \sum_{i=1}^{n} d_i^2 \left[ \frac{1}{2} \right] \left( \beta + \lambda P + \frac{\xi}{2} t \right) P T_i \]

\[- (c_T + s_T + c, E[f_0]) P T_i \]

\[- \frac{h_m}{2} \left[ \left( P - \sum_{i=1}^{n} d_i^2 \right) \left( 1 - \beta + \lambda P \right) P T_i - \left( 1 - \delta \right) \frac{\xi}{2} \right] P T_i \]

\[- \sum_{i=1}^{n} s_i \left[ d_i^2 \left( t_2 + \frac{1}{d_i} \left( e^{-\beta T_i} - 1 \right) \right) - \frac{d_i}{d_i} \right] \left( 1 - e^{-\delta (T_2 - T_i)} \right) \]

and \( E[\overline{P}(t; \tilde{M}, \tilde{n})] = sm \sum_{i=1}^{n} d_i^2 \left[ \frac{1}{2} \right] \left( \beta + \lambda P + \frac{\xi}{2} t \right) P T_i \)

\[- (c_T + s_T + c, E[f_0]) P T_i \]

\[- \frac{h_m}{2} \left[ \left( P - \sum_{i=1}^{n} d_i^2 \right) \left( 1 - \beta + \lambda P \right) P T_i - \left( 1 - \delta \right) \frac{\xi}{2} \right] P T_i \]

\[- \sum_{i=1}^{n} s_i \left[ d_i^2 \left( t_2 + \frac{1}{d_i} \left( e^{-\beta T_i} - 1 \right) \right) - \frac{d_i}{d_i} \right] \left( 1 - e^{-\delta (T_2 - T_i)} \right) \]

\[- \sum_{i=1}^{n} s_i \left[ d_i^2 \left( (T_i - T) \left( e^{-\beta T_i} - 1 \right) \right) - \frac{d_i}{d_i} \right] \left( 1 - e^{-\delta (T_i - T)} \right) \]

\[- \sum_{i=1}^{n} s_i \left[ d_i^2 \left( (T_i - T) \left( e^{-\beta T_i} - 1 \right) \right) - \frac{d_i}{d_i} \right] \left( 1 - e^{-\delta (T_i - T)} \right) \]

\[- \sum_{i=1}^{n} s_i \left[ d_i^2 \left( (T_i - T) \left( e^{-\beta T_i} - 1 \right) \right) - \frac{d_i}{d_i} \right] \left( 1 - e^{-\delta (T_i - T)} \right) \]

273
11.4 Solution Procedure

The optimum values production time (\(t_1\)) and expected total profit for the fuzzy stochastic model are obtained through algorithm.

**Step 1:** For the random variable \(\tilde{y}\) with p.d.f \(g(\tilde{y})\) evaluate the expected value of integrated total profit \((\tilde{I}_{TP}^{(1)}(t_1; \tilde{M}, \tilde{\hat{n}})), (\tilde{I}_{TP}^{(2)}(t_1; \tilde{M}, \tilde{\hat{n}}))\) and \((\tilde{I}_{TP}^{(3)}(t_1; \tilde{M}, \tilde{\hat{n}}))\) using the definition \(E[\tilde{I}_{TP}^{(1)}(t_1; \tilde{M}, \tilde{\hat{n}})] = \int_{-\infty}^{\infty} \tilde{I}_{TP}^{(1)}(t_1; \tilde{M}, \tilde{\hat{n}})g(\tilde{y})d(\tilde{y})\). Similarly obtained \(E[\tilde{I}_{TP}^{(2)}(t_1; \tilde{M}, \tilde{\hat{n}})]\) and \(E[\tilde{I}_{TP}^{(3)}(t_1; \tilde{M}, \tilde{\hat{n}})]\).

**Step 2:** For a given triangular fuzzy number (TFN) \(\tilde{M} = (M - \Delta_1, M, M + \Delta_2)\), the fuzzy expressions of \(E[I_{TP}^{(1)}(M)] = E[I_{TP}^{(1)}] = \left(E[I_{TP}^{(1)}], E[I_{TP}^{(2)}], E[I_{TP}^{(3)}]\right)\),
\[E[I_{TP}^{(2)}] = E[I_{TP}^{(2)}] = \left(E[I_{TP}^{(2)}], E[I_{TP}^{(2)}], E[I_{TP}^{(2)}]\right)\] and
\[E[I_{TP}^{(3)}] = E[I_{TP}^{(3)}] = \left(E[I_{TP}^{(3)}], E[I_{TP}^{(3)}], E[I_{TP}^{(3)}]\right)\] are obtained using fuzzy extension principal.

**Step 3:** Then from the fuzzy expressions \(E[I_{TP}^{(1)}(t_1; \tilde{M}, \tilde{\hat{n}})], E[I_{TP}^{(2)}(t_1; \tilde{M}, \tilde{\hat{n}})], E[I_{TP}^{(3)}(t_1; \tilde{M}, \tilde{\hat{n}})]\) obtained the centroid values \(CE[I_{TP}^{(1)}] = C[CE[I_{TP}^{(1)}]], CE[I_{TP}^{(2)}] = C[CE[I_{TP}^{(2)}]], CE[I_{TP}^{(3)}] = C[CE[I_{TP}^{(3)}]]\) respectively, using the definition presents in § 4, which is the process of defuzzification.

**Step 4:** Finally, maximized each \(CE[I_{TP}^{(1)}(t_1)], CE[I_{TP}^{(2)}(t_1)], CE[I_{TP}^{(3)}(t_1)]\) with respect to the decision variable \(t_1\) by using LINGO Solver 12.0 for particular input data.

11.5 Numerical Illustrations

In this section, we illustrate some numerical examples to study the feasibility of the proposed imperfect production inventory model.

**Example 11.1.** we consider a production-inventory supply chain model with the following characteristics:

\[P = 42 \text{ units, } \beta_1 = 0.10, \lambda = 0.001, \xi = 0.02, \Delta_1 = 0.01, \Delta_2 = 0.02, \delta = 0.70, c_f =$32 per unit, \(c_i = $92\) per unit, \(r_m = $101\) per unit, \(c_L = $5\) per unit, \(h_m = $4\) per unit per unit time, \(h_r^{(1)} = $45.5\) per unit per unit time, \(h_r^{(2)} = $48\) per unit per unit time, \(A^{(m)} = $8140, A^{(1)} = $8140, A^{(2)} = $8130, s_m = $8140\) per unit, \(s_r^{(1)} = $260\) per unit, \(s_r^{(2)} = $250\) per unit, \(c_r = $2.5\) per unit, \(s_{min} = $220, s_{max} = $280, d_r^{(1)} = 17\) unit, \(d_r^{(2)} = 18\)
### 11.5. Numerical Illustrations

unit, $d^{(1)}_{10} = 9.84$ unit, $d^{(2)}_{10} = 9.54$ unit, $d^{(1)}_{21} = 1.8$, $d^{(2)}_{21} = 2$, $d^{(1)}_{1} = 7$, $d^{(2)}_{1} = 6$, $\mu^{(1)} = 7$, $\mu^{(2)} = 6$. The carbon-emission rates $\eta$, $\eta'$, $\eta''$ for production process, rework process and disposal units respective followed a Beta distribution with parameters $\nu$, $\nu$ i.e., the p.d.f. of $\eta$ is

$$
g(\eta) = \begin{cases} 
\frac{\eta^{\nu-(\nu+1)\eta-1}}{\Gamma(\nu, \nu)}, & 0 \leq \eta \leq 1 \\
0, & \text{otherwise}
\end{cases}
$$

$E[\eta'] = \frac{\mu^{(1)}}{\nu+\mu^{(1)}}$, $E[\eta''] = \frac{\mu^{(2)}}{\nu+\mu^{(2)}}$, $E[\eta'] = \frac{\mu^{(1)}}{\nu+\mu^{(1)}}$ for $\nu = 1$, $\nu = 2$ and $\nu = 3$. For $\nu = 8$, $\nu = 3$: $E[\eta'] = 0.727$, $E[\eta''] = 0.545$ and $E[\eta'] = 0.119$ numerical computation, we consider applying the proposed computational algorithm yields the results shown in Table 11.2 for different cases.

<table>
<thead>
<tr>
<th>Cases</th>
<th>Retailers' credit period ($X^1$, $X^2$)</th>
<th>Manufacturer credit period ($\bar{M}$)</th>
<th>Production Period $t_1$</th>
<th>Period of Manufacturers</th>
<th>Period of Retailers'</th>
<th>Expected Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case-I</td>
<td>(0.12, 0.10)</td>
<td>(0.19, 0.20, 0.22)</td>
<td>0.528</td>
<td>0.605</td>
<td>(0.734, 0.730)</td>
<td>4178.54</td>
</tr>
<tr>
<td>Case-II</td>
<td>(0.12, 0.10)</td>
<td>(0.19, 0.20, 0.22)</td>
<td>0.531</td>
<td>0.610</td>
<td>(0.739, 0.732)</td>
<td>3512.53</td>
</tr>
<tr>
<td>Case-III</td>
<td>(0.65, 0.62)</td>
<td>(0.69, 0.70, 0.72)</td>
<td>0.538</td>
<td>0.618</td>
<td>(0.726, 0.720)</td>
<td>4128.05</td>
</tr>
</tbody>
</table>

In manager’s point of view, 2nd case gives minimum profit. Since, in that case manufacturer lost maximum opportunity of credit period, whereas the customers receive maximum benefit from the management system. The first and last cases are nearly same profitable, since in the first case both the members offer lower credit period and in the last case both the members offer higher credit periods. More over, in third case, as retailer offers higher credit period, so demand of the customers become high but the quantity transferred from manufacturer to retailer is same, therefore, period of consumptions of the retailer is reduced. As the demand of the customers are linked with the credit period offered by the retailer, so high demand of the customers reduced the time period of the retailers.

**Example 11.2.** Evaluate the optimal policy of the decision maker when the defective rate of produce item depends on production rate only (i.e., $\xi = 0$) and the remain parameters of the system are unaltered.

Following table shows the optimum policies of the decision maker when the defective rate of the produced item is of the form $\theta = \beta + \lambda P$. In comparison of the cases, Example 11.2 reveal same decisions as Example 11.1. More-over, when defective rate does not depend on time, the defective units are quite less, i.e., fresh units are more than that of Example 11.1. So, business periods are larger than the Example 11.1, which also yield more profit than Example 11.1 for each cases.

275
### Table 11.3: Optimal results of illustrated model when $\theta = \beta + \lambda P$

<table>
<thead>
<tr>
<th>Cases</th>
<th>Retailers' credit period $(Y_1, Y_2)$</th>
<th>Manufacturer credit period $(\hat{Y}_1)$</th>
<th>Production time $t_1$</th>
<th>Period of Manufacturer $t_2$</th>
<th>Period of Retailers $(T_1, T_2)$</th>
<th>Expected Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>(0.10, 0.12)</td>
<td>(0.19, 0.20, 0.22)</td>
<td>0.543</td>
<td>0.622</td>
<td>(0.763, 0.757)</td>
<td>4370.80</td>
</tr>
<tr>
<td>Case II</td>
<td>(0.12, 0.10)</td>
<td>(0.69, 0.70, 0.72)</td>
<td>0.548</td>
<td>0.637</td>
<td>(0.767, 0.761)</td>
<td>3872.79</td>
</tr>
<tr>
<td>Case III</td>
<td>(0.65, 0.62)</td>
<td>(0.69, 0.70, 0.72)</td>
<td>0.558</td>
<td>0.642</td>
<td>(0.755, 0.748)</td>
<td>4327.64</td>
</tr>
</tbody>
</table>

### Example 11.3. Find the optimal policies of the decision maker when the defective rate of produce item depend on time only (i.e., $\lambda = 0$) and the remain parameters of the system are unalter.

When the defective rate of produce item does not depend on production rate but depend on time only i.e., $\theta = \beta + \xi t$, then the optimum results are shown in the following table.

### Table 11.4: Optimal results of illustrated model when $\theta = \beta + \xi t$

<table>
<thead>
<tr>
<th>Cases</th>
<th>Retailers' credit period $(Y_1, Y_2)$</th>
<th>Manufacturer credit period $(\hat{Y}_1)$</th>
<th>Production time $t_1$</th>
<th>Period of Manufacturer $t_2$</th>
<th>Period of Retailers $(T_1, T_2)$</th>
<th>Expected Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>(0.10, 0.12)</td>
<td>(0.19, 0.20, 0.22)</td>
<td>0.728</td>
<td>0.848</td>
<td>(1.020, 1.010)</td>
<td>6074.09</td>
</tr>
<tr>
<td>Case II</td>
<td>(0.12, 0.10)</td>
<td>(0.69, 0.70, 0.72)</td>
<td>0.731</td>
<td>0.851</td>
<td>(1.023, 1.013)</td>
<td>6833.38</td>
</tr>
<tr>
<td>Case III</td>
<td>(0.65, 0.62)</td>
<td>(0.69, 0.70, 0.72)</td>
<td>0.739</td>
<td>0.858</td>
<td>(0.998, 0.991)</td>
<td>6095.70</td>
</tr>
</tbody>
</table>

If defective rate does not depend on the production rate, then production time is much larger. So manufacturer produced more quantity of items. For this reason, the periods of manufacturer and retailers are more than the scenarios when $\theta = \beta + \lambda P + \xi t$ or $\theta = \beta + \lambda P$. So the optimum profit is more than the other two scenarios, as expected.

### Example 11.4. When the defective rate of produce item is constant (i.e., $\lambda = 0$ and $\xi = 0$) then evaluate the optimal profit of the decision maker (the remain parameter of the system remain unalter).

### Table 11.5: Optimal results of illustrated model when $\theta = \beta$

<table>
<thead>
<tr>
<th>Cases</th>
<th>Retailers' credit period $(Y_1, Y_2)$</th>
<th>Manufacturer credit period $(\hat{Y}_1)$</th>
<th>Production time $t_1$</th>
<th>Period of Manufacturer $t_2$</th>
<th>Period of Retailers $(T_1, T_2)$</th>
<th>Expected Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>(0.10, 0.12)</td>
<td>(0.19, 0.20, 0.22)</td>
<td>0.783</td>
<td>0.913</td>
<td>(1.080, 1.081)</td>
<td>6593.80</td>
</tr>
<tr>
<td>Case II</td>
<td>(0.12, 0.10)</td>
<td>(0.69, 0.70, 0.72)</td>
<td>0.787</td>
<td>0.918</td>
<td>(1.098, 1.099)</td>
<td>7664.13</td>
</tr>
<tr>
<td>Case III</td>
<td>(0.65, 0.62)</td>
<td>(0.69, 0.70, 0.72)</td>
<td>0.794</td>
<td>0.924</td>
<td>(1.033, 1.045)</td>
<td>6638.49</td>
</tr>
</tbody>
</table>

276
From Table 11.5, it is decided that when defective rate is fixed then the yields optimum profits are maximum for each cases. This is due to the less amount of defective units. The other conclusions regarding the comparison of the three cases remain same as in Example 11.1.

### 11.5.1 Sensitivity Analysis

In this section, we examine the effects of changes in the system parameters. A Sensitivity analysis is performed by changing some of the parameters as follows. On the basis of the results calculated the following observations can be made.

**Sensitivity analysis 11.1.** In this example, we use the same data as in Example 11.1 except the production rate on the optimal solution. The results in Table 11.6 given below:

<table>
<thead>
<tr>
<th>Production rate (P)</th>
<th>Cases</th>
<th>Defective rate (θ_1)</th>
<th>Production time (θ_2)</th>
<th>Manufacturer business period (θ_3)</th>
<th>Retailers’ business period (θ_4, θ_5)</th>
<th>Expected Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>Case-I</td>
<td>0.141</td>
<td>0.561</td>
<td>0.612</td>
<td>(0.747, 0.741)</td>
<td>4257.68</td>
</tr>
<tr>
<td></td>
<td>Case-II</td>
<td>0.145</td>
<td>0.564</td>
<td>0.616</td>
<td>(0.748, 0.741)</td>
<td>3622.90</td>
</tr>
<tr>
<td></td>
<td>Case-III</td>
<td>0.151</td>
<td>0.572</td>
<td>0.625</td>
<td>(0.735, 0.728)</td>
<td>4189.38</td>
</tr>
<tr>
<td>42</td>
<td>Case-I</td>
<td>0.153</td>
<td>0.528</td>
<td>0.605</td>
<td>(0.734, 0.730)</td>
<td>4178.54</td>
</tr>
<tr>
<td></td>
<td>Case-II</td>
<td>0.155</td>
<td>0.531</td>
<td>0.610</td>
<td>(0.739, 0.732)</td>
<td>3512.53</td>
</tr>
<tr>
<td></td>
<td>Case-III</td>
<td>0.159</td>
<td>0.538</td>
<td>0.618</td>
<td>(0.726, 0.720)</td>
<td>4128.05</td>
</tr>
<tr>
<td>45</td>
<td>Case-I</td>
<td>0.156</td>
<td>0.484</td>
<td>0.597</td>
<td>(0.725, 0.719)</td>
<td>4090.69</td>
</tr>
<tr>
<td></td>
<td>Case-II</td>
<td>0.159</td>
<td>0.490</td>
<td>0.601</td>
<td>(0.728, 0.723)</td>
<td>3347.35</td>
</tr>
<tr>
<td></td>
<td>Case-III</td>
<td>0.165</td>
<td>0.495</td>
<td>0.607</td>
<td>(0.714, 0.707)</td>
<td>4036.89</td>
</tr>
</tbody>
</table>

Here, we conclude that increasing rate of production (P) increases the rate of defectiveness(θ) as it is a increasing function of P, it also reduced value of production time, manufacturing time as well as expected profit due to the increase of production with fixed demand expand the inventory / holding cost. Again, increasing defective amount reduced the amount of fresh unit that caused lower business period and lower profit.

**Sensitivity analysis 11.2.** This example outlines the effects of changes in the different values of trade credit periods $\bar{M}$ and $M$’s for case-II. The results given in Table 11.7.

Table 11.7 shows that, more spread of fuzzy credit period gives lower profit of the system as well as lower interest paid by the retailer (as expected). And increasing of down stream credit period (offered by the retailer to the customers) of the system yields more earn of the retailer due to the increasing of demand which consequently decreases the holding cost also.

277
### 11.6 Conclusion

This chapter develops an integrated production inventory model involving manufacturer, retailer and customers with up-stream and down stream credit periods. It will provide the following decision making:

- If a manufacturer produces an item with defective quality also, then the rate of defective may depend on production rate and/or time. And the effect of these dependencies are shown here numerically with efficient cause.
- The duration of upstream credit period may fluctuate due to different causes, in this regard here, an imprecise nature of upstream credit period is considered and analyzed numerically. More over, its effects are justified by a sensitivity analysis.
- Here, an unavoidable circumstance of carbon-emission is taken into account for a good gesture of society and this emission rate is not fixed through the cycle time, so it is formulated with random nature.
- Furthermore, we discuss some special cases of credit periods to show their effects on the management.
- Finally, we run several numerical examples and sensitivity analysis to illustrate the problem and provide some managerial insights.
Part VI

Summary and Extension of the Thesis