Chapter 8

A fuzzy imperfect production inventory model based on fuzzy differential and fuzzy integral method

8.1 Introduction

The fuzzy set concept has been used to treat the uncertainty in the classical inventory model. This set theory, originally introduced by Zadeh [239], provides a feasible approach to deal with the fuzzy uncertainty problem. The considerable attention has been attracted to fuzzy circumstances in the literature. For example, Park [164] used fuzzy inventory cost in economic order quantity model. Chang [24] discussed how to obtain the economic production quantity, when the quantity of demand is uncertain. Chen and Hsieh [29] established a fuzzy economic production model to treat the inventory problem with all the parameters and variables being fuzzy numbers. Hsieh [93], Lee and Yao [128], and Lin and Yao [137] also wrote some papers about the fuzzy production model. Except these there exist many other papers such as Das et al. [51] in which uncertainties have been solved using fuzzy set theory.

Traditional EPQ models assume that all items made are of perfect quality. However, in real world manufacturing systems, due to manufacturing operator, machine-component and/or other factors, generation of defective items is inevitable. It is more realistic to assume that all industries produce a certain percent of imperfect quality items. Such a production process is called imperfect production (cf. Salameh and Jaber [184], Yoo et al. [237]). Among other researchers, Salameh and Jaber [184] developed an inventory model which accounted for imperfect quality items using the EPQ/EOQ formulae. They assumed defective items are sold as a single batch at the end of the total screening process. Moreover,
PORTER [166] assumed that the probability of a shift from the 'in-control' state to the 'out-of-control' state has a given value for each production item. Rosenblatt and Lee [181] proposed an EPQ model that deals with imperfect quality. They assumed that at some random point in time the process might shift from an in-control to an out-of-control state, and a fixed percentage of defective items are produced. Cheng [32] examined an economic order quantity model with demand-dependent unit production cost and imperfect production processes. Several researchers have applied fuzzy or rough sets theory to solve production inventory problems. Recently, Cai et al. [40], Jaber [100], Lin et al. [138], Chung and Hou [42], Lee [129], Lo et al. [145], Das et al. [51] have been carried out to address the issues of imperfect production and reduction of its corresponding quality costs. From the previous researchers, we can find that some papers discussed fuzzy costs, fuzzy demand etc. but till now no one has considered fuzzy defective rate, fuzzy time on which imperfect production to be started. Therefore, to study Fuzzy Economic Production Quantity (FEPQ) model considering the above lacunas with imperfect products which cannot be repaired is very important in vague environment.

Recently, it is noted that most of the consumers are increasingly demanding good-quality products. With a rise in consumer consciousness, providing defective products to customers not only increases service and replacement costs, but it may also cause inestimable damage to the credibility of the company. Regarding these imperfect systems, an understanding of the relationship among production, inventory, inspection, and maintenance can assist a manager to perform operation control and quality assurance in a more effective manner. In actual production processes, the process begins in a controlled state, but it may change to an out-of-control state as production proceeds, and some non-conforming items may appear. The conditions of a production system are tracked through inspection to ensure that consumers do not receive defective products. The purpose of the inspection is to determine the state of the production system and of product quality. The quality costs must be balanced with inspection costs when deciding on the frequency of the inspections to be performed. Therefore, the schedule of the inspections is essential. Wang and Shen [210] generalized the model of Porter [166] introducing a product inspection policy. Wang [214] extended Kim and Hong's [118] work considering a product inspection policy only at the end of the production run, instead of full inspections during a production run.

The classical inventory models developed for constant demand rate can be applied to both manufacturing and sales environment. In the case of certain consumer products, the consumption rate may be influenced by the stock levels. This phenomenon induces some researchers to consider stock dependent demand rate. The traditional literature dealing with inventory model usually assumed the market demand to be constant, stock dependent, time dependent and stochastic etc. in many real situations. However, it is very difficult to estimate the probability distribution of market demand due to the lack of historical data. Given the situations, they can only use linguistic terms, such as the market demand is about $d_{LF}$, but not less than $d_{L}$ and not larger than $d_{U}$, to describe the fuzzy market demand. In this case, the demand quantity is approximately specified based on the experience. Some papers have dealt
with this case by applying fuzzy theory. Petrovic and Vujosevic [165] first proposed a newsvendor problem with discrete fuzzy demand. Dutta et al. [67] presented a single-period inventory problem in an imprecise and uncertain mixed environment, and introduced demand as a fuzzy random variable. Zhen and Xiayu [242] considered the multi-product newsvendor problem with fuzzy demands under budget constraint. Kao and Hsu [107] constructed a single-period inventory model with fuzzy demand. Li et al. [133] established two single-period inventory models in fuzzy environment, one of which assuming demand is stochastic while the holding and shortage costs are fuzzy, the other assuming the costs are deterministic but the demand is fuzzy. Lee and Yao [128] discussed the production inventory problems for fuzzy demand quantity. Taleizadeh et al. [201] extended an uncertain EOQ model for joint replenishment strategy with incremental discount policy and fuzzy rough demand.

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In this chapter, an imperfect production model has been considered with fuzzy defective rate. Here production starts with a constant production rate up to a variable time. At the beginning of a production process, the system is assumed to be in a controlled state and perfect items are produced. During production-run-time, the manufacturing process may shift to an ‘out-of-control’ state after certain time that follows a fuzzy number. In ‘out-of-control’ state, a percent of produced items is defective. The defective items are sold at a single lot after end of the production at a reduced cost. Then, two profit functions have been formulated and optimized through some numerical illustrations.
8.2 Notations and Assumptions

To formulate the proposed model, we have used the following notations and assumptions.

### 8.2.1 Notations

The following notations are used throughout the chapter.

- \( q_i(t) \): On hand inventory at any time \( t \) in crisp environment for perfect quality item.
- \( q^*_i(t) \): On hand inventory at any time \( t \) in fuzzy environment for perfect quality item.
- \( q^*_i(t) \): On hand inventory at any time \( t \) in crisp environment for imperfect quality item.
- \( q^*_i(t) \): On hand inventory at any time \( t \) in fuzzy environment for imperfect quality item.
- \( D \): Demand rate of perfect quality items in crisp environment.
- \( \tilde{D} \): Fuzzy demand rate of perfect quality items in fuzzy environment.
- \( P \): Production rate \( (P > D) \).
- \( \tilde{\beta} \): Fuzzy percentage of imperfect quality items per unit time with its \( \alpha \)-cut \( [\beta^L, \beta^U] \).
- \( T \): Fuzzy time from which the production system shifts from ‘in-control’ state to the ‘out-of-control’ state with its \( \alpha \)-cut \( [T^L, T^U] \).
- \( t_1 \): Duration of production run time.
- \( c_p \): Production cost per unit item.
- \( c_{sc} \): Screening cost per unit item.
- \( h_p \): Inventory holding cost per unit for perfect item in production center per unit time.
- \( h_i \): Inventory holding cost per unit for imperfect item in production center per unit time.
- \( s \): Selling price of perfect item per unit.
- \( s' \): Selling price of imperfect item per unit.
- \( T \): Length of business period.
- \( \sim \): Symbol is used on the top of notations to represent fuzzy parameters.

### 8.2.2 Assumptions

The proposed model based on the following assumptions:

(i) The model is developed only for a single item manufacturer which produces the items at the rate of \( P \).

(ii) Practically, it is seen that any production concern initially produces items to be perfect since all resources are fresh i.e., the production system initially is ‘in-control-state’. After some times, the production system produces perfect items as well as imperfect items also since with the increase of time, the manufacturing system gradually breakdowns i.e., the system enters in out-of-control state. Therefore, the production system may be either in-control state or out-of-control state. In this proposed model, it is assumed that the production system is initially being a controlled state upto time \( \tilde{T} \) which is consider as fuzzy after that the production system goes to ‘out-of-control’ state.
(iii) Normally the rate of defectiveness is not be a constant, because it may vary in a production system due to many factors such as production rate, machine component etc. So it should be taken a uncertain quantity. In this proposed model it is assumed that the defective rate (\( \tilde{\delta} \)) has been considered as fuzzy.

(iv) It is a single item and single period (\( T \)) inventory model which is decision variable.

(v) Production rate (\( P \)) is constant.

(vi) In practical business world, sometimes it is seen that the demand of a retailer changes due to various factors according to his/her business policy. So in nature, it is vague and imprecise. For this reason, it has been considered a fuzzy demand of the retailer from the manufacturer. Now in fuzzy set theory [244], there are some standard fuzzy numbers such as triangular, trapezoidal, parabolic, general etc. which are considered for illustration.

(vii) All imperfect items which are produced in out-of-control state, are sold altogether at the end of production period at a reduced price.

8.3 Mathematical Formulation of the Proposed Model

In this paper, we consider an imperfect production inventory problem in which production starts at time \( t = 0 \) at the rate of \( P \). Initially up to time \( \tilde{T} \) the system produces perfect item. Then it produces both good and defective items during \( [\tilde{T}, t_1] \). At time \( t_1 \) production stop. After that, from the stock demand of the customer is fulfilled up to time \( T \). According to assumptions \( \tilde{T} \), \( \tilde{\delta} \) and \( \tilde{D} \) are taken as fuzzy numbers. Due to existence of fuzzy parameters, the inventory level at any time \( t \) is also fuzzy in nature. Since there exist productions of perfect items and imperfect items, hence here two separate inventories have been considered.

8.3.1 Formulation for Perfect Quality Items

In this case, the initially stock of the product of perfect items is zero then it starts production at the rate \( P \). The system produces good quality items during \( [0, \tilde{T}] \) and it produces both good and defective items during \( [\tilde{T}, t_1], \tilde{T} \in (0, t_1) \). The total good items produced during \( [0, t_1] \) are used to meet the demand of perfect item up to time \( T \). Production of the cycle stops at time \( t_1 \).
CHAPTER 8. A FUZZY IMPERFECT PRODUCTION INVENTORY MODEL BASED ON FUZZY DIFFERENTIAL AND FUZZY INTEGRAL METHOD

![Graphical representation of inventory over time]

Figure 8.1: Pictorial representation of manufacturer’s inventory of perfect quality item

Under such consideration the inventory level of perfect item $\tilde{q}_1(t)$ at time $t$ satisfies the following differential equations:

$$\frac{d\tilde{q}_1(t)}{dt} = \begin{cases} P - \tilde{D}, & 0 \leq t \leq \tilde{\tau} \\ (P - \tilde{D}) - \beta \tilde{P}, & \tilde{\tau} \leq t \leq t_1 \\ -\tilde{\beta}, & t_1 \leq t \leq T \end{cases}$$  \hspace{1cm} (8.1)

subject to the conditions that, $\tilde{q}_1(0) = 0$, $\tilde{q}_1(T) = 0$.

To solve the fuzzy differential equations (8.1) first, we find out the solution $q_1(t)$ in crisp differential environment according to Chaleco-Cano and Roman-Flores [19] is as follows:

$$\frac{dq_1(t)}{dt} = \begin{cases} P - D, & 0 \leq t \leq \tau \\ (P - D) - \beta P, & \tau \leq t \leq t_1 \\ -\beta, & t_1 \leq t \leq T \end{cases}$$  \hspace{1cm} (8.2)

subject to the conditions that, $q_1(0) = 0$, $q_1(T) = 0$.

The solution of the above differential equations are

$$q_1(t) = \begin{cases} (P - D)t, & 0 \leq t \leq \tau \\ (P - D)t + \beta Pt - \beta Pt_1, & \tau \leq t \leq t_1 \\ D(T - t), & t_1 \leq t \leq T \end{cases}$$  \hspace{1cm} (8.3)

Lemma 8.1. Manufacturer’s production rate $(P)$, Demand rate $(D)$, production time period $(t_1)$ and business time period $(T)$ must satisfy the condition

$$(1 - \beta)Pt_1 + \beta P\tau = DT$$  \hspace{1cm} (8.4)
8.3. MATHEMATICAL FORMULATION OF THE PROPOSED MODEL

Proof. From the continuity condition of \( q_0(t) \) at \( t = t_1 \), the following is obtained

\[
(P - D)t_1 + \beta \bar{P} \tau - \beta Pt_1 = -D(t_1 - T)
\]

i.e.,

\[
(1 - \beta)Pt_1 + \beta \bar{P} \tau = DT
\]

Hence the proof. \( \square \)

As \( q(t) \) is continuous for \( t \geq 0 \), the unique fuzzy solution of equation (8.1) according to Chalco-Cano and Roman-Flores [19] is given by

\[
\bar{q}_i(t) = \begin{cases} 
(P - \bar{D})t, & 0 \leq t \leq \bar{\tau} \\
(P - \bar{D})t + \bar{P} \bar{\tau} - \bar{P}t, & \bar{\tau} \leq t \leq t_1 \\
\bar{D}(T - t), & t_1 \leq t \leq T
\end{cases}
\]

with the condition \((1 - \beta)Pt_1 + \bar{\beta} \bar{P} \bar{\tau} = DT\).

**Theorem 8.1.** If \( \tilde{\tau} \) be a fuzzy number then (i) \( t \leq \tau_0^L \) for all \( \alpha \), provided that \( t \leq \bar{\tau} \).

(ii) \( t \geq \tau_0^R \) for all \( \alpha \), provided that \( t \geq \bar{\tau} \).

Proof. (i) Let \( \tilde{t} \) and \( \bar{\tau} \) be two fuzzy numbers and \( \tilde{t} \leq \bar{\tau} \). Then the \( \alpha \)-cuts of \( \tilde{t} \) and \( \tau \) must satisfy

\[
|t_0^L, t_0^R| \leq |\tau_0^L, \tau_0^R|
\]

which implies that \( t_0^L \leq \tau_0^L \) and \( t_0^R \leq \tau_0^R \).

But since \( \tilde{t} \) is precise so \( t_0^L = t_0^R \). Then \( t \leq \tau_0^L \). Hence the proof.

(ii) Let \( \tilde{\tau} \) and \( \bar{\tau} \) be two fuzzy numbers and \( \tilde{\tau} \geq \bar{\tau} \). Then \(|t_0^L, t_0^R| \geq |\tau_0^L, \tau_0^R|\), which implies that

\[
t_0^L \geq \tau_0^L \quad \text{and} \quad t_0^R \geq \tau_0^R.
\]

But since \( \tilde{\tau} \) is precise so \( t = t_0^L = t_0^R \). Then \( t \geq \tau_0^R \). Hence the proof. \( \square \)

Hence, \( \alpha \)-cut of above equation (8.5) is given by

\[
\bar{q}_i(t)[\alpha] = |q_i^L(\alpha, t), q_i^R(\alpha, t)|, \quad (8.6)
\]

where

\[
q_i^L(\alpha, t) = \begin{cases} 
(P - D^R_0)t, & 0 \leq t \leq \tau_0^L \\
(P - D^L_0)t + P_0^L \tau_0^L - P_0^L t, & \tau_0^L \leq t \leq t_1 \\
D^L_0(T - t), & t_1 \leq t \leq T
\end{cases}
\]

and

\[
q_i^R(\alpha, t) = \begin{cases} 
(P - D^R_0)t, & 0 \leq t \leq \tau_0^R \\
(P - D^L_0)t + P_0^R \tau_0^R - P_0^R t, & \tau_0^R \leq t \leq t_1 \\
D^R_0(T - t), & t_1 \leq t \leq T
\end{cases}
\]

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Lemma 8.2. In fuzzy environment manufacturer’s production time period \( t_1 \) and business time period \( T \) must satisfy the condition either

\[
\begin{align*}
&i) \left\{(1 - \beta_0^R)P + D_0^L - D_0^R \right\} t_1 + P \beta_0^L \tau_0 = D_0^R T \\
&or \quad ii) \left\{(1 - \beta_0^L)P + D_0^R - D_0^L \right\} t_1 + P \beta_0^R \tau_0 = D_0^R T.
\end{align*}
\]

Proof. From the continuity conditions of \( q_1^L(\alpha, t) \) and \( q_1^R(\alpha, t) \) at \( t = t_i \), the followings are obtained respectively

\[
\begin{align*}
i) (P - D_0^R)t_1 + P \beta_0^L \tau_0 - P \beta_0^R t_1 &= D_0^R (T - t_1) \\
\text{i.e., } \left\{(1 - \beta_0^R)P + D_0^L - D_0^R \right\} t_1 + P \beta_0^L \tau_0 &= D_0^R T
\end{align*}
\]

or, \( ii) (P - D_0^L)t_1 + P \beta_0^R \tau_0 - P \beta_0^L t_1 = D_0^R (T - t_1) \)

\[
\begin{align*}
\text{i.e., } \left\{(1 - \beta_0^L)P + D_0^R - D_0^L \right\} t_1 + P \beta_0^R \tau_0 &= D_0^R T
\end{align*}
\]

Now from Lemma 8.1, it is seen that there exist variabilities of \( t_1 \) and \( T \) for crisp value of \( \beta, \tau \) and \( D \). But in fuzzy environment, two relations are obtained. If \( t_1 \) and \( T \) satisfy both these two relations simultaneously, then there will be loss of variability of \( t_1 \) and \( T \). Therefore to maintain variabilities of \( t_1 \) and \( T \), they must satisfy either \( \left\{(1 - \beta_0^R)P + D_0^L - D_0^R \right\} t_1 + P \beta_0^L \tau_0 = D_0^R T \) or \( \left\{(1 - \beta_0^L)P + D_0^R - D_0^L \right\} t_1 + P \beta_0^R \tau_0 = D_0^R T \). Hence the proof. \( \square \)

### 8.3.2 Formulation for Imperfect Quality Items

At the end of the screening process, the imperfect quality items are sold as a single lot. The inventory level \( \tilde{q}_2(t) \) at time \( t \) satisfies the following differential equation:

\[
\frac{d\tilde{q}_2(t)}{dt} = \beta P, \quad \tilde{\tau} \leq t \leq t_1
\]

subject to the condition that, \( \tilde{q}_2[\tilde{\tau}] = 0 \).

According to Chalcó-Cano and Roman-Flores [19], first find out the solution \( q_2(t) \) of the crisp differential equation

\[
\frac{dq_2(t)}{dt} = \beta P, \quad \tau \leq t \leq t_1
\]

subject to the condition that, \( q_2[\tau] = 0 \).

The solution of the above differential equation is given by

\[
q_2(t) = \beta P(t - \tau), \quad \tau \leq t \leq t_1
\]

[19]
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As $\mathcal{C}(t)$ is continuous for each $t \geq 0$, the unique fuzzy solution (according to Chaleo-Cano and RomanFlores [19] of equation (8.7) is given by

$$\tilde{q}_2(t) = \tilde{y}_i(t - \tilde{\tau}), \quad \tilde{\tau} \leq t \leq t_1$$

(8.10)

Hence, $\alpha$-cut of above equation (8.10) is given by

$$\tilde{q}_2(t)[\alpha] = [q_2^L(\alpha, t), q_2^R(\alpha, t)]$$

(8.11)

where

$$q_2^L(\alpha, t) = P(\mathcal{C}(t - \tau^R_\alpha), \quad \tau^R_\alpha \leq t \leq t_1$$

(8.12)

$$q_2^R(\alpha, t) = P(\mathcal{C}(t - \tau^L_\alpha), \quad \tau^L_\alpha \leq t \leq t_1$$

(8.13)

8.3.3 The Profit Function of the Proposed Model

The total production cost ($PC$) in the production system during the cycle $(0, T)$ is given by

$$PC = c_0 \int_0^{t_1} P \, dt = c_0 P t_1$$

Total screening cost ($SC$) in the production system during the cycle $(0, T)$ is given by

$$SC = c_{sr} \int_0^{t_1} P \, dt = c_{sr} P t_1$$

The total set up cost in the production system during the cycle $(0, T) = A_{on}$

Theorem 8.2. Let $\tilde{f}(x)$ be a bounded and closed-fuzzy-valued function defined on the closed fuzzy real number system $(\mathbb{R}_R \times \mathbb{R}_R)$ and $\tilde{f}(x)$ be induced by $\hat{f}(x)$. Suppose that $\hat{b} \geq \hat{a}$ and there exists a fuzzy number $\tilde{\xi}$ such that $\hat{a} \preceq \tilde{\xi} \preceq \hat{b}$.

(i) If $\tilde{f}(x)$ is non-negative as well as $f_2^L(x)$ and $f_2^R(x)$ be Riemann-integrable $\alpha$-cut of $\tilde{f}(x)$ on $[a^L_\alpha, b^L_\alpha]$ and $[a^R_\alpha, b^R_\alpha]$ respectively for all $\alpha$ then

$$\left( \int_{\hat{a}}^{\hat{b}} \tilde{f}(x) \, dx \right)[\alpha] = \left[ \int_{a^L_\alpha}^{b^L_\alpha} f_2^L(x) \, dx + \int_{a^R_\alpha}^{b^R_\alpha} f_2^R(x) \, dx \right]$$

Proof. If there exists a fuzzy number $\tilde{\xi}$ such that $\hat{a} \preceq \tilde{\xi} \preceq \hat{b}$ and $\tilde{f}(x)$ is Riemann-integrable on $[\hat{a}, \hat{b}]$, for all $\alpha$ then

$$\left( \int_{\hat{a}}^{\hat{b}} \tilde{f}(x) \, dx \right)[\alpha] = \left( \int_{\tilde{\xi}}^{\hat{b}} \tilde{f}(x) \, dx + \int_{\tilde{\xi}}^{\hat{a}} \tilde{f}(x) \, dx \right)[\alpha]$$

(8.14)
Now, \( \left( \int_{\alpha}^{\beta} f(x) \, dx \right) [\alpha] = \left[ \int_{\alpha}^{\beta} f^L(x) \, dx \right] \left[ \int_{\alpha}^{\beta} f^R(x) \, dx \right] \)
and \( \left( \int_{\alpha}^{\beta} f(x) \, dx \right) [\alpha] = \left[ \int_{\alpha}^{\beta} f^L(x) \, dx \right] \left[ \int_{\alpha}^{\beta} f^R(x) \, dx \right] \)
Therefore,
\[
\left( \int_{\alpha}^{\beta} f(x) \, dx \right) [\alpha] + \left( \int_{\alpha}^{\beta} f(x) \, dx \right) [\alpha] = \left[ \int_{\alpha}^{\beta} f^L(x) \, dx \right] \left[ \int_{\alpha}^{\beta} f^R(x) \, dx \right] \\
+ \left[ \int_{\alpha}^{\beta} f^L(x) \, dx \right] \left[ \int_{\alpha}^{\beta} f^R(x) \, dx \right]
\]
\[
\Rightarrow \left( \int_{\alpha}^{\beta} f(x) \, dx \right) [\alpha] + \left( \int_{\alpha}^{\beta} f(x) \, dx \right) [\alpha] = \left[ \int_{\alpha}^{\beta} f^L(x) \, dx \right] \left[ \int_{\alpha}^{\beta} f^R(x) \, dx \right] \\
\quad + \left[ \int_{\alpha}^{\beta} f^L(x) \, dx \right] \left[ \int_{\alpha}^{\beta} f^R(x) \, dx \right]
\]
Hence the proof.

\[\square\]

**Theorem 8.3.** Let \( \tilde{f}(\tilde{x}) \) be a bounded and closed-fuzzy-valued function defined on the closed fuzzy real number system \( (\mathbb{R}_f, \sim) \) and \( \tilde{f}(\tilde{x}) \) be induced by \( \tilde{f}(\tilde{x}) \). Suppose that \( \alpha \geq b \) and there exists a fuzzy number \( \tilde{\xi} \) such that \( \alpha \leq \tilde{\xi} \leq b \).

(i) If \( \tilde{f}(\tilde{x}) \) is non-negative and \( f^L(x) \) and \( f^R(x) \) are Riemann-integrable \( \alpha \)-cut of \( \tilde{f}(\tilde{x}) \) on \( [\alpha, \tilde{\xi}^L] \) and \( [\tilde{\xi}^R, b] \), respectively, for all \( \alpha \) then
\[
\left( \int_{\alpha}^{\beta} \tilde{f}(\tilde{x}) \, d\tilde{x} \right) [\alpha] = \left[ \int_{\alpha}^{\beta} f^L(x) \, dx \right] \left[ \int_{\alpha}^{\beta} f^R(x) \, dx \right] + \left[ \int_{\alpha}^{\beta} f^L(x) \, dx \right] \left[ \int_{\alpha}^{\beta} f^R(x) \, dx \right]
\](8.15)

**Proof.** Similar proof of above theorem. \( \square \)

Now \( \alpha \)-cut of the total holding cost \( HC \) in the production system during the cycle \( (0, T) \) is given by
\[
HC[\alpha] = \left( h_m \int_{0}^{T} q^L(t, \alpha) \, dt + h_m \int_{0}^{T} q^R(t, \alpha) \, dt \right) [\alpha] \\
= \left[ h_m \int_{0}^{T} q^L(t, \alpha) \, dt + h_m \int_{0}^{T} q^R(t, \alpha) \, dt \right] \left[ h_m \int_{0}^{T} q^L(t, \alpha) \, dt + h_m \int_{0}^{T} q^R(t, \alpha) \, dt \right] \\
= \left[ HC^L_{\alpha}, HC^R_{\alpha} \right] (say).
\]

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where

\[ HC^L_\alpha = h_m \int_0^T q_1^L(t, \alpha) \, dt + h_m' \int_{\tau_R}^{\tau_1} q_2^L(t, \alpha) \, dt \]

\[ = h_m \left[ \int_0^{\tau_R} q_1^L(t, \alpha) \, dt + \int_{\tau_R}^{\tau_1} q_1^L(t, \alpha) \, dt + \int_{\tau_1}^T q_1^L(t, \alpha) \, dt \right] + h_m' \int_{\tau_R}^{\tau_1} q_2^L(t, \alpha) \, dt \]

\[ = \frac{h_m}{2} \left[ (P - D^R_\alpha) l^2_1 + (\tau^L_\alpha)^2 - (\tau^R_\alpha)^2 \right] + 2P \beta^R_\alpha \tau^R_\alpha (t_1 - \tau^R_\alpha) - P \beta^R_\alpha \{ l^2_1 - (\tau^L_\alpha)^2 \} + D^R_\alpha (T - t_1)^2 \]

\[ HC^R_\alpha = h_m \int_0^T q_1^R(t, \alpha) \, dt + h_m' \int_{t_1}^{\tau_1} q_2^R(t, \alpha) \, dt \]

\[ = h_m \left[ \int_0^{t_1} q_1^R(t, \alpha) \, dt + \int_{t_1}^{\tau_1} q_1^R(t, \alpha) \, dt + \int_{\tau_1}^T q_1^R(t, \alpha) \, dt \right] + h_m' \int_{t_1}^{\tau_1} q_2^R(t, \alpha) \, dt \]

\[ = \frac{h_m}{2} \left[ (P - D^R_\alpha) l^2_1 + (\tau^L_\alpha)^2 - (\tau^R_\alpha)^2 \right] + 2P \beta^R_\alpha \tau^R_\alpha (t_1 - \tau^L_\alpha) - P \beta^R_\alpha \{ l^2_1 - (\tau^L_\alpha)^2 \} + D^R_\alpha (T - t_1)^2 \]

The \( \alpha \)-cut of total revenue (\( SR \)) in the production system during the cycle \((0, T)\) is given by

\[ SR[\alpha] = \left( s \int_0^T \tilde{D} \, dt + s' P(t_1 - \tau^L_\alpha) \right) \{ \alpha \}
\]

\[ = \left[ s \int_0^T D^L_\alpha \, dt + s' \beta^L_\alpha P(t_1 - \tau^R_\alpha), s \int_0^T D^R_\alpha \, dt + s' \beta^R_\alpha P(t_1 - \tau^R_\alpha) \right] \]

\[ = \left[ s D^L_\alpha T + s' \beta^L_\alpha P(t_1 - \tau^R_\alpha), s D^R_\alpha T + s' \beta^R_\alpha P(t_1 - \tau^R_\alpha) \right] \]

\[ = [SR^L_\alpha, SR^R_\alpha] \text{ (say)}, \]

where

\[ SR^L_\alpha = s \int_0^T D^L_\alpha \, dt + s' P \beta^L_\alpha (t_1 - \tau^R_\alpha) = s D^L_\alpha T + s' P \beta^L_\alpha (t_1 - \tau^R_\alpha), \]

\[ SR^R_\alpha = s \int_0^T D^R_\alpha \, dt + s' P \beta^R_\alpha (t_1 - \tau^R_\alpha) = s D^R_\alpha T + s' P \beta^R_\alpha (t_1 - \tau^R_\alpha) \]

The \( \alpha \)-cut of the total Profit (\( TP \)) in the production system during the cycle \((0, T)\) is given by

\[ TP[\alpha] = [TP^L_\alpha(t_1, T), TP^R_\alpha(t_1, T)] \{ \text{say} \}, \]

where

\[ TP^L_\alpha(t_1, T) = \frac{1}{T} \left[ SR^L_\alpha - HC^R_\alpha - PC - SC - A_m \right], \]

\[ TP^R_\alpha(t_1, T) = \frac{1}{T} \left[ SR^R_\alpha - HC^L_\alpha - PC - SC - A_m \right] \]
Finally, the model becomes:

\[
\text{Max } TP^B_{B_0}(t_1, T) = \frac{1}{T} \left[ \frac{\bar{S}R_{\bar{\alpha}}^B - H C_{\bar{\alpha}}^B - PC - SC - A_m}{\bar{\alpha}} \right],
\]

\[
\text{Max } TP^L_{B_0}(t_3, T) = \frac{1}{T} \left[ \frac{\bar{S}R_{\bar{\alpha}}^L - H C_{\bar{\alpha}}^L - PC - SC - A_m}{\bar{\alpha}} \right]
\]

such that: \(0 < \tau^L_{\bar{\alpha}} < t_1, \quad 0 < t_1 < T\),

and \(\left\{ (1 - \sigma^L_{\bar{\alpha}}) P + D^L_{\bar{\alpha}} - D^R_{\bar{\alpha}} \right\} t_1 + P \sigma^L_{\bar{\alpha}} \tau^L_{\bar{\alpha}} = D^L_{\bar{\alpha}} T\)

or \(\left\{ (1 - \sigma^L_{\bar{\alpha}}) P + D^L_{\bar{\alpha}} - D^R_{\bar{\alpha}} \right\} t_1 + P \sigma^R_{\bar{\alpha}} \tau^R_{\bar{\alpha}} = D^R_{\bar{\alpha}} T\).

### 8.4 Solution Procedure

To get the optimum value of the production run time \((t_1)\), business period \((T)\) and average profit in the proposed model, the following steps are necessary.

**Step-1:** Input the suitable values of crisp and fuzzy parameters of \(TP^L_{B_0}(t_1, T)\) and \(TP^R_{B_0}(t_1, T)\).

**Step-2:** Compute the left \(\alpha\)-cut \((\tau^L_{\bar{\alpha}})\) and right \(\alpha\)-cut \((\tau^R_{\bar{\alpha}})\) of fuzzy parameter \(\bar{\tau}\) as follows:

(i) If \(\bar{\tau}\) be TFN such as \(\bar{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2)\) then \(\tau^L_{\bar{\alpha}} = (\tau_0 - \Delta_1 + \alpha \Delta_1)\) and \(\tau^R_{\bar{\alpha}} = (\tau_0 + \Delta_2 - \alpha \Delta_2)\).

(ii) If \(\bar{\tau}\) be TrFN such as \(\bar{\tau} = (\tau_0 - \Delta_1, \tau_0 - \Delta_2, \tau_0 + \Delta_3, \tau_0 + \Delta_4)\) then \(\tau^L_{\bar{\alpha}} = (\tau_0 - \Delta_1 + \alpha \Delta_1 - \Delta_2)\) and \(\tau^R_{\bar{\alpha}} = (\tau_0 + \Delta_3 - \alpha \Delta_3)\).

(iii) If \(\bar{\tau}\) be PFN such as \(\bar{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2)\) respectively then \(\tau^L_{\bar{\alpha}} = \tau_0 - \sqrt{\alpha} \Delta_1\) and \(\tau^R_{\bar{\alpha}} = \tau_0 + \sqrt{\alpha} \Delta_2\).

(iv) If \(\bar{\tau}\) be GFN such that \(\bar{\tau} = (\tau_0 - \Delta_1, \tau_0 - \Delta_2, \tau_0 + \Delta_3, \tau_0 + \Delta_4)\) then \(\tau^L_{\bar{\alpha}} = (\tau_0 - \Delta_1 + \sqrt{\alpha} \Delta_1 - \Delta_2)\) and \(\tau^R_{\bar{\alpha}} = (\tau_0 + \Delta_4 - \sqrt{\alpha} \Delta_4)\).

Similarly, compute left and right \(\alpha\)-cuts of other two fuzzy parameters \(\bar{\beta}\) and \(\bar{d}\) for TFN, TrFN, PFN and GFN.

**Step-3:** Maximize the profit \(TP^B_{B_0}(t_1, T)\) and obtain the optimal values of \(t_1, T, TP^L_{B_0}(t_1, T)\) and \(TP^R_{B_0}(t_1, T)\) for different values of \(\alpha\) using the standard LINGO software.

**Step-4:** Maximize the profit \(TP^R_{B_0}(t_1, T)\) and obtain the optimal values of \(t_1, T, TP^L_{B_0}(t_1, T)\) and \(TP^R_{B_0}(t_1, T)\) for different values of \(\alpha\) using the standard LINGO software.

**Step-5:** Maximize the both \(TP^L_{B_0}(t_1, T)\) and \(TP^R_{B_0}\) by Fuzzy Programming Technique (FPT) and obtain the optimal values of \(t_1, T\) and average profit for different values of \(\alpha\) using the standard LINGO software as follows:

**Step-6:** From the results of step-3 and step-4, the following pay off matrix can be constructed:

\[
\begin{pmatrix}
TP^B_{B_0}(t_1, T) & TP^R_{B_0}(t_1, T) \\
TP^L_{B_0}(t_1, T) & TP^L_{B_0}(t_1, T)
\end{pmatrix}
\]

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8.5. Numerical Illustrations

Step-7: From this payoff matrix, two values $U_j$ and $L_j$ are defined such that they are the upper and lower bounds of the $j$-th objective for each $j=1, 2$ respectively. Here, $L_j$ is higher acceptable level of achievement, $U_j$ is aspired level of achievement for maximization, which are computed as follows:

$$U_1 = \max\{TP^{L_i}_\alpha(t_1, T)\}, \quad U_2 = \max\{TP^{R_i}_\alpha(t_1, T)\}$$
$$L_1 = \min\{TP^{L_i}_\alpha(t_1, T)\}, \quad L_2 = \min\{TP^{R_i}_\alpha(t_1, T)\}$$

Step-8: Then the membership functions $\mu_1(TP^{L_i}_\alpha(t_1, T))$ and $\mu_2(TP^{R_i}_\alpha(t_1, T))$ corresponding to the objective functions of $TP^{L_i}_\alpha(t_1, T)$ and $TP^{R_i}_\alpha(t_1, T)$ are constructed linearly as follows:

$$\mu_1(TP^{L_i}_\alpha(t_1, T)) = \begin{cases} 0, & \text{if } TP^{L_i}_\alpha(t_1, T) \leq L_1 \\ \frac{TP^{L_i}_\alpha(t_1, T) - L_1}{U_1 - L_1}, & \text{if } L_1 \leq TP^{L_i}_\alpha(t_1, T) \leq U_1 \\ 1, & \text{if } TP^{L_i}_\alpha(t_1, T) \geq U_1 \end{cases}$$

$$\mu_2(TP^{R_i}_\alpha(t_1, T)) = \begin{cases} 0, & \text{if } TP^{R_i}_\alpha(t_1, T) \leq L_2 \\ \frac{TP^{R_i}_\alpha(t_1, T) - L_2}{U_2 - L_2}, & \text{if } L_2 \leq TP^{R_i}_\alpha(t_1, T) \leq U_2 \\ 1, & \text{if } TP^{R_i}_\alpha(t_1, T) \geq U_2 \end{cases}$$

Step-9: Finally, according to the Zimmermann [244] method, the multi-objective programming problem is reduced to the following single objective programming problem:

Max $\lambda$

such that

$\mu_1(TP^{L_i}_\alpha(t_1, T)) \geq \lambda,$

$\mu_2(TP^{R_i}_\alpha(t_1, T)) \geq \lambda,$

$0 < \tau^{R_i}_\alpha < t_1, \quad 0 < t_1 < T, \quad \lambda \in [0, 1],$$

and $\left\{(1 - \beta^{L_i}_\alpha)P + D^{L_i}_\alpha - D^{R_i}_\alpha\right\}t_1 + P\beta^{L_i}_\alpha\tau^{L_i}_\alpha = D^{L_i}_\alpha T$

or $\left\{(1 - \beta^{R_i}_\alpha)P + D^{R_i}_\alpha - D^{L_i}_\alpha\right\}t_1 + P\beta^{R_i}_\alpha\tau^{R_i}_\alpha = D^{R_i}_\alpha T.$

8.5 Numerical Illustrations

To illustrate numerically the proposed model, according to Lemma 8.2 we have two relations

$$\left\{(1 - \beta^{L_i}_\alpha)P + D^{L_i}_\alpha - D^{R_i}_\alpha\right\}t_1 + P\beta^{L_i}_\alpha\tau^{L_i}_\alpha = D^{L_i}_\alpha T$$

and

$$\left\{(1 - \beta^{R_i}_\alpha)P + D^{R_i}_\alpha - D^{L_i}_\alpha\right\}t_1 + P\beta^{R_i}_\alpha\tau^{R_i}_\alpha = D^{R_i}_\alpha T$$
one of which is taken to compute \( P_1 \) and \( T \) optimizing objective functions. At first considering the relation

\[
\left\{ (1 - \beta_0^P)P + D_0^P - D_0^T \right\}t_1 + P_1 \beta_0^R \gamma_0^R = D_0^RT
\]

the proposed model is optimized for the following examples and then the other relation

\[
\left\{ (1 - \beta_0^P)P + D_0^P - D_0^T \right\}t_1 + P_1 \beta_0^R \gamma_0^R = D_0^RT
\]

has been considered to get the optimum solution but in this case latter it has been shown that this relation is not acceptable due to some in-feasibility of the solution.

**Example 8.1.** The following parametric values are used to illustrate the model:

\[ C_p = 830, C_{sr} = 82, A_{m} = 85200, s = 859, s' = 835, h_{m} = 8100, h'_{m} = 80.5, P = 3380. \]

Here, fuzzy parameters are considered as triangular fuzzy number (TFN) and their different values are given below. \( \tilde{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2) = (2,3,5), \Delta_1 = 1, \Delta_2 = 0.5, \tilde{\beta} = (\beta_1 - \sigma_1, \beta_1, \beta_1 + \sigma_2) = (0.07, 0.10, 0.15), \sigma_1 = 0.03, \sigma_2 = 0.05, \tilde{D} = (D_1 - \rho_1, D_0, D_0 + \rho_2) = (2500, 2550, 2631), \rho_1 = 59, \rho_2 = 72. \] This example is solved using LINGO-12.0. Table 8.2, 8.3 represent the optimum results when \( TP_0^P \), \( TP_0^R \) are maximized separately and Table 8.4 represent the optimum results when \( TP_0^P \) and \( TP_0^R \) are maximized simultaneously. For the above parametric values, the optimum business time period \( (T^*) \), optimum production time period \( (t_1^*) \), and profit interval \( ([TP_0^P, TP_0^R]) \) are obtained. Obtained results for different values of \( \alpha \) are presented in Table 8.2, 8.3 and 8.4.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Production period ( (t_1^*) )</th>
<th>Business period ( (T^*) )</th>
<th>( TP_0^P ) (Max)</th>
<th>( TP_0^R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>8.445339</td>
<td>9.452128</td>
<td>47447.09</td>
<td>69007.36</td>
</tr>
<tr>
<td>0.25</td>
<td>7.884611</td>
<td>9.066641</td>
<td>52386.18</td>
<td>68234.01</td>
</tr>
<tr>
<td>0.50</td>
<td>7.100508</td>
<td>8.402024</td>
<td>57183.69</td>
<td>67540.85</td>
</tr>
<tr>
<td>0.75</td>
<td>5.982998</td>
<td>7.315060</td>
<td>61904.18</td>
<td>66988.16</td>
</tr>
<tr>
<td>0.99</td>
<td>4.345648</td>
<td>5.555799</td>
<td>66499.80</td>
<td>66701.16</td>
</tr>
</tbody>
</table>
Table 8.3: Optimum results of Example 8.1 for maximizing $TP^L_\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Production period ($t^*_1$)</th>
<th>Business period ($T^*$)</th>
<th>$TP^L_\alpha$</th>
<th>$TP^R_\alpha$ (Max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>10.71312</td>
<td>11.93943</td>
<td>47248.61</td>
<td>69111.75</td>
</tr>
<tr>
<td>0.25</td>
<td>9.064714</td>
<td>10.38858</td>
<td>52327.64</td>
<td>68270.08</td>
</tr>
<tr>
<td>0.50</td>
<td>7.526072</td>
<td>8.888575</td>
<td>57175.25</td>
<td>67546.87</td>
</tr>
<tr>
<td>0.75</td>
<td>5.976583</td>
<td>7.307580</td>
<td>61904.18</td>
<td>66988.16</td>
</tr>
<tr>
<td>0.99</td>
<td>4.332328</td>
<td>5.539977</td>
<td>66499.78</td>
<td>66701.17</td>
</tr>
</tbody>
</table>

Table 8.4: Optimum results of Example 8.1 for maximizing both $TP^L_\alpha$ and $TP^R_\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Production period ($t^*_1$)</th>
<th>Business period ($T^*$)</th>
<th>$TP^L_\alpha$ (Max)</th>
<th>$TP^R_\alpha$ (Max)</th>
<th>Average profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>9.513103</td>
<td>10.62325</td>
<td>47397.64</td>
<td>69085.74</td>
<td>58241.69</td>
</tr>
<tr>
<td>0.25</td>
<td>8.454594</td>
<td>9.705132</td>
<td>52371.56</td>
<td>68261.07</td>
<td>60316.32</td>
</tr>
<tr>
<td>0.50</td>
<td>7.310260</td>
<td>8.641835</td>
<td>57181.58</td>
<td>67545.36</td>
<td>62363.47</td>
</tr>
<tr>
<td>0.75</td>
<td>5.982997</td>
<td>7.315060</td>
<td>61904.18</td>
<td>66988.16</td>
<td>64446.17</td>
</tr>
<tr>
<td>0.99</td>
<td>4.339744</td>
<td>5.548787</td>
<td>66499.79</td>
<td>66701.17</td>
<td>66600.48</td>
</tr>
</tbody>
</table>

As expected, the left and right optimum profits increases and decreases respectively with the increase of $\alpha$. At $\alpha = 0.99$, the above profit values are almost same.

Example 8.2. Here, fuzzy parameters are considered as trapezoidal fuzzy number (TrFN) and their different values are taken as:

$$\tilde{\tau} = (\tau_0 - \Delta_1, \tau_0 - \Delta_2, \tau_0 + \Delta_3, \tau_0 + \Delta_4) = (2, 2.5, 3.5, 5), \Delta_1 = 1.25, \Delta_2 = 0.75, \Delta_3 = 0.25, \Delta_4 = 1.75, \tilde{\beta} = (\beta_0 - \sigma_1, \beta_0 - \sigma_2, \beta_0 + \sigma_3, \beta_0 + \sigma_4) = (0.07, 0.10, 0.13, 0.15), \sigma_1 = 0.05, \sigma_2 = 0.02, \sigma_3 = 0.01, \sigma_4 = 0.03, \tilde{D} = (d_0 - \rho_1, d_0 - \rho_2, d_0 + \rho_3, d_0 + \rho_4) = (2500, 2549, 2571, 2631), \rho_1 = 65, \rho_2 = 16, \rho_3 = 6, \rho_4 = 66. Other parametric values are same as in Example 8.1. This example is solved using LINGO-12.0. Table 8.5, 8.6 represent the optimum results when $TP^L_\alpha, TP^R_\alpha$ are maximized separately and Table 8.7 represent the optimum results when $TP^L_\alpha$ and $TP^R_\alpha$ are maximized simultaneously.
### Table 8.5: Optimum results of Example 8.2 for maximizing $TP_\alpha^{P*}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Production period ($t^*_P$)</th>
<th>Business period ($t^*_B$)</th>
<th>$TP_\alpha^{P*}$ (Max)</th>
<th>$TP_\alpha^{R*}$ (Max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>8.445339</td>
<td>9.452128</td>
<td>47447.09</td>
<td>69007.36</td>
</tr>
<tr>
<td>0.25</td>
<td>8.211077</td>
<td>9.327867</td>
<td>50801.53</td>
<td>68435.14</td>
</tr>
<tr>
<td>0.50</td>
<td>7.869975</td>
<td>9.079349</td>
<td>54132.44</td>
<td>67864.09</td>
</tr>
<tr>
<td>0.75</td>
<td>7.394903</td>
<td>8.672050</td>
<td>57450.46</td>
<td>67305.73</td>
</tr>
<tr>
<td>0.99</td>
<td>6.772804</td>
<td>8.081556</td>
<td>60639.97</td>
<td>66795.06</td>
</tr>
</tbody>
</table>

### Table 8.6: Optimum results of Example 8.2 for maximizing $TP_\alpha^{R*}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Production period ($t^*_P$)</th>
<th>Business period ($t^*_B$)</th>
<th>$TP_\alpha^{R*}$ (Max)</th>
<th>$TP_\alpha^{R*}$ (Max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>10.71312</td>
<td>11.93943</td>
<td>47248.61</td>
<td>69111.75</td>
</tr>
<tr>
<td>0.25</td>
<td>9.666378</td>
<td>10.94183</td>
<td>50716.17</td>
<td>68484.49</td>
</tr>
<tr>
<td>0.50</td>
<td>8.696047</td>
<td>10.00548</td>
<td>54103.70</td>
<td>67882.37</td>
</tr>
<tr>
<td>0.75</td>
<td>7.772620</td>
<td>9.100049</td>
<td>57444.15</td>
<td>67310.16</td>
</tr>
<tr>
<td>0.99</td>
<td>6.907287</td>
<td>8.235475</td>
<td>60639.13</td>
<td>66795.72</td>
</tr>
</tbody>
</table>

### Table 8.7: Optimum results of Example 8.2 for maximizing both $TP_\alpha^{P*}$ and $TP_\alpha^{R*}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Production period ($t^*_P$)</th>
<th>Business period ($t^*_B$)</th>
<th>$TP_\alpha^{P*}$ (Max)</th>
<th>$TP_\alpha^{R*}$ (Max)</th>
<th>Average profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>9.513103</td>
<td>10.62325</td>
<td>47397.64</td>
<td>69085.74</td>
<td>58241.69</td>
</tr>
<tr>
<td>0.25</td>
<td>8.909673</td>
<td>10.10263</td>
<td>50780.23</td>
<td>68472.17</td>
<td>59626.20</td>
</tr>
<tr>
<td>0.50</td>
<td>8.273025</td>
<td>9.531221</td>
<td>54125.26</td>
<td>67877.80</td>
<td>61001.53</td>
</tr>
<tr>
<td>0.75</td>
<td>7.581600</td>
<td>8.883601</td>
<td>57448.88</td>
<td>67309.05</td>
<td>62378.97</td>
</tr>
<tr>
<td>0.99</td>
<td>6.840087</td>
<td>8.158563</td>
<td>60639.76</td>
<td>66795.55</td>
<td>63717.66</td>
</tr>
</tbody>
</table>

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8.5. NUMERICAL ILLUSTRATIONS

For the above parametric values, the optimum business time period \( (T^b) \), optimum production time period \( (t^*_p) \), and profit interval \( (TP^L_\alpha, TP^R_\alpha) \) are obtained. Obtained results for different values of \( \alpha \) are presented in Table 8.5, 8.6, and 8.7. As expected, the left and right optimum profits increases and decreases respectively with the increase of \( \alpha \).

Example 8.3. In this example fuzzy parameters are considered as parabolic fuzzy number (PFN) type and their different values are:

\[
\tilde{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2) = (2, 3, 5), \Delta_1 = 1, \Delta_2 = 2, \tilde{\omega} = (\beta_0 - \sigma_1, \beta_0, \beta_0 + \sigma_2) = (0.07, 0.10, 0.15), \sigma_1 = 0.03, \sigma_2 = 0.05, \tilde{D} = (D_0 - \rho_1, D_0, D_0 + \rho_2) = (2500, 2559, 2631), \rho_1 = 50, \rho_2 = 72. \]

Other parametric values are same as in Example 8.1. This example is solved using LINGO-12.0. Table 8.8 and 8.9 represent the optimum results when \( TP^L_\alpha, TP^R_\alpha \) are maximized separately and Table 8.10 represent the optimum results when \( TP^L_\alpha, TP^R_\alpha \) are maximized simultaneously.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Production period ( (t^*_p) )</th>
<th>Business period ( (T^b) )</th>
<th>( TP^L_\alpha ) (Max)</th>
<th>( TP^R_\alpha ) (Max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>8.445339</td>
<td>9.452128</td>
<td>47447.09</td>
<td>69007.36</td>
</tr>
<tr>
<td>0.25</td>
<td>8.168350</td>
<td>9.274599</td>
<td>50114.31</td>
<td>68585.50</td>
</tr>
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<td>0.50</td>
<td>7.768098</td>
<td>8.975275</td>
<td>53218.01</td>
<td>68108.13</td>
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<td>0.75</td>
<td>7.100508</td>
<td>8.402024</td>
<td>57183.69</td>
<td>67540.85</td>
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<td>0.99</td>
<td>5.055331</td>
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<td>64752.49</td>
<td>66769.40</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Production period ( (t^*_p) )</th>
<th>Business period ( (T^b) )</th>
<th>( TP^L_\alpha ) (Max)</th>
<th>( TP^R_\alpha ) (Max)</th>
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</thead>
<tbody>
<tr>
<td>0.00</td>
<td>10.71312</td>
<td>11.93943</td>
<td>47248.61</td>
<td>69111.75</td>
</tr>
<tr>
<td>0.25</td>
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<tr>
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<td>8.888575</td>
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<td>67546.87</td>
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<tr>
<td>0.99</td>
<td>4.980989</td>
<td>6.248691</td>
<td>64752.15</td>
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</tr>
</tbody>
</table>
Table 8.10: Optimum results of Example 8.3 for maximizing both \( TP_{\alpha}^{L} \) and \( TP_{\alpha}^{R} \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Production period ( (t_{p}) )</th>
<th>Business period ( (T^{*}) )</th>
<th>( TP_{\alpha}^{L} ) (Max)</th>
<th>( TP_{\alpha}^{R} ) (Max)</th>
<th>Average profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>9.513103</td>
<td>10.62325</td>
<td>47397.64</td>
<td>69085.74</td>
<td>58241.69</td>
</tr>
<tr>
<td>0.25</td>
<td>8.951570</td>
<td>10.14348</td>
<td>50087.23</td>
<td>68632.21</td>
<td>59359.72</td>
</tr>
<tr>
<td>0.50</td>
<td>8.266637</td>
<td>9.535721</td>
<td>53206.74</td>
<td>68129.54</td>
<td>60668.14</td>
</tr>
<tr>
<td>0.75</td>
<td>7.310260</td>
<td>8.641835</td>
<td>57181.58</td>
<td>67545.36</td>
<td>62363.47</td>
</tr>
<tr>
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<td>64752.40</td>
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<td>65761.02</td>
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</table>

For the above parametric values, the optimum business time period \( (T^{*}) \), optimum production time period \( (t_{p}) \), and profit interval \( (TP_{\alpha}^{L}, TP_{\alpha}^{R}) \) are obtained. Obtained results for different values of \( \alpha \) are presented in Table 8.8, 8.9 and 8.10. As expected, the left and right optimum profits increases and decreases respectively with the increase of \( \alpha \). At \( \alpha = 0.99 \), the above profit values are almost same.

**Example 8.4.** Here, fuzzy parameters are considered as general fuzzy number (GFN) and their different values are given.

\[ \tilde{\tau} = (\tau_{0} - \Delta_{1}, \tau_{0} - \Delta_{2}, \tau_{0} + \Delta_{3}, \tau_{0} + \Delta_{4}) = (2.2, 3.5, 5), \Delta_{1} = 1, \Delta_{2} = 0.25, \Delta_{3} = 0.75, \Delta_{4} = 2, \tilde{\beta} = (\beta_{0} - \sigma_{1}, \beta_{0} - \sigma_{2}, \beta_{0} + \sigma_{3}, \beta_{0} + \sigma_{4}) = (0.07, 0.10, 0.13, 0.15), \sigma_{1} = 0.05, \sigma_{2} = 0.02, \sigma_{3} = 0.01, \sigma_{4} = 0.03, \tilde{D} = (d_{0} - \rho_{1}, d_{0} - \rho_{2}, d_{0} + \rho_{3}, d_{0} + \rho_{4}) = (2500, 2510, 2571, 2631), \rho_{1} = 65, \rho_{2} = 16, \rho_{3} = 6, \rho_{4} = 66. \] Other parametric values are same as in Example-1. This example is solved using LINGO-12.0. Table 8.11, 8.12 represent the optimum results when \( TP_{\alpha}^{L}, TP_{\alpha}^{R} \) are maximized separately and Table 8.13 represent the optimum results when \( TP_{\alpha}^{L}, TP_{\alpha}^{R} \) are maximized simultaneously.

For the above parametric values, the optimum business time period \( (T^{*}) \), optimum production time period \( (t_{p}) \), and profit interval \( (TP_{\alpha}^{L}, TP_{\alpha}^{R}) \) are obtained. Obtained results for different values of \( \alpha \) are presented in Table 8.11, 8.12 and 8.13.
### Table 8.11: Optimum results of Example 8.4 for maximizing $\Pi_{\alpha}^{L}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Production period ($t^*_1$)</th>
<th>Business period ($T^*$)</th>
<th>$\Pi_{\alpha}^{L}$ (Max)</th>
<th>$\Pi_{\alpha}^{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>8.445339</td>
<td>9.452128</td>
<td>47447.09</td>
<td>69007.36</td>
</tr>
<tr>
<td>0.25</td>
<td>7.869975</td>
<td>9.079349</td>
<td>54132.44</td>
<td>67864.09</td>
</tr>
<tr>
<td>0.50</td>
<td>7.487523</td>
<td>8.755217</td>
<td>56881.46</td>
<td>67400.04</td>
</tr>
<tr>
<td>0.75</td>
<td>7.117747</td>
<td>8.415065</td>
<td>58990.51</td>
<td>67054.82</td>
</tr>
<tr>
<td>0.99</td>
<td>6.757742</td>
<td>8.066704</td>
<td>60706.47</td>
<td>66784.82</td>
</tr>
</tbody>
</table>

### Table 8.12: Optimum results of Example 8.4 for maximizing $\Pi_{\alpha}^{R}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Production period ($t^*_1$)</th>
<th>Business period ($T^*$)</th>
<th>$\Pi_{\alpha}^{L}$ (Max)</th>
<th>$\Pi_{\alpha}^{R}$ (Max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>10.71312</td>
<td>11.93943</td>
<td>47248.61</td>
<td>69111.75</td>
</tr>
<tr>
<td>0.25</td>
<td>8.696047</td>
<td>10.00548</td>
<td>54103.70</td>
<td>67882.37</td>
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<td>0.50</td>
<td>7.928822</td>
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<td>67405.93</td>
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<tr>
<td>0.75</td>
<td>7.353021</td>
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<td>67056.67</td>
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<td>6.889372</td>
<td>8.217389</td>
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</tbody>
</table>

### Table 8.13: Optimum results of Example 8.4 for maximizing both $\Pi_{\alpha}^{L}$ and $\Pi_{\alpha}^{R}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Production period ($t^*_1$)</th>
<th>Business period ($T^*$)</th>
<th>$\Pi_{\alpha}^{L}$ (Max)</th>
<th>$\Pi_{\alpha}^{R}$ (Max)</th>
<th>Average profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>9.513103</td>
<td>10.62325</td>
<td>47397.64</td>
<td>69085.74</td>
<td>58241.69</td>
</tr>
<tr>
<td>0.25</td>
<td>8.273025</td>
<td>9.531221</td>
<td>54125.26</td>
<td>67877.80</td>
<td>61001.53</td>
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<tr>
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<td>67404.46</td>
<td>62141.89</td>
</tr>
<tr>
<td>0.75</td>
<td>7.234143</td>
<td>8.547598</td>
<td>58989.88</td>
<td>67056.21</td>
<td>63023.05</td>
</tr>
<tr>
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<td>6.823119</td>
<td>8.141546</td>
<td>60706.27</td>
<td>66785.29</td>
<td>63745.78</td>
</tr>
</tbody>
</table>
CHAPTER 8. A FUZZY IMPERFECT PRODUCTION INVENTORY MODEL BASED ON FUZZY DIFFERENTIAL AND FUZZY INTEGRAL METHOD

As expected, the left and right optimum profits increases and decreases respectively with the increase of $\alpha$.

**Example 8.5.** When \( (1 - \beta) P + D^R - D^L \) \( t_1 + P \beta^R P^R = D^R T \), the following parametric values are used to illustrate the model:

\[
C_P = 8.30, \quad C_{sr} = 82, \quad A_m = 85200, \quad s = 85, \quad s' = 835, \quad h_m = 81, \quad h_m' = 80.50, \quad P = 3380.
\]

Here, fuzzy parameters are considered as triangular fuzzy number (TFN) and their different values are given below. \( \tilde{F} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2) = (2, 3, 5), \Delta_1 = 1, \Delta_2 = 2, \tilde{\beta} = (\beta_1 - \sigma_1, \beta_0, \beta_0 + \sigma_2) = (0.07, 0.10, 0.15), \sigma_1 = 0.03, \sigma_2 = 0.05, \tilde{D} = (D_0 - \rho_1, D_0, D_0 + \rho_2) = (2500, 2559, 2631), \rho_1 = 59, \rho_2 = 72. \) This example is solved using LINGO-12.0. Table 8.14, 8.15 represent the optimum results when \( TP_\alpha^L, TP_\alpha^R \) are maximized separately.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Production period ($t_1$)</th>
<th>Business period ($T$)</th>
<th>( TP_\alpha^L )</th>
<th>( TP_\alpha^R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.904205</td>
<td>3.333382</td>
<td>72465.08</td>
<td>91202.27</td>
</tr>
<tr>
<td>0.25</td>
<td>1.992095</td>
<td>3.252411</td>
<td>70387.39</td>
<td>84903.91</td>
</tr>
<tr>
<td>0.50</td>
<td>1.834233</td>
<td>2.883574</td>
<td>68885.25</td>
<td>78970.41</td>
</tr>
<tr>
<td>0.75</td>
<td>1.735770</td>
<td>2.604554</td>
<td>67219.03</td>
<td>72486.40</td>
</tr>
<tr>
<td>0.99</td>
<td>4.184291</td>
<td>5.377229</td>
<td>66696.28</td>
<td>66897.66</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Production period ($t_1$)</th>
<th>Business period ($T$)</th>
<th>( TP_\alpha^R )</th>
<th>( TP_\alpha^L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.904205</td>
<td>3.333382</td>
<td>72465.08</td>
<td>91202.27</td>
</tr>
<tr>
<td>0.25</td>
<td>1.992095</td>
<td>3.252411</td>
<td>70387.39</td>
<td>84903.91</td>
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<tr>
<td>0.50</td>
<td>1.834233</td>
<td>2.883574</td>
<td>68885.25</td>
<td>78970.41</td>
</tr>
<tr>
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<td>1.468407</td>
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<td>72582.51</td>
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<td>5.359706</td>
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<td>66897.67</td>
</tr>
</tbody>
</table>

Now from Table 8.14 and 8.15, it is observed that when $\alpha = 0.00$ then the optimum profit interval is \( [TP_\alpha^L, TP_\alpha^R] = [72465.08, 91202.27] \). Again when $\alpha = 0.25$ then the optimum
profit interval is \([TP_{10}^L, TP_{10}^H] = [70387.39, 84903.91]\). But here \([TP_{125}^L, TP_{125}^H] \supset \[TP_{100}^L, TP_{100}^H]\) does not satisfied. This infeasibility is also shown when \(t\) further is increasing. Therefore this relation has no roll to give the optimum solution of the model.

8.5.1 Comparison between the Optimum Average Profit by Fisher’s t-test

Comparison between the optimum average profit due to TFN and TrFN by Fisher’s t-test. In the fuzzy EPQ model, two optimum average profit have been obtained using TFN and TrFN. Now question is that does there exist any significance difference between these two values? If exists, then how much? To get this answer, it can be tested that the null hypothesis \(H_0\): 
\[\overline{AP}_{TFN} \text{ (mean of values of average profit for TFN)} = \overline{AP}_{TrFN} \text{ (mean of values of average profit for TrFN)}\]

against the alternative hypothesis \(H_1\): \(\overline{AP}_{TFN} \neq \overline{AP}_{TrFN}\) on the basis of the results presented in Tables 8.4 and 8.7. This hypothesis can be tested using t-distribution. The test statistic is
\[t = \frac{\overline{AP}_{TrFN} - \overline{AP}_{TFN}}{s \sqrt{(1/n_1) + (1/n_2)}}\]
which follows t-distribution with \((n_1 + n_2 - 2)\) degrees of freedom, where
\[s^2 = \frac{n_1 s_{TFN}^2 + n_2 s_{TrFN}^2}{n_1 + n_2 - 2}\]

Here \(n_1 = 5, n_2 = 5, \overline{AP}_{TFN} = 62393.63, \overline{AP}_{TrFN} = 60993.21, s_{TFN}^2 = 10866309.90, s_{TrFN}^2 = 4695651.613\). Therefore, degrees of freedom \((n_1 + n_2 - 2) = 8\) and value of \(t = 0.2840\). Since the evaluated value of \(t\) < the tabulated value of \(t_{0.05}\), we accept the null hypothesis \(H_0\) with 95% confidence limit and conclude that there is no significant difference between the mean average profit (\(AP\)) for TFN and TrFN.

8.5.2 Sensitivity Analysis

To study the sensitivity analysis of the proposed model with respect to key parameters, the Example 8.1 has been considered. The optimum results of the model with the changes in the parameters \(\Delta_1, \Delta_2, \sigma_1, \sigma_2, \rho_1, \rho_2, P\) and \(s\) are given in Table 8.16 taking \(\alpha = 0.5\).
### Table 8.16: Sensitivity analysis on Example 8.1 w.r.t. $\Delta_1$, $\Delta_2$, $\sigma_1$, $\sigma_2$, $\rho_1$, $\rho_2$, $P$ and $S$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>value</th>
<th>Production period $d_f$</th>
<th>Production period (T)</th>
<th>$TP_1$</th>
<th>$TP_2$</th>
<th>Average profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_1$</td>
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<td>1.09</td>
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<td>67548.36</td>
<td>62363.47</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
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<td>62363.47</td>
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<tr>
<td>$s$</td>
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<td>8.641835</td>
<td>44534.88</td>
<td>54570.36</td>
<td>49552.22</td>
</tr>
<tr>
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<td>67548.36</td>
<td>62363.47</td>
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<td>8.641835</td>
<td>69829.08</td>
<td>80529.30</td>
<td>64188.79</td>
</tr>
</tbody>
</table>

Now, from Table 8.16 the following features of the proposed model have been observed:
- When $\Delta_1$ increases, the production run time and business period also increase. But, the average profit decreases with increasing of $\Delta_1$.
- When $\Delta_2$ increases, the production run time and business period increase. But, the average profit decreases with increasing of $\Delta_2$.
- When $\sigma_1$ increases, the production run time and business period also increase. But, the
average profit decreases with increasing of $\sigma_1$.

- When $\sigma_2$ increases, the production run time and business period also increase. But, the average profit decreases with increasing of $\sigma_2$.
- When $\rho_1$ increases, the production run time and business period also decrease. But, the average profit increases with increasing of $\rho_1$.
- When $\rho_2$ increases, the production run time and business period also increases. But, average profit increases with increasing of $\rho_2$.
- When the production rate $\langle P \rangle$ increases, the inventory increases as well as the production run time and business period reduce. But, the average profit increases with the production rate.
- When selling price of perfect item per unit $\langle s \rangle$ increases, the production run time and business period are not change. But, the average profit increases with the increasing of selling price $\langle s \rangle$.

### 8.5.3 Discussion

The optimum results of the proposed model are obtained from Table 8.4, 8.7, 8.10 and 8.13 when the fuzzy numbers $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{D}$ have been considered as TFN, TrFN, PFN and GFN in Example 8.1, 8.2, 8.3 and 8.4 respectively. From these tables it is shown that when $\alpha$ increases then $\left(\frac{\Delta T \bar{I}}{\Delta \alpha}\right)$ also increases but $\left(\frac{\Delta T \bar{I}}{\Delta \alpha}\right)$ decreases. Again, from Figure 8.2, it is observed that the rate of increase left optimum profit $\left(\frac{\Delta T \bar{I}}{\Delta \alpha}\right)$ is more than rate of decrease of right optimum profit $\left(\frac{\Delta T \bar{I}}{\Delta \alpha}\right)$. Hence forth, the average profit $\langle AP \rangle$ i.e., $\frac{1}{2}(TP_{\alpha}^L + TP_{\alpha}^R)$ is not same for all $\alpha$. Ultimately, the average profit rises with the increase of $\alpha$.

![Graphical representation of average profit with \( \alpha \)](image)

Figure 8.2: Graphical representation of average profit with $\alpha$

From this Figure 8.2 it is also observed that the TFNs gives the maximum average profit among others, i.e., the ordering of optimum average profit is $AP_{TFN} \leq AP_{TrFN} \leq AP_{PFN} \leq AP_{GFN}$ where $AP_{FN}$ indicates the optimum average profit for fuzzy number $FN$, though all fuzzy numbers have same spread.
8.6 Conclusion

The main contribution of this chapter is to develop a FEPQ model with fuzzy demand for perfect quality items with inspection of imperfect items. In the production run-time, the manufacturing system produces perfect items in the "in-control" state. Many research papers relating to the manufacturing process may shift to an ‘out-of-control’ state after certain time that follows constant/random. But, first time a fuzzy production inventory model has been developed, where the manufacturing process may shift to an ‘out-of-control’ state after certain time that follows a fuzzy number. During ‘out-of-control’ state, the process starts to produce defective items. The defective rate is considered as fuzzy number. Using fuzzy differential equation and fuzzy Riemann integration, an approach has been proposed, where α-cut of fuzzy profit is optimized to get optimal decision. It proposes a strategy for maximizing profit in an fuzzy imperfect production. The model is optimized for the production run time($t_1$), business period($T_2$) and profit interval. Here, four fuzzy numbers TFN, TriFN, PFN, and GNF have been used to illustrate the model for fuzzy parameters. Finally, Fuzzy Programming Technique(FPT) has been used to get the optimal solution. These models are applicable in the factory like steel, plastic etc. The present models can be extended to the rough, fuzzy-rough, random, fuzzy-random environment taking constant part of screening cost, holding cost, set-up cost, etc.