Chapter 7

Three layers supply chain in an imperfect production inventory model with two storage facilities under fuzzy rough environment

7.1 Introduction

Business organizations all over the world are striving hard to evolve strategies to survive in the era of competition ushered in by globalization. Supply chain management (SCM) is one such strategy. It is an effective methodology and presents an integrated approach to resolve issues in sourcing customer service, demand flow and distribution. The focus is on the customer. The results are in the form of reduced operational costs, improved flow of supplies, reduction in delays of production and increased customer satisfaction. While the goal of supply chain management is to reduce cost of producing and reaching the finished products to the customers, inventory control is the means to achieve the goal. Researchers as well as practitioners in manufacturing industries have given importance to develop inventory control problems in supply chain management. All steps from supply of raw materials to finished products can be included into a supply chain, connecting raw materials supplier, manufacturer, retailer and finally customers. Recent reviews on supply chain management are provided by Wong [223], Munson and Rosenblatt [157], Yang and Wee [230], Khonja [117], Yao et al. [234], Chahrazooghi et al. [16], Wang et al. [216] and others.

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CHAPTER 7. THREE LAYERS SUPPLY CHAIN IN AN IMPERFECT PRODUCTION INVENTORY MODEL WITH TWO STORAGE FACILITIES UNDER FUZZY ROUGH ENVIRONMENT

Now a days, it is common to all industries that a certain percentage of produced or ordered items are mixed of perfect and imperfect quality. It is also important to a supply manager of any organization to control and maintain the inventories of perfect and imperfect quality items. Salameh and Jaber [184] developed an inventory model for imperfect quality items using the EPQ/EOQ formulae and assumed that inferior quality items are sold as a single batch at the end of the total screening process. Goyal and Cardenas-Barron [78] extended the idea of Salameh and Jaber’s [184] model and proposed a practical approach to determine EPQ for items with imperfect quality. Yu et al. [238] generalized the models of Salameh and Jaber [184], incorporating deterioration and partial back ordering. Liu and Yang [143] investigated a single stage production system with imperfect process delivering two types of defects: reworkable and non-reworkable items. The reworkable items are sent for reworking, whereas non-reworkable items are immediately discarded from the system. Panda and Maiti [163] represented a geometric programming approach for multi-item inventory models with price dependent demand under flexibility and reliability with imprecise space constraint. Ma et al. [146] considered the effects of imperfect production processes and the decision on whether and when to implement a screening process for defective items generated during a production run. Sana [187] develops two inventory models in an imperfect production system and showed that the inferior quality items could be reworked at a cost where overall production inventory costs could be reduced significantly. Sana [188] extended the idea of imperfect production process in three layer supply chain management system.

Warehousing is an integral part of every logistics system. We can define warehousing as that part of a firm's logistics system that stores products (raw materials, parts, goods in process, finished goods). The two warehouse model for a finite and infinite time horizon is developed by several researchers (cf. Harrel [90], Pakkala and Achary [161], Bhunia and Maiti [11], Kar et al. [111] and others) already discussed in Section 6.1.

Dubois and Prade [63] first studied the fuzzification problem of rough sets. Furthermore, Morai and Yakout [156] defined the upper and lower approximations of the fuzzy sets with respect to a fuzzy min-similarity relation. Additionally, Radzikowska and Kerre [167], Xu and Zhou [228], Liu and Sa [139], Chen [33] and others generalized the above definitions of the fuzzy rough set to a more general case. Different types of uncertainty such as randomness, fuzziness and roughness are common factors in any production inventory problem. But some problems in production inventory system occur both fuzziness and roughness simultaneously. In many cases, it is found that some inventory parameters involve both the fuzzy and rough uncertainties. For example, the inventory related costs holding cost, set-up cost, idle costs, etc. depend on several factors such as bank interest, inflation, etc. which are uncertain in fuzzy rough sense. To be more specific, the inventory holding cost is sometimes represented by a fuzzy number and it depends on the storage amount which may be imprecise and range within an interval due to several factors such as scarcity of storage space, market fluctuation, human estimation thought process i.e. it may be represented by a rough set.

In this chapter, a supply chain model consisting of supplier, manufacturer and retailer has been considered. Here supplier receives the raw materials in a lot and then the superior
quality items of the raw materials are sold at a higher price to the manufacturer after the screening the imperfect raw materials as well as inferior quality items of the raw materials are also sold to another manufacturer at a reduced price in a single batch by the end of cent percent screening process. A mixture of perfect and imperfect quality items are produced by the manufacturer. After some rework, some repairable portion of imperfect quality items is transformed into perfect quality items and some of non repairable portion of imperfect items are sold with reduced price to the retailer. Retailer purchases both perfect and imperfect quality items and sells both items to the customers through his/her respective showrooms of finite capacities at a market place. Here customers' demand is stock dependent and selling price dependent for the perfect quality items and less perfect items respectively. Since the storage space of the showroom for perfect quality items is limited due to space problem and the demand of the corresponding items is stock dependent, hence a secondary warehouse is hired by the retailer on rental basis to store the excess amount of perfect quality items and these items are continuously transferred to the showroom concerned. The literature suggests that the holding cost of secondary warehouse per unit item per unit time is more than the holding cost of showroom due to the preservation cost for maintaining the quality of the product and other costs related to holding large quantity of the product in the secondary warehouse. But in this chapter it has been considered that the holding cost of perfect quality items in the secondary warehouse is less than the holding cost of the showroom as the nature of the items are non-deteriorating and so having no preservation cost. Here, transportation cost is also incurred to transport both quality items at the respective showrooms from the production center. Due to complexity of environment, inventory holding costs, idle costs, set-up costs and transportation costs are considered as fuzzy rough type and these are reduced to crisp ones using fuzzy rough expectation. In order to optimize the production rate and raw material order size (the decision variables), the average profit function of the manufacturer is maximized as the manufacturer acts as a leader (Stackelberg approach) and the supplier as well as retailer are the followers of that chain. The decision variables are also optimized by maximizing the integrated average profit function of the chain. Finally, a comparative study has been made between both approaches Stackelberg and integrated. A numerical example is provided to illustrate the feasibility of the model.

7.2 Notations and Assumptions

The following notations and assumptions have been used to develop the proposed model:

7.2.1 Notations

The following notations have been used to developed the model.

- $R$: Replenishment lot size of the supplier.
- $P$: Production rate for the manufacturer which is also the demand rate of supplier.
- $x$: Screening rate of supplier.
CHAPTER 7. THREE LAYERS SUPPLY CHAIN IN AN IMPERFECT PRODUCTION INVENTORY MODEL WITH TWO STORAGE FACILITIES UNDER FUZZY ROUGH ENVIRONMENT

\( \theta \) : Percentage of inferior quality items in each lot received by the supplier.
\( t_1 \) : Cycle length of supplier.
\( t' \) : Total Screening time of \( R \) units order per cycle.
\( c_s \) : Set up cost of supplier.
\( c_s' \) : Purchase cost per unit item of supplier.
\( h_s \) : Holding cost per unit item for per unit time for supplier.
\( I_{cs} \) : Cost per unit idle time of supplier.
\( s_c \) : Screening cost per unit item.
\( w_s \) : Selling price per unit of superior quality item for supplier.
\( w_s' \) : Selling price per unit of inferior quality item for supplier.
\( \beta \) : Percentage of imperfect quality items suitable for rework to make perfect items.
\( \gamma \) : Percentage of imperfect items which are suitable for sale through reduction.
\( q_{1m}(t) \) : Inventory level of perfect quality items for the manufacturer at any time \( t \).
\( q_{2m}(t) \) : Inventory level of less perfect quality items for the manufacturer at any time \( t \).
\( D_s \) : Demand rate of the retailer for perfect quality items.
\( D_s' \) : Demand rate of the retailer for less perfect quality items.
\( A_m \) : Set up cost of manufacturer.
\( h_{cm} \) : Holding cost per unit item for per unit time for perfect item in manufacturer.
\( h_{cm}' \) : Holding cost per unit item for per unit time for imperfect item in manufacturer.
\( I_{cm} \) : Cost per unit idle time of manufacturer.
\( I_{sm} \) : Inspection cost per unit item for manufacturer.
\( r_{cm} \) : Reworking cost per unit item for manufacturer.
\( C(P) \) : Production cost per unit item.
\( s_m \) : Selling price per unit of perfect quality items for manufacturer.
\( s_m' \) : Selling price per unit less perfect quality items for manufacturer.
\( q_{1r}(t) \) : Inventory level of perfect quality items for the retailer at any time \( t \).
\( q_{2r}(t) \) : Inventory level of less perfect quality items for the retailer at any time \( t \).
\( A_r \) : Set up cost for perfect quality items of retailer.
\( A_r' \) : Set up cost for less perfect quality items of retailer.
\( h_r \) : Holding cost per unit item for per unit time of perfect quality items in \( PW_1 \) for retailer.
\( h_{rs} \) : Holding cost per unit item for per unit time of perfect quality items for the retailer in secondary warehouse.
\( h_r' \) : Holding cost per unit item for per unit time of less perfect quality items in \( PW_2 \) for retailer.
\( s_r \) : Selling price per unit of perfect quality items for retailer \( PW_1 \).
\( s_r' \) : Selling price per unit of less perfect quality items for retailer \( PW_2 \).
\( c_r \) : Transportation cost perfect item for retailer.
\( c_r' \) : Transportation cost of less perfect quality items for retailer.
\( S \) : Capacity of secondary warehouse.
\( W_j \) : Capacity of \( PW_j (j=1,2) \).
\( D_c \) : Demand rate of customer for perfect quality items \( PW_1 \).
\( D_c' \) : Demand rate of customer for less perfect quality items \( PW_2 \).
\( \approx \) : Denotes the fuzzy rough parameters.
7.2.2 Assumptions

(i) Joint effect of supplier, manufacture and retailer is considered in a supply chain management.

(ii) Model is developed for single item products and lead time is negligible.

(iii) Production rate is a decision variable.

(iv) Demand for perfect quality items is deterministic and function of current stock level.

(v) Replenishment rate of manufacture is instantaneously infinite but its size is finite.

(vi) Unit production cost $C(P)$ per unit item is considered as $C(P) = L + \frac{G}{T} + HF$, where $G$ be the total labor cost for manufacturing the items, $L$ and $H$ are respectively the material cost and tool/die cost per unit item.

(vii) The manufacturer has ignored the machine breakdown.

(viii) Cost of idle times of supplier and manufacturer are taken into account.

(ix) Showrooms $PW_1$ and $PW_2$ of retailer are adjacent.

7.3 Mathematical Formulation of the Proposed Model

Block diagram and pictorial representation of the proposed supply chain production inventory model are respectively depicted in Figure 7.1 and Figure 7.2. Formulation of the model for supplier, manufacturer and retailer are given in the subsections 7.3.1, 7.3.2 and 7.3.3 respectively.

![Figure 7.1: Block diagram representation of the proposed model](image-url)

173
7.3.1 Formulation of the Supplier

Here $R$ be the lot-size received by the supplier at $t = 0$. A screening process of the lot is conducted at a rate of $x$ units per unit time and $\ell$ be the total screening time of $R$ units. Defective items are kept in stock and sold prior to receiving the next shipment as a single batch at a discounted price of $v' s$ per unit. $R\theta$ is the number of inferior quality items withdrawn from inventory and $t_1$ is the cycle length of the supplier. The number of superior quality items in each lot, denoted by $N(R, \theta)$, is given by

$$N(R, \theta) = (R - R\theta). \quad (7.1)$$

Supplier supplies the superior quality items as a raw materials to the manufacturer at a rate $P$ up to the time $t_1$ and to avoid shortages, it is assumed that the number of superior quality items $N(r, \theta)$ is at least equal to the demand during screening time $\ell$, i.e.,

$$N(R, \theta) \geq \ell P \quad (7.2)$$

Substituting equation(7.1) in equation(7.2) and replacing $\ell$ by $\frac{R}{x}$, the value of $R$ is restricted to $R \leq 1 - \frac{P}{x}$.

Sales revenue from superior quality items per cycle = $w_s(R - \theta R)$.
Sales revenue from inferior quality items per cycle = $v'sR\theta$.
Procurement cost for the supplier per cycle = (Setup cost + Purchasing cost) = $A_s + c_sR$.
Screening cost per cycle = $sR$.
Holding cost during $(0, t_1) = h_s\left[\frac{(R - \theta R)x}{2}\right] + \frac{R\theta}{x}$.
Idle time cost per cycle = $I_s(T - t_1)$.
7.3. MATHEMATICAL FORMULATION OF THE PROPOSED MODEL

Therefore, the average profit (APS) of the Supplier during \((0, T)\) is given by

\[
APS = \frac{1}{T} \left[ \left( u_s(1-\theta) + u_s^0\theta - c_s - s_s \right) R - A_s - h_s R \left\{ \frac{(1-\theta) t_1}{2} + \frac{R \theta}{x} \right\} - I_s(T - t_1) \right] \\
= \frac{1}{T} \left[ -Z_{r_s} + Z_{r_s} R + Z_{r_s} \frac{R}{P} - Z_{r_s} \frac{R^2}{2P} - Z_{r_s} R^2 \right],
\]

where \( t_1 = \frac{R - R_0}{P} \) and \( Z_{r_s}, \ i = 0, 1, 2, 3, 4 \) are independent of \( R \) and \( P \). (See Appendix E)

7.3.2 Formulation of the Manufacturer

It is consider that a manufacturer produces some perfect and imperfect quality items at the rate of \( P \) units during the period \((0, t_1)\), receiving the raw material from the supplier at the same rate \( P \) during the period \((0, t_1)\). \( P e^{-\alpha t} \) and \( P(1 - e^{-\alpha t}) \) are respectively the expected quantity of perfect and imperfect quality items at any time \( t \), where \( \alpha \) be the reliability parameter given by \( \alpha = \frac{\text{number of failures}}{\text{total units of operating hours}} \). Among the imperfect quality items, only \( \beta P(1 - e^{-\alpha t}) \) units per unit time become perfect quality after reworking and the portion \( \gamma (1 - \beta) P(1 - e^{-\alpha t}) \) are less perfect quality which are soled at a reduced price to the retailer. Here \( D_r \) and \( D_i \) denote the demand rates of a retailer for perfect quality and less perfect quality items which are met by manufacturer during \((0, t_2)\) and \((0, t_2)\) respectively.

For Perfect Quality Items of Manufacturer

The rate of change of inventory level of manufacturer for perfect quality items can be represented by the following differential equations:

\[
\frac{dq_{pm}}{dt} = \begin{cases} 
P e^{-\alpha t} + \beta P(1 - e^{-\alpha t}) - D_r, & 0 \leq t \leq t_1 \\
-D_r, & t_1 \leq t \leq t_2
\end{cases}
\]

with boundary conditions \( q_{pm}(0) = 0 \) at \( t = 0 \) and \( t = t_2 \).

The solution of above differential equations are given by

\[
q_{pm}(t) = \begin{cases} 
\frac{P}{\alpha}(1 - \beta)(1 - e^{-\alpha t}) + (P\beta - D_r)t, & 0 \leq t \leq t_1 \\
D_r(t_2 - t), & t_1 \leq t \leq t_2
\end{cases}
\]

(7.5)

From continuity at \( t = t_1 \), following condition is obtain

\[
P \frac{1}{\alpha}(1 - \beta)(1 - e^{-\alpha t_1}) + (P\beta - D_r)t_1 = D_r(t_2 - t_1)
\]

which implies \( t_2 = \frac{P}{D_r} \left[ \frac{1}{\alpha}(1 - \beta)(1 - e^{-\alpha t_1}) + \beta t_1 \right]. \)

(7.6)

Now holding cost \( (HCM_1) \) for perfect quality items for manufacture is given by

\[
HCM_1 = h_m \int_0^{t_1} q_{pm}(t) \, dt + h_m \int_{t_1}^{t_2} q_{pm}(t) \, dt
\]

\[= h_m \frac{P}{\alpha}(1 - \beta)t_1 - \frac{P h_m}{\alpha^2}(1 - \beta)(1 - e^{-\alpha t_1}) + h_m P \beta \frac{t_2^2}{2} - D_r h_m \frac{t_2^2}{2}, \]

175
and reworking Cost(RCM) for Manufacture

\[ RCM = r_cm \int_0^{t_1} P_\beta(1 - e^{-\alpha t}) \, dt = r_cmP_{\beta}[t_1 - \frac{1}{\alpha}(1 - e^{-\alpha t})]. \]

**For Less Perfect Quality Items of Manufacturer**

The rate of change of inventory level of less perfect quality items for manufacturer can be represented by the following differential equations:

\[
\frac{dq_{2m}}{dt} = \begin{cases} 
\gamma(1 - \beta)P(1 - e^{-\alpha t}) - D_f, & 0 \leq t \leq t_1 \\
-D_f, & t_1 \leq t \leq t'_2 
\end{cases}
\tag{7.7}
\]

with boundary conditions \( q_{2m}(t) = 0 \) at \( t = 0 \) and \( t = t'_2 \).

The solution of above differential equations are given by

\[
q_{2m}(t) = \begin{cases} 
\frac{-P}{\alpha}\gamma(1 - \beta)(1 - e^{-\alpha t}) + \left[\gamma(1 - \beta)P - D_f\right]t, & 0 \leq t \leq t_1 \\
D_f(t'_2 - t), & t_1 \leq t \leq t'_2 
\end{cases}
\tag{7.8}
\]

From continuity at \( t = t_1 \), following condition is obtain

\[
t'_2 = \frac{1}{D_f}\left[\frac{-P}{\alpha}\gamma(1 - \beta)(1 - e^{-\alpha t_1}) + \gamma(1 - \beta)P t_1\right].
\tag{7.9}
\]

Now holding cost(HCM2) for less perfect quality items for manufacture

\[
HCM2 = h'_m \int_0^{t_1} q_{2m}(t) \, dt + h'_m \int_{t_1}^{t'_2} q_{2m}(t) \, dt = h'_m \frac{P \gamma}{\alpha} (1 - \beta) \left[(1 - e^{-\alpha t_1}) - \alpha t_1\right] + h'_m \left[P \gamma(1 - \beta) - D_f\right] \frac{t_1^2}{2} + h'_m \frac{D_f}{2} (t'_2 - t_1)^2.
\]

Production cost for the manufacturer = \( C(P)Pt_1 \),

Inspection cost = \( I_{sm}Pt_1 \),

Holding cost for the manufacturer = \( |HCM1 + HCM2| \),

Set up cost of the manufacturer = \( A_m \),

Idle time cost for the manufacturer = \( I_{sm}(T - t_2) \),

Revenue of perfect quality items for the manufacturer = \( s_m \int_0^{t_2} D_q \, dt = s_m D_q t_2 \),

Revenue of less perfect quality items for the manufacturer = \( s'_m \int_{t_1}^{t'_2} D_q \, dt = s'_m D_q \left(t'_2 - t_1\right) \).
7.3. MATHEMATICAL FORMULATION OF THE PROPOSED MODEL

Average Profit of Manufacturer

Average profit (APM) of manufacturer during the period \((0, T)\) is given by

\[
APM = \frac{1}{T} \left[ (s_m D_1 T_2 + s_m D_1 T_2') - \left\{ a_k + C(P) + I_{sm} \right\} P_t T_1 - A_m - t_{cm}(T - t_2) \\
- \frac{P_m P}{\alpha} \left\{ \alpha t_1 - (1 - e^{-\alpha t_1}) \right\} - h_m \left\{ \frac{P}{\alpha} (1 - \beta) \{ \alpha t_1 - (1 - e^{-\alpha t_1}) \} + P \beta \frac{t_1^3}{3} - D_1 \frac{t_1^2}{2} \right\} \\
- h_m \left\{ \gamma (1 - \beta) \{ (1 - e^{-\alpha t_1}) - \alpha t_1 \} + \{ \gamma (1 - \beta) P - D_1' \} \frac{t_1^2}{2} + \frac{D_1'}{2} (t_1 - t_2) \gamma^2 \right\} \right]
\]

where \(Z_{im}, \ i = 0, 1, 2, ..., 6, \) are independent of \(R\) and \(P.\) (see Appendix E)

7.3.3 Formulation of the Retailer

Customers’ demand for both perfect and less perfect quality items are met by the retailer through the adjacent showrooms \(PW_1\) and \(PW_2\) respectively. Retailer has a secondary warehouse \(SW\) to store the excess perfect quality items which are continuously transferred to the showroom \(PW_1.\) Less perfect quality items are directly transferred to the showroom \(PW_2.\) Transportation cost is taken into account to transfer each items from production center to the showrooms.

For Perfect Quality Items of Retailer

In this case the demand rate \((D_r)\) of customers at \(PW_1\) has consider as stock dependent as the following form

\[
D_r = \begin{cases} \\
\alpha_1 + \beta_1 q_{ir}, & 0 \leq t \leq t_3 \\
\alpha_1 + \beta_1 W_1, & t_3 \leq t \leq t_1 \\
\alpha_1 + \beta_1 q_{ir}, & t_1 \leq t \leq T \\
\end{cases}
\]

Now the corresponding rate of change of on hand inventory of perfect quality items are given by

\[
\frac{dq_{ir}}{dt} = \begin{cases} \\
D_r - D_1, & 0 \leq t \leq t_2 \\
-D_1, & t_2 \leq t \leq T \\
\end{cases}
\]

with boundary conditions

\[
q_{ir}(t) = \begin{cases} \\
0, & t = 0 \\
W_1, & t = t_3 \\
S_1, & t = t_2 \\
W_1, & t = t_1 \\
0, & t = T \\
\end{cases}
\]

177
Therefore the solutions of above differential equations are given by

\[
q_{t_3}(t) = \begin{cases} 
\frac{D_r - \alpha_1}{\beta_1} (1 - e^{-\beta_1 t}), & 0 \leq t \leq t_3 \\
W_1 + \left[ D_r - (\alpha_1 + \beta_1 W_1) \right] (t - t_3), & t_3 \leq t \leq t_2 \\
S - \left[ (\alpha_1 + \beta_1 W_1) \right] (t - t_2), & t_2 \leq t \leq t_4 \\
\frac{-\alpha_1}{\beta_1} \left[ 1 - e^{-\beta_1 (t - T)} \right], & t_4 \leq t \leq T 
\end{cases}
\]  

(7.10)

Now, \(q_{t_3}(t_3) = W_1 \Rightarrow t_3 = \frac{1}{\beta_1} \log\left(1 - \frac{W_1}{D_r - \alpha_1}\right)\)  

(7.11)

\[
q_{t_4}(t_2) = S \text{ gives } t_4 = t_2 + \frac{S - W_1}{\alpha_1 + \beta_1 W_1}
\]  

(7.12)

and \(W_1 + \left[ D_r - (\alpha_1 + \beta_1 W_1) \right] (t_2 - t_3) = S\)  

(7.13)

\[
q_{t_4}(t_4) = W_1 \Rightarrow T = t_4 + \frac{1}{\beta_1} \log\left(1 - \frac{D_r}{\alpha_1}(1 - e^{-\beta_1 t_3})\right)
\]  

(7.14)

Holding cost (\(HC\,RW\)) of the secondary warehouse SW is given by

\[
HC\,RW = h_{rs} \int_{t_3}^{t_2} \{ q_{t_3}(t) - W_1 \} \, dt + h_{rs} \int_{t_4}^{t_3} \{ q_{t_4}(t) - W_1 \} \, dt
\]

\[
= \frac{h_{rs}}{2} \left[ (D_r - (\alpha_1 + \beta_1 W_1)) (t_2 - t_3)^2 - (\alpha_1 + \beta_1 W_1) (t_2 - t_3)^2 \right].
\]

Holding cost (\(HC\,RS_1\)) of the showroom PW is given by

\[
HC\,RS_1 = h_r \int_{t_3}^{t_3} q_{t_3}(t) \, dt + h_r \int_{t_3}^{t_4} W_1 \, dt + \int_{t_4}^{T} q_{t_4}(t) \, dt
\]

\[
= h_r \frac{D_r - \alpha_1}{\beta_1} \left[ t_3 + \frac{e^{-\beta_1 t_3}}{\beta_1} - \frac{1}{\beta_1} \right] + W_1 h_r (t_4 - t_3)
\]

\[
- h_r \frac{\alpha_1}{\beta_1} \left[ (T - t_4) + \frac{1}{\beta_1} - \frac{e^{-\beta_1 (T - t_4)}}{\beta_1} \right].
\]

Transportation cost (\(TC\,PR\)) for perfect quality items of retailer is given by

\[
TC\,PR = \alpha_r \int_{0}^{T} D_e \, dt
\]

\[
= \alpha_r \left[ (\alpha_1 + \beta_1 W_1) (t_4 - t_3) + D_r (t_4 - t_3) \frac{D_r - \alpha_1}{\beta_1} e^{-\beta_1 t_3} + \frac{\alpha_1}{\beta_1} e^{-\beta_1 (T - t_4)} \right].
\]

178
Revenue from selling perfect quality items is given by

\[ R_{PR} = s_r \int_0^T D_r \, dt \]

\[ = s_r \left[ (\alpha_1 + \beta_1 W_1)(t_4 - t_3) + D_r t_3 - \frac{D_r}{\beta_1} - \frac{D_r}{\beta_1} e^{-\beta_1 t_3} + \frac{\alpha_1}{\beta_1} e^{-\beta_1 (t_4 - t)} \right]. \]

Hence the total profit of the retailer from the perfect quality items is given by

\[ TPR_1 = s_r \left[ (\alpha_1 + \beta_1 W_1)(t_4 - t_3) + D_r t_3 - \frac{D_r}{\beta_1} - \frac{D_r}{\beta_1} e^{-\beta_1 t_3} + \frac{\alpha_1}{\beta_1} e^{-\beta_1 (t_4 - T)} \right] \]

\[ - \alpha_r \left[ (\alpha_1 + \beta_1 W_1)(t_4 - t_3) + D_r t_3 - \frac{D_r}{\beta_1} - \frac{D_r}{\beta_1} e^{-\beta_1 t_3} + \frac{\alpha_1}{\beta_1} e^{-\beta_1 (t_4 - T)} \right] \]

\[ - \frac{h_r}{2} \left[ (D_r - (\alpha_1 + \beta_1 W_1))(t_2 - t_3)^2 - (\alpha_1 + \beta_1 W_1)(t_2 - t_4)^2 \right] - A_r \]

\[ - h_r \left[ \frac{D_r}{\beta_1} \left( t_3 + \frac{\beta_1}{\alpha_1} \right) + W_1(t_4 - t_3) - \frac{\alpha_1}{\beta_1} \left( (T - t_4) \right) \frac{1}{\beta_1} \left( \frac{\beta_1}{\alpha_1} \right) \right] \]

\[ = s_r \left[ \frac{P_1}{\alpha_1} (1 - \beta) \{ \alpha t_1 - (1 - e^{-\alpha t_1}) \} + P_2 \frac{t^2}{2} - D_r \frac{t^2}{2} \right]. \]

For Less Perfect Quality Items of Retailer

In this case demand rate (\( D'_r \)) of customers at \( PW_2 \) is assumed as selling price dependent defined as \( D'_r = (a - b \bar{x}_r) \) and corresponding change of on hand inventory of less perfect quality items are given by

\[ \frac{dq_{2r}}{dt} = \begin{cases} 
D'_r - D'_c, & 0 \leq t \leq t'_2 \\
-D'_c, & t'_2 \leq t \leq T' 
\end{cases} \]

with boundary conditions \( q_{2r}(t) = 0 \) at \( t = 0 \) and \( t = T' \).

The solution of above differential equations are given by

\[ q_{2r}(t) = \begin{cases} 
(D'_r - D'_c)t, & 0 \leq t \leq t'_2 \\
-D'_c(T - T'), & t'_2 \leq t \leq T' 
\end{cases} \]

(7.15)

From continuity condition at \( t = t'_2, T' = \frac{1}{\alpha_1} \left[ \gamma (1 - \beta)Pt_1 - \frac{P_1}{\alpha_1} \gamma (1 - \beta) (1 - e^{-\alpha t_1}) \right] \).

Also \( q_{2r}(t'_2) = W_2 \) i.e., \( W_2 = (1 - \frac{D'_c}{\alpha_1}) \left[ - \frac{h_r}{2} \gamma (1 - \beta) (1 - e^{-\alpha t_1}) + \gamma (1 - \beta)Pt_1 \right] \).

Holding cost \( HC_{RS_2} \) of the showroom \( PW_2 \) is given by

\[ HC_{RS_2} = h_r\int_0^{t'_2} q_{2r}(t) \, dt + h_r\int_{t'_2}^{T'} q_{2r}(t) \, dt \]

\[ = \frac{h_r}{2} \left[ (D'_r - D'_c)t'_2^2 + D'_c(T - T')^2 \right]. \]
Transportation cost \( TCHR \) for less perfect quality items of retailer is given by

\[
TCHR = c_r \int_0^T D'_r \, dt = c_r \int_0^T D'_r \, t^r.
\]

Revenue \( REIR \) from selling less perfect quality items of retailer is given by

\[
REIR = s'_r \int_0^T D'_r \, dt = s'_r \int_0^T D'_r \, t^r.
\]

Total Profit \( TPBR_2 \) of Retailer for less perfect quality items is given by

\[
TPBR_2 = \{s'_r - c'_r\} D'_r T^r - \frac{h'_r}{2} \left[ (D'_r - D'_s)^2 + D'_r (t'_2 - T)^2 \right] - A'_r \\
- s'_m \left[ \frac{P'_2}{\alpha^2} (1 - \beta) \left\{ \left( 1 - e^{-t_0} \right) + \alpha t_1 \right\} + (P\gamma(1 - \beta) - D'_s) \frac{t_1^2}{2} + \frac{D'_r}{2} (t_1 - t'_2)^2 \right].
\]

### Average Profit of Retailer

Average Profit \( APR \) of Retailer during \((0, T)\) is

\[
APR = \frac{1}{T} [TPBR_1 + TPBR_2]
\]

\[
= \frac{1}{T} \left[ s_r \left\{ \left( \alpha_1 + \beta_1 W_1 \right) (t_1 - t_3) + D_3 \, t_3 - \frac{D_3}{\beta_1} + \frac{D_3 - \alpha_1}{\beta_1} e^{-\beta_1 t_3} + \frac{\alpha_1}{\beta_1} e^{-\beta_1 (t_4 - T)} \right\} \\
- c_r \left\{ \left( \alpha_1 + \beta_1 W_1 \right) (t_1 - t_3) + D_3 \, t_3 - \frac{D_3}{\beta_1} + \frac{D_3 - \alpha_1}{\beta_1} e^{-\beta_1 t_3} + \frac{\alpha_1}{\beta_1} e^{-\beta_1 (t_4 - T)} \right\} \\
- \frac{h'_r}{2} \left\{ (D'_r - (\alpha_1 + \beta_1 W_1)) (t_2 - t_3)^2 - (\alpha_1 + \beta_1 W_1) (t_2 - t_4)^2 \right\} + s'_r \int_0^T D'_r \, t^r \\
- c'_r D'_r T^r - \frac{h'_r}{2} \left[ (D'_r - D'_s)^2 + D'_r (t'_2 - T)^2 \right] - h'_r \left\{ \frac{D_3 - \alpha_1}{\beta_1} (t_3 + \frac{e^{-\beta_1 t_3}}{\beta_1}) \right\} \\
- \frac{1}{\beta_1} + W_1 (t_4 - t_3) - \frac{\alpha_1}{\beta_1} \left\{ (T - t_4) + \frac{1}{\beta_1} - e^{-\beta_1 (T - t_4)} \right\} \right] - s'_m \left\{ \frac{P}{\alpha^2} (1 - \beta) t_1 \right\} \\
- \frac{P'_2}{\alpha^2} (1 - \beta) (1 - e^{-\alpha_1 t_1}) + P'_2 (\frac{t_1^2}{2}) - D'_s \frac{t_2^2}{2} \right\} - s'_m \left\{ \frac{P'_2}{\alpha^2} (1 - \beta)(1 - e^{-\alpha_1 t_1}) \\
- \frac{P'_2}{\alpha^2} (1 - \beta) t_1 + (P\gamma(1 - \beta) - D'_s) \frac{t_1^2}{2} + \frac{D'_r}{2} (t_1 - t'_2)^2 \right\} - A'_r \\
= \frac{1}{T} \left[ Z_{w_1} + Z_{w_2} R + Z_{w_3} \frac{R^2}{2R} + Z_{w_4} R^2 + Z_{w_5} \frac{R^3}{2R} + Z_{w_6} \frac{R^4}{2R^2} \right],
\]

where \( Z_{w_i}, i = 0, 1, ..., 5 \) are independent of \( R \) and \( P \) (See Appendix E).
7.3. Mathematical Formulation of the Proposed Model

7.3.4 Integrated Average Profit

Average Profit (\(IAP\)) for the Integrated Model during \((0, T)\) is

\[
IAP = |APS + APM + APR|
\]

\[
= \frac{1}{T} \left[ Z_0 + Z_1 \frac{R}{P} + Z_2 R - Z_2 \frac{R^2}{2P} + Z_3 \frac{R^2}{P^2} - Z_5 P R^2 - Z_6 R^2 + Z_8 \frac{R^3}{P} + Z_8 \frac{R^4}{P^2} \right]
\]

where \(Z_i, i = 0, 1, ..., 8\) are independent of \(R\) and \(P\) (See Appendix E).

7.3.5 Model in Fuzzy Rough Environment

In this environment, all holding cost, idle cost, setup cost and transportation cost have been considered fuzzy-rough parameters. Then the corresponding fuzzy-rough objective functions for supplier, manufacturer and retailer are given by

\[
\tilde{A}\tilde{P}S = \frac{1}{T} \left[ -\tilde{Z}_{o0} + \tilde{Z}_{o1} R + \tilde{Z}_{o2} \frac{R}{P} - \tilde{Z}_{o3} \frac{R^2}{2P} - \tilde{Z}_{o4} R^2 \right]
\]

\[
\tilde{A}\tilde{P}M = \frac{1}{T} \left[ -\tilde{Z}_{m0} + \tilde{Z}_{m1} R + \tilde{Z}_{m2} \frac{R}{P} + \tilde{Z}_{m3} \frac{R^2}{2P} + \tilde{Z}_{m4} R^2 - \tilde{Z}_{m5} P R^2 \right]
\]

and \(\tilde{A}\tilde{P}R\)

\[
= \frac{1}{T} \left[ \tilde{Z}_{r0} + \tilde{Z}_{r1} R + \tilde{Z}_{r2} \frac{R^2}{2P} + \tilde{Z}_{r3} R^2 + \tilde{Z}_{r4} \frac{R^3}{P} + \tilde{Z}_{r5} \frac{R^4}{P^2} \right]
\]

Also the fuzzy-rough objective functions for integrated model is given by

\[
I\tilde{A}P = |A\tilde{P}S + A\tilde{P}M + A\tilde{P}R|
\]

\[
= \frac{1}{T} \left[ Z_0 + \tilde{Z}_{r1} \frac{R}{P} + \tilde{Z}_{r2} R - \tilde{Z}_{r3} \frac{R^2}{2P} + \tilde{Z}_{r4} \frac{R^3}{P^2} - \tilde{Z}_{r5} P R^2 - \tilde{Z}_{r6} R^2 + \tilde{Z}_{r7} \frac{R^3}{P} + \tilde{Z}_{r8} \frac{R^4}{P^2} \right]
\]

where fuzzy rough parameters \(\bar{h}_s, \bar{h}_m, \bar{h}_r, \bar{h}_s, \bar{h}_m, \bar{h}_r, \bar{A}_s, \bar{A}_m, \bar{A}_r, \bar{I}_s, \bar{I}_m, \bar{I}_r, \hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\) are defined as follows,

\(\bar{h}_s = (\bar{h}_{s1}, \bar{h}_{s2}, \bar{h}_{s3}, \bar{h}_{s4})\) with \(\bar{h}_{s1} = (h_{s11}, h_{s12}, h_{s13}, h_{s14}), 0 \leq h_{s11} \leq h_{s12} < h_{s13} \leq h_{s14}\),

\(\bar{h}_m = (\bar{h}_{m1}, \bar{h}_{m2}, \bar{h}_{m3}, \bar{h}_{m4})\) with \(\bar{h}_{m1} = (h_{m11}, h_{m12}, h_{m13}, h_{m14}), 0 \leq h_{m11} \leq h_{m12} < h_{m13} \leq h_{m14}\),

\(\bar{h}_r = (\bar{h}_{r1}, \bar{h}_{r2}, \bar{h}_{r3}, \bar{h}_{r4})\) with \(\bar{h}_{r1} = (h_{r11}, h_{r12}, h_{r13}, h_{r14}), 0 \leq h_{r11} \leq h_{r12} < h_{r13} \leq h_{r14}\),

\(\bar{A}_s = (\bar{A}_{s1}, \bar{A}_{s2}, \bar{A}_{s3}, \bar{A}_{s4})\) with \(\bar{A}_{s1} = (A_{s11}, A_{s12}, A_{s13}, A_{s14}), 0 \leq A_{s11} \leq A_{s12} < A_{s13} \leq A_{s14}\),

\(\bar{A}_m = (\bar{A}_{m1}, \bar{A}_{m2}, \bar{A}_{m3}, \bar{A}_{m4})\) with \(\bar{A}_{m1} = (A_{m11}, A_{m12}, A_{m13}, A_{m14}), 0 \leq A_{m11} \leq A_{m12} < A_{m13} \leq A_{m14}\),

\(\bar{A}_r = (\bar{A}_{r1}, \bar{A}_{r2}, \bar{A}_{r3}, \bar{A}_{r4})\) with \(\bar{A}_{r1} = (A_{r11}, A_{r12}, A_{r13}, A_{r14}), 0 \leq A_{r11} \leq A_{r12} < A_{r13} \leq A_{r14}\),

\(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\) are defined as follows.

181
CHAPTER 7. THREE LAYERS SUPPLY CHAIN IN AN IMPERFECT PRODUCTION INVENTORY MODEL WITH TWO STORAGE FACILITIES UNDER FUZZY ROUGH ENVIRONMENT

\[ \tilde{A}_r = (\tilde{A}_{r1}, \tilde{A}_{r2}, \tilde{A}_{r3}, \tilde{A}_{r4}) \text{ with } \tilde{A}_{r1} \triangleq (|A_{r21}, A_{r31}|, |A_{r22}, A_{r32}|), \ 0 \leq A_{r11} \leq A_{r21} < A_{r31} \leq A_{r41}, \]

\[ \tilde{I}_{es} = (\tilde{I}_{es1}, \tilde{I}_{es2}, \tilde{I}_{es3}, \tilde{I}_{es4}) \text{ with } \tilde{I}_{es1} \triangleq (|I_{es2}, I_{es3}|, |I_{es1}, I_{es4}|), \ 0 \leq I_{es1} \leq I_{es2} < I_{es3} \leq I_{es4}, \]

\[ \tilde{I}_{em} = (\tilde{I}_{em1}, \tilde{I}_{em2}, \tilde{I}_{em3}, \tilde{I}_{em4}) \text{ with } \tilde{I}_{em1} \triangleq (|I_{em2}, I_{em3}|, |I_{em1}, I_{em4}|), \ 0 \leq I_{em1} \leq I_{em2} < I_{em3} \leq I_{em4}. \]

\[ \tilde{c}_{tp} = (\tilde{c}_{tp1}, \tilde{c}_{tp2}, \tilde{c}_{tp3}, \tilde{c}_{tp4}) \text{ with } \tilde{c}_{tp1} \triangleq (|c_{tp2}, c_{tp3}||c_{tp1}, c_{tp4}|), \ 0 \leq c_{tp1} \leq c_{tp2} < c_{tp3} \leq c_{tp4}. \]

\[ \tilde{c}_{tp}' = (\tilde{c}_{tp1}', \tilde{c}_{tp2}', \tilde{c}_{tp3}', \tilde{c}_{tp4}') \text{ with } \tilde{c}_{tp1}' \triangleq (|c_{tp2}', c_{tp3}'||c_{tp1}', c_{tp4}'|), \ 0 \leq c_{tp1}' \leq c_{tp2}' < c_{tp3}' \leq c_{tp4}'. \]

7.3.6 Model in Equivalent Crisp Environment

In this environment, using Lemma 2.4 and Theorems 2.2 & 2.3, the fuzzy rough objective functions for supplier, manufacturer and retailer are given by

\[ EAPS = E[A\tilde{P}S] = \frac{1}{T} \left[ -E[\tilde{Z}_{s1}] + E[\tilde{Z}_{s2}]R + E[\tilde{Z}_{s3}]\frac{R^2}{P} - E[\tilde{Z}_{s4}]R - E[\tilde{Z}_{s6}]R^2 \right] \]

\[ EAPM = E[A\tilde{P}M] = \frac{1}{T} \left[ -E[\tilde{Z}_{m1}] + E[\tilde{Z}_{m2}]\frac{R^2}{P} + E[\tilde{Z}_{m3}]R + E[\tilde{Z}_{m4}]\frac{R^2}{P} \right. \]

\[ \left. + E[\tilde{Z}_{m5}]\frac{R^2}{P} + E[\tilde{Z}_{m6}]R^2 - E[\tilde{Z}_{m7}]PR^2 \right] \]

and \[ EAPR = E[A\tilde{P}R] \]

\[ = \frac{1}{T} \left[ E[\tilde{Z}_{r1}] + E[\tilde{Z}_{r2}]R + E[\tilde{Z}_{r3}]\frac{R^2}{P} + E[\tilde{Z}_{r4}]\frac{R^2}{P} + E[\tilde{Z}_{r5}]\frac{R^4}{P^2} + E[\tilde{Z}_{r6}]\frac{R^4}{P^2} \right] \]

Also the objective functions for integrated model is given by

\[ EIAP = E[I\tilde{A}P] = \left[ E[A\tilde{P}S] + E[A\tilde{P}M] + E[A\tilde{P}R] \right] \]

\[ = \frac{1}{T} \left[ E[\tilde{Z}_{s1}] + E[\tilde{Z}_{s2}]\frac{R^2}{P} + E[\tilde{Z}_{s3}]R + E[\tilde{Z}_{s4}]\frac{R^2}{P} + E[\tilde{Z}_{s5}]\frac{R^2}{P} + E[\tilde{Z}_{s6}]R^2 \right. \]

\[ \left. + E[\tilde{Z}_{r1}]\frac{R^3}{P} + E[\tilde{Z}_{r2}]\frac{R^3}{P} \right] \]

where \[ \hat{E}[\hat{h}_{ts}] = \frac{1}{16} \sum_{t=1}^{4} \sum_{k=1}^{4} h_{ts}, \quad \hat{E}[\hat{h}_{tm}] = \frac{1}{16} \sum_{t=1}^{4} \sum_{k=1}^{4} h_{tm}, \quad \hat{E}[\hat{h}_{mn}] = \frac{1}{16} \sum_{t=1}^{4} \sum_{k=1}^{4} h_{mn} \]

\[ \hat{E}[\hat{h}_{tr}] = \frac{1}{16} \sum_{t=1}^{4} \sum_{k=1}^{4} h_{tr}, \quad \hat{E}[\hat{h}_{rs}] = \frac{1}{16} \sum_{t=1}^{4} \sum_{k=1}^{4} h_{rs}, \quad \hat{E}[\hat{h}_{mr}] = \frac{1}{16} \sum_{t=1}^{4} \sum_{k=1}^{4} h_{mr} \]

\[ \hat{E}[\hat{h}_{sr}] = \frac{1}{16} \sum_{t=1}^{4} \sum_{k=1}^{4} h_{sr}, \quad \hat{E}[\hat{h}_{ra}] = \frac{1}{16} \sum_{t=1}^{4} \sum_{k=1}^{4} h_{ra}, \quad \hat{E}[\hat{h}_{as}] = \frac{1}{16} \sum_{t=1}^{4} \sum_{k=1}^{4} h_{as}. \]

182
7.3. MATHEMATICAL FORMULATION OF THE PROPOSED MODEL

\[ E[\tilde{r}_{is}] = \frac{1}{16} \sum_{l=1}^{4} \sum_{k=1}^{4} I_{slk}, \quad E[\tilde{r}_{im}] = \frac{1}{16} \sum_{l=1}^{4} \sum_{k=1}^{4} I_{simk}, \quad E[\tilde{g}_{ip}] = \frac{1}{16} \sum_{l=1}^{4} \sum_{k=1}^{4} c_{iplk}, \]
\[ E[\hat{c}_{ip}^2] = \frac{1}{16} \sum_{l=1}^{4} \sum_{k=1}^{4} c_{iplk}^2. \]

7.3.7 Stakelberg Approach (Leader-follower Relationship)

In this case manufacturer is leader, supplier and retailer are followers. Also, optimum values of the average profit of supplier and retailer are obtained by putting the optimum value of the decision variables, which are obtained by optimizing the average profit of manufacturer. Using the equation (7.6), (7.11), (7.12), (7.13) and \((1 - e^{-\alpha T}) \approx \alpha T, \) in the equation (7.14), the relation \(T = \frac{1}{\alpha^2} R + t_0 \) is obtained where \(t_0\) is given by \(t_0 = \frac{\delta P}{\alpha^2}, \frac{1}{\alpha} \log \left(1 - \frac{\delta P}{\alpha} \right) \) and it is independent of \(R\) and \(P\) and.

When both \(R\) and \(P\) are Decision Variables:

The expected average profit of manufacturer is given by

\[ E(AM)(R,P) = \frac{1}{T} \left[ -E[Z_{am}] - \frac{1}{P} E[Z_{2m}] R + E[Z_{2m}] \frac{R^2}{P^2} + E[Z_{3m}] \frac{R^2}{P^2} + E[Z_{3m}] \frac{R^2}{P^2} \right] \]

where \(E[Z_{im}], i=0,1,2,..,6, \) are independent of \(R\) and \(P\) (see Appendix E).

The necessary conditions for maximum value of \(E(AM)(R,P)\) are \(\frac{\partial^2}{\partial R^2}(E(AM)) = 0 \) and \(\frac{\partial^2}{\partial P^2}(E(AM)) = 0 \) which gives respectively

\[ \frac{(1 - \theta)}{D_i T} E[Z_{2m}] - \left\{ 1 - \frac{(1 - \theta) R}{D_i T} \right\} \left\{ \frac{1}{P} E[Z_{1m}] - E[Z_{2m}] \right\} + \frac{2R}{P^2} - \frac{(1 - \theta) R^2}{D_i T P} \]
\[ \times \left\{ \frac{1}{P} E[Z_{2m}] + E[Z_{4m}] \right\} + \left\{ 2R - \frac{(1 - \theta) R^2}{D_i T} \right\} \left\{ E[Z_{5m}] - P E[Z_{6m}] \right\} = 0 \] (7.16)

and

\[ E[Z_{1m}] \frac{R}{P^2} - 2E[Z_{2m}] \frac{R^2}{P^3} - E[Z_{4m}] \frac{R^2}{P^3} - E[Z_{6m}] \frac{R^2}{P^3} = 0 \] (7.17)

Solving (7.16) and (7.17), we can obtain the optimum value of \(R\) and \(P,\) say \(R^*\) and \(P^*\).

If \(\left\{ \frac{\partial^2}{\partial R^2}(E(AM)) \right\} \left\{ \frac{\partial^2}{\partial P^2}(E(AM)) \right\} - \left\{ \frac{\partial^2}{\partial R \partial P}(E(AM)) \right\}^2 > 0, \frac{\partial^2}{\partial R^2}(E(AM)) < 0, \) and \(\frac{\partial^2}{\partial P^2}(E(AM)) < 0 \) holds for \(R = R^*\) and \(P = P^*\) then EAPM \((R^*, P^*)\) is maximum.
CHAPTER 7. THREE LAYERS SUPPLY CHAIN IN AN IMPERFECT PRODUCTION INVENTORY MODEL WITH TWO STORAGE FACILITIES UNDER FUZZY ROUGH ENVIRONMENT

Now \( \frac{\partial^2}{\partial P^2} (EAPM) \mid_{a_t \left[R^*, P^* \right]} < 0 \)

i.e., \( - \left( \frac{1 - \theta}{D_t} \right) \frac{1 - \theta^2 R^*}{D_t^2 T^2} E[\hat{Z}_{2m}] + \left\{ \frac{1 - \theta}{D_t} \right\} \frac{1 - \theta^2 R^*}{D_t^2 T^2} E[\hat{Z}_{4m}] + \left( \frac{1 - \theta}{D_t} \right) \left( \frac{1 - \theta^2 R^*}{D_t^2 T^2} P_n \right) \}

\( - \left( 1 - \theta \right) \left\{ \frac{1 - \theta^2 R^*}{D_t^2 T^2} P_n \right\} E[\hat{Z}_{2m}] + \left( \frac{1 - \theta}{D_t} \right) \frac{1 - \theta^2 R^*}{D_t^2 T^2} P_n \}

\( + \frac{1}{P_n} \left\{ \left( \frac{1 - \theta}{D_t} \right) \right\} \frac{1 - \theta^2 R^*}{D_t^2 T^2} P_n \}

\( - \left\{ \frac{1 - \theta}{D_t} \right\} \frac{1 - \theta^2 R^*}{D_t^2 T^2} P_n \}

(7.18)

and \( \frac{\partial^2}{\partial P^2} (EAPM) \mid_{a_t \left[R^*, P^* \right]} < 0 \)

i.e., \( - E[\hat{Z}_{2m}] \frac{P^*}{P_n} + 3 E[\hat{Z}_{2m}] \frac{P^2}{P_n^2} + E[\hat{Z}_{3m}] \frac{P^2}{P_n^2} < 0 \)

(7.19)

and \( \left\{ \left\{ \frac{\partial^2}{\partial P^2} (EAPM) \right\} \right\} \left\{ \left\{ \frac{\partial^2}{\partial P^2} (EAPM) \right\} \right\} \mid_{a_t \left[R^*, P^* \right]} > 0 \)

(7.20)

Therefore, EAPM\( (R^*, P^*) \) is maximum if the relations (7.18), (7.19) and (7.20) holds and the corresponding optimum average profit of supplier and retailer are

\( EAPS (R^*, P^*) = T \left[ - E[\hat{Z}_{2m}] + E[\hat{Z}_{3m}] R^* + E[\hat{Z}_{4m}] \frac{P^*}{P_n} \right] \)

\( EAPR (R^*, P^*) = T \left[ E[\hat{Z}_{2m}] + E[\hat{Z}_{3m}] R^* + E[\hat{Z}_{4m}] \frac{P^2}{P_n^2} \right] \)

\( + E[\hat{Z}_{3m}] \frac{P^*}{P_n} \) where \( T = \frac{R^*}{D_t} R^* + t_0 \)

When \( P \) is only Decision Variable

The necessary conditions for maximum value of \( EAPM(P) \) is \( \frac{d}{dP} (EAPM) = 0 \)

i.e., \( E[\hat{Z}_{2m}] \frac{P}{P_n^2} - 2 E[\hat{Z}_{3m}] \frac{P^2}{P_n^2} - E[\hat{Z}_{4m}] \frac{P^2}{P_n^2} - E[\hat{Z}_{3m}] R^* = 0 \)

(7.21)

which gives the optimum value of \( P \), say \( P^{**} \).

If \( \frac{d}{dP} (EAPM) < 0 \) hold for \( P = P^{**} \) then EAPM\( (P^{**}) \) is maximum.

Now \( \frac{d^2}{dP^2} (EAPM) \mid_{a_t \left[R^*, P^{**} \right]} < 0 \) gives \( E[\hat{Z}_{2m}] - 3 E[\hat{Z}_{2m}] \frac{P^*}{P_n} - E[\hat{Z}_{3m}] R^* < 0 \)

(7.22)

Therefore, EAPM\( (P^{**}) \) is maximum if the relation (7.22) holds and corresponding optimum average profit of supplier and retailer are respectively

\( EAPS (P^{**}) = \frac{1}{T} \left[ - E[\hat{Z}_{3m}] + E[\hat{Z}_{2m}] R^* + E[\hat{Z}_{3m}] \frac{P^*}{P_n} \right] \)

\( EAPR (P^{**}) = \frac{1}{T} \left[ E[\hat{Z}_{2m}] + E[\hat{Z}_{2m}] R^* + E[\hat{Z}_{3m}] R^* + E[\hat{Z}_{3m}] \frac{P^*}{P_n} \right] \)

\( + E[\hat{Z}_{3m}] \frac{P^*}{P_n} \) where \( T = \frac{R^*}{D_t} R^* + t_0 \)

184
7.3.8 Integrated Approach

When both $R$ and $P$ are Decision Variables:
The necessary conditions for maximum value of $EIAP(R, P)$ are $\frac{\partial}{\partial R}(EIAP) = 0$ and $\frac{\partial}{\partial P}(EIAP) = 0$ which gives respectively

\[
\begin{align*}
\left\{ 1 - \frac{(1 - \theta)R}{D,T} \right\} \left\{ \frac{1}{P}E[\tilde{Z}_1] + E[\tilde{Z}_2] \right\} &- \left\{ 1 + \frac{(1 - \theta)R}{2D,T} \right\} \left\{ \frac{R}{P}E[\tilde{Z}_3] \right. \\
+ 2RBE[\tilde{Z}_3] + 2BE[\tilde{Z}_4] \} - \frac{(1 - \theta)R}{D,T} E[\tilde{Z}_3] + \left\{ \frac{2R}{P} - \frac{(1 - \theta)R^2}{D,T} \right\} E[\tilde{Z}_4] \\
+ \left\{ \frac{3R^2}{P} - \frac{(1 - \theta)R^3}{D,T} \right\} E[\tilde{Z}_5] \right. \\
+ \left\{ \frac{AP^3}{P^2} - \frac{(1 - \theta)R^4}{D,T} \right\} E[\tilde{Z}_8] & = 0
\end{align*}
\] (7.23)

and
\[
\begin{align*}
E[\tilde{Z}_1] + E[\tilde{Z}_3] \frac{R}{P} & + 2E[\tilde{Z}_4] \frac{R}{P} + E[\tilde{Z}_5] R P^2 + E[\tilde{Z}_7] R^2 + 2E[\tilde{Z}_8] \frac{R^2}{P^2} & = 0
\end{align*}
\] (7.24)

Solving (7.23) and (7.24), we can obtain the optimum value of $R$ and $P$, say $R^*$ and $P^*$.

If $\left\{ \frac{\partial}{\partial R}(EIAP) \right\} - \left\{ \frac{\partial}{\partial P}(EIAP) \right\}^2 > 0$, $\frac{\partial^2}{\partial R^2}(EIAP) < 0$, and $\frac{\partial^2}{\partial P^2}(EIAP) < 0$ holds for $R = R^*$ and $P = P^*$ then $EIAP(R^*, P^*)$ is maximum.

Now $\left( \frac{\partial^2}{\partial P^2}(EIAP) \right)_{R=R^*,P=P^*} < 0$

\[
\begin{align*}
i.e., & \frac{2(1 - \theta)^2}{D,T^2} E[\tilde{Z}_3] + \frac{2(1 - \theta)}{D,T^2} \left\{ \frac{(1 - \theta)R^*}{D,T^2} - 1 \right\} \left\{ \frac{1}{P}E[\tilde{Z}_1] + E[\tilde{Z}_2] \right\} \\
& - \left\{ 1 + \frac{2(1 - \theta)^2 R^*}{D,T^2} - \frac{2(1 - \theta)R^*}{D,T^2} \right\} E[\tilde{Z}_3] + \left\{ \frac{2}{P^2} + \frac{2(1 - \theta)^2 R^*}{D,T^2} \right\} P^2 \\
& - \frac{4(1 - \theta)R^*}{D,T^2} \} E[\tilde{Z}_4] - \left\{ 2P^2 + \frac{2(1 - \theta)^2 P^2 R^*}{D,T^2} - \frac{4(1 - \theta)P^2 R^*}{D,T^2} \right\} E[\tilde{Z}_5] \\
& - \left\{ 2 + \frac{2(1 - \theta)^2 R^*}{D,T^2} - \frac{4(1 - \theta)R^*}{D,T^2} \right\} E[\tilde{Z}_6] + \left\{ \frac{6R^*}{P^2} + \frac{2(1 - \theta)^2 R^*}{D,T^2} \right\} P^2 \\
& - \frac{6(1 - \theta)R^*}{D,T^2} \} E[\tilde{Z}_7] + \left\{ \frac{12R^*}{P^2} + \frac{2(1 - \theta)^2 R^*}{D,T^2} \right\} P^2 \right\} E[\tilde{Z}_8] < 0
\end{align*}
\] (7.25)

and $\left( \frac{\partial^2}{\partial F^2}(EIAP) \right)_{R=R^*,P=P^*} < 0$

\[
\begin{align*}
i.e., \left\{ 2E[\tilde{Z}_1] \frac{R}{P^3} + E[\tilde{Z}_3] \frac{R^2}{P^3} + 6E[\tilde{Z}_4] \frac{R^2}{P^3} + 2E[\tilde{Z}_5] \frac{R^3}{P^3} + 6E[\tilde{Z}_6] \frac{R^4}{P^3} \right\} < 0
\end{align*}
\] (7.26)

and
\[
\left( \frac{\partial^2}{\partial R^2}(EIAP) \right) - \left( \frac{\partial^2}{\partial P^2}(EIAP) \right) \right\}_{R=R^*,P=P^*} > 0
\] (7.27)
Therefore, \( IAP(R_1, P^*) \) is maximum if the relations (7.25), (7.26) and (7.27) holds and corresponding optimum integrated average profit of the supply chain is

\[
IAP(R_1, P^*) = \frac{1}{T} \left[ E[\tilde{Z}_0] + E[\tilde{Z}_1]\frac{R_1}{P_1} + E[\tilde{Z}_2]R_1 - Z_0\frac{R_1^2}{P_1} + E[\tilde{Z}_3]\frac{R_2}{P_2} \right.

\[
- E[\tilde{Z}_4]P_2^2 + E[\tilde{Z}_5] R_1^2 + E[\tilde{Z}_6]\frac{R_1}{P_1} + E[\tilde{Z}_7]\frac{R_1^2}{P_1} \left. \right] .
\]

where \( T = \frac{1}{D_r} R_1 + t_0 \)

When \( P \) is only Decision Variable:
The necessary conditions for maximum value of \( IAP(P) \) is \( \frac{d}{dP} (IAP) = 0 \)

i.e., \( E[\tilde{Z}_1]\frac{R}{P_1} - E[\tilde{Z}_3]\frac{R^2}{P_1} + 2E[\tilde{Z}_2]\frac{R}{P_1} + E[\tilde{Z}_4]R_1^3 - E[\tilde{Z}_5] R_1^2 + 2E[\tilde{Z}_6] \frac{R_1}{P_1} = 0 \)

which gives the optimum value of \( P \), say \( P^{**} \).

If \( \frac{d^2}{dP^2}(IAP) < 0 \) hold for \( P = P^{**} \) then \( IAP(P^{**}) \) is maximum.

Now \( \frac{d^2}{dP^2}(IAP)|_{P = P^{**}} < 0 \) gives

\[
-2E[\tilde{Z}_4]\frac{R}{P^{**}} + E[\tilde{Z}_3]\frac{R^2}{P^{**}} - 6E[\tilde{Z}_5] \frac{R^{3}}{P^{**}} - 2E[\tilde{Z}_6] \frac{R^3}{P^{**}} - 6E[\tilde{Z}_7] \frac{R^3}{P^{**}} = 0 \quad (7.28)
\]

Therefore, \( IAP(P^{**}) \) is maximum if the relation (7.28) hold and corresponding maximum integrated average profit of the supply chain is

\[
IAP(P^{**}) = \frac{1}{T} \left[ E[\tilde{Z}_0] + E[\tilde{Z}_1]\frac{R}{P^{**}} + Z_0\frac{R^2}{2P^{**}} + E[\tilde{Z}_3]\frac{R_2}{P_2^{**}} \right.

\[
- Z_5P_2R_2^2 + E[\tilde{Z}_6] R_1^2 + E[\tilde{Z}_7]\frac{R_1}{P^{**}} + E[\tilde{Z}_8]\frac{R_1^2}{P^{**}} \left. \right] .
\]

7.4 Numerical Illustration

To illustrate the proposed production inventory model, we consider the following numerical data in Table 7.1 and 7.2. The optimal values of the decision variables and corresponding profits are given in Table 7.3 and 7.4. Also sensitivity analysis has been performed of the profits, production rate (\( P \)) and inventory level (\( R \)) of supplier with respect to different parameters are shown in Figure 7.3 to Figure 7.12.
7.4. NUMERICAL ILLUSTRATION

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>Parameters</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.11</td>
<td>$r_{cm}$</td>
<td>2</td>
<td>$C_s$</td>
<td>40</td>
<td>$S$</td>
<td>335</td>
</tr>
<tr>
<td>$x$</td>
<td>20</td>
<td>$s_{cm}$</td>
<td>158</td>
<td>$w_s$</td>
<td>90</td>
<td>$W_1$</td>
<td>45</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.10</td>
<td>$s'_m$</td>
<td>108</td>
<td>$w'_s$</td>
<td>55</td>
<td>$W_2$</td>
<td>16</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.44</td>
<td>$s_p$</td>
<td>176</td>
<td>$R_1$</td>
<td>16</td>
<td>$a$</td>
<td>51</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.78</td>
<td>$s'_p$</td>
<td>135</td>
<td>$H$</td>
<td>0.02</td>
<td>$b$</td>
<td>0.17</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>20</td>
<td>$S_c$</td>
<td>0.35</td>
<td>$G$</td>
<td>10</td>
<td>$D_f$</td>
<td>35</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.2</td>
<td>$D_f$</td>
<td>36</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Fu-Ro parameters</th>
<th>Fu-Ro value</th>
<th>Input values</th>
<th>Expected value</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{F}_s$</td>
<td>near roughly(0.5)</td>
<td>(149, 150, 151, 152) with ([0.04, 0.04], [0.08, 0.08])</td>
<td>$E(\tilde{F}_s)$</td>
<td>0.505</td>
</tr>
<tr>
<td>$\tilde{F}_{cm}$</td>
<td>near roughly(1.7)</td>
<td>(115, 116, 117, 118) with ([0.2, 0.2], [0.4, 0.4])</td>
<td>$E(\tilde{F}_{cm})$</td>
<td>1.65</td>
</tr>
<tr>
<td>$\tilde{F}_{cm}$</td>
<td>near roughly(1.5)</td>
<td>(135, 136, 137, 138) with ([0.16, 0.18], [0.36, 0.38])</td>
<td>$E(\tilde{F}_{cm})$</td>
<td>1.45</td>
</tr>
<tr>
<td>$\tilde{F}_y$</td>
<td>near roughly(1.6)</td>
<td>(125, 126, 127, 128) with ([0.12, 0.14], [0.24, 0.26])</td>
<td>$E(\tilde{F}_y)$</td>
<td>1.61</td>
</tr>
<tr>
<td>$\tilde{F}_y$</td>
<td>near roughly(1.3)</td>
<td>(115, 116, 117, 118) with ([0.11, 0.13], [0.22, 0.24])</td>
<td>$E(\tilde{F}_y)$</td>
<td>1.26</td>
</tr>
<tr>
<td>$\tilde{F}_{cm}$</td>
<td>near roughly(1.8)</td>
<td>(135, 136, 137, 138) with ([0.13, 0.15], [0.26, 0.28])</td>
<td>$E(\tilde{F}_{cm})$</td>
<td>1.78</td>
</tr>
<tr>
<td>$\tilde{F}_o$</td>
<td>near roughly(2.0)</td>
<td>(125, 126, 127, 128) with ([0.13, 0.15], [0.26, 0.28])</td>
<td>$E(\tilde{F}_o)$</td>
<td>1.45</td>
</tr>
<tr>
<td>$\tilde{F}_o$</td>
<td>near roughly(5.00)</td>
<td>(135, 136, 137, 138) with ([0.29, 0.31], [0.53, 0.55])</td>
<td>$E(\tilde{F}_o)$</td>
<td>4.91</td>
</tr>
<tr>
<td>$\tilde{F}_p$</td>
<td>near roughly(0.40)</td>
<td>(135, 136, 137, 138) with ([0.3, 0.35], [0.7, 0.75])</td>
<td>$E(\tilde{F}_p)$</td>
<td>4.32</td>
</tr>
<tr>
<td>$\tilde{F}_p$</td>
<td>near roughly(1.00)</td>
<td>(135, 136, 137, 138) with ([0.4, 0.45], [0.9, 0.95])</td>
<td>$E(\tilde{F}_p)$</td>
<td>1.00</td>
</tr>
</tbody>
</table>

where value of $\tilde{F}_s$ is near roughly $(0.5) = (139, 140, 141, 142)$ with oscillation $([-0.04, 0.04], [-0.08, 0.08])$ means that $0.49 \pm ([0.15, 0.13], [0.49, 0.51])$, $0.50 \pm ([0.46, 0.48], [0.54, 0.56])$, $0.51 \pm ([0.47, 0.49], [0.53, 0.59])$ and $0.52 \pm ([0.48, 0.52], [0.44, 0.60])$.
CHAPTER 7. THREE LAYERS SUPPLY CHAIN IN AN IMPERFECT PRODUCTION INVENTORY MODEL WITH TWO STORAGE FACILITIES UNDER FUZZY ROUGH ENVIRONMENT

Table 7.3: Optimal result when $P$ and $R$ are decision variables

<table>
<thead>
<tr>
<th>Approach</th>
<th>Total profit</th>
<th>EAPS</th>
<th>EAPM</th>
<th>EAPR</th>
<th>$R$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stakelberg</td>
<td>5126.616</td>
<td>1543.341</td>
<td>1814.635</td>
<td>1656.160</td>
<td>3217.589</td>
<td>64.9821</td>
</tr>
<tr>
<td>Integrated</td>
<td>5176.606</td>
<td>1596.751</td>
<td>1872.695</td>
<td>1707.160</td>
<td>3189.629</td>
<td>69.8521</td>
</tr>
</tbody>
</table>

Table 7.4: Optimal result when $P$ is decision variable and $R = 3225$

<table>
<thead>
<tr>
<th>Approach</th>
<th>Total profit</th>
<th>EAPS</th>
<th>EAPM</th>
<th>EAPR</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stakelberg</td>
<td>5012.614</td>
<td>1512.752</td>
<td>1714.346</td>
<td>1751.426</td>
<td>63.64570</td>
</tr>
<tr>
<td>Integrated</td>
<td>5049.634</td>
<td>1582.912</td>
<td>1759.146</td>
<td>1707.576</td>
<td>67.37870</td>
</tr>
</tbody>
</table>

Figure 7.3: Reliability parameter ($\alpha$) vs $P$

Figure 7.4: Reliability parameter ($\alpha$) vs $R$

Figure 7.5: Reliability parameter vs profit

Figure 7.6: Reliability parameter ($\alpha$) vs IAP

188
7.4. NUMERICAL ILLUSTRATION

Figure 7.7: Defective rate (θ) vs R

Figure 7.8: Defective rate (θ) vs APS

Figure 7.9: Defective rate (θ) vs IAP

Figure 7.10: Defective rate (θ) vs APS, IAP

Figure 7.11: Screening rate (x) vs APS

Figure 7.12: Screening rate (x) vs IAP
CHAPTER 7. THREE LAYERS SUPPLY CHAIN IN AN IMPERFECT PRODUCTION INVENTORY MODEL WITH TWO STORAGE FACILITIES UNDER FUZZY ROUGH ENVIRONMENT

7.4.1 Discussion

From Table 7.3 and 7.4, it is observed that profits under the integrated approach is greater than the Stakelberg approach and hence the former approach is better than the later approach. The sensitivity analysis in section 7.1 shows that with increase of the reliability parameter $\alpha$, (i) the profits of the supplier and retailer are slightly increasing (Figure 7.5), (ii) the values of $P$ and $R$ are gradually increasing (Figure 7.3 & 7.4) but (iii) both profits of the manufacturer (APM) and the integrated profit (IAP) are gradually decreasing (Figure 7.5 & 7.6). From Figure 7.7 to 7.10 it is also seen that the values of APS and IAP are decreasing but initial amount of inventory level of supplier is increasing with increase of $\theta$. It is also noted that the values of APS and IAP increase with the screening rate ($x$) of the supplier.

7.5 Conclusion

This chapter develops a three layer supply chain production inventory model involving supplier, manufacturer and retailer as the members of the chain who are responsible for performing the raw materials into finished product and make them available to satisfy customers’ demand in time. In comparing with the existing literature on the supply chain, the followings are the main contributions in the proposed model:

- Inspection cost is incurred during the production run time and manufacturer continuously inspects as well as separates the perfect quality items, less perfect quality items, repairable items which are transformed into perfect quality items after some rework, and rejected items.
- Here, reworked cost is considered by the manufacturer to repair a certain percent of imperfect quality items. Demand rate of customers for perfect quality items and less perfect quality items are respectively assumed to be stock dependent and selling price dependent. Here retailer have two showrooms $PW_1$ and $PW_2$ of finite capacities at busy market place and the market demands of perfect and less perfect quality items are respectively met through the showrooms $PW_1$ and $PW_2$. Retailer has a secondary warehouse SW of infinite capacity, away from busy market place, to store the excess amount perfect quality items from where the items are continuously transferred to the showroom. It is considered that the holding cost per unit per unit time at SW is less than the holding cost at $PW_1$ per unit per unit time. The repairing costs of corrective and preventive maintenance should also be considered, as these costs increase the unit production cost. Inventory and production decisions are made at the supplier, manufacturer and retailer levels. Actually in this chapter, the coordination between production and inventory decisions has been established across the supply chain so that integrated average profit of the chain is maximum.