CHAPTER 6

GENERALIZED FIBONACCI GRACEFUL GRAPHS

6.1 INTRODUCTION

A graph lableing is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Given a graph $G = (V, E)$, the set N of non-negative integers and a commutative binary operation $* : N \times N \to N$, every vertex function $f : V(G) \to N$ induces an edge function $f^* : E(G) \to N$ such that $f^*(uv) = f(u) * f(v)$ for all $uv \in E(G)$.

Rosa [30] introduced a (p, q) graph $G = (V, E)$ to be graceful if f is an injection from the vertices of G to the set $\{0, 1, 2, 3, \ldots, q\}$ such that, when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct.

Acharya and Hegde [1] generalized graceful labeling to (k, d)-graceful labeling by permitting the vertex labels to belong to $\{0, 1, 2, \ \ldots, k + (q - 1)d\}$ and requiring the set of edge labels induced by the absolute difference of labels of adjacent vertices to be $\{k, k + d, \ldots, k + (q - 1)d\}$ where k and d are positive integers.

Graham and Sloane defined a graph G with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that, when each xy is assigned the label $(f(x) + f(y))(mod\ q)$, the resulting edge labels are distinct.

A generalization of harmonious labelings are felicitous which was introduced by Lee, Schmeichel and Shee [25].

An injective function f from the vertices of a graph G with q edges to the set $\{0, 1, 2, \ldots, q\}$ is called felicitous if the edge labels induced by $(f(x) + f(y))(mod\ q)$ for each edge xy are distinct.

Kathiresan and Amutha [21] introduced Fibonacci graceful labeling f of the vertices of G with distinct integers from the set $\{0, 1, 2, \ldots, F_q\}$ where F_q is the q^{th} Fibonacci number (that is $F_1 = 1, F_2 = 2, F_3 = 3$ etc) such that the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$.

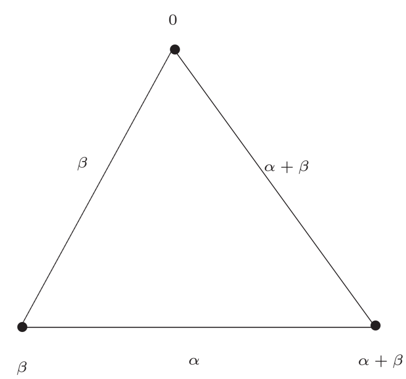
Motivated by these definitions, we generalized the above concept as the new type labeling called the generalized Fibonacci graceful labeling.

Definition 6.1.1. Let $G(p, q)$ be a graph. The function $f : V(G) \rightarrow \{0, \alpha, \beta, \alpha + \beta, \ldots, s_q\}$ where s_q is the q^{th} number of the sequence s_n (that is $s_1 = \alpha, s_2 = \beta, s_n = s_{n-1} + s_{n-2}, n > 2$) is said to be generalized Fibonacci graceful if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{s_1, s_2, s_3, \ldots, s_q\}$.

If a graph $G(p, q)$ admits a generalized Fibonacci graceful labeling then G is called ıgeneralized Fibonacci graceful.

Remark 6.1.2. If $\alpha = 1, \beta = 2$ in the above definition, we will get the Fibonacci graceful graphs.

The following is the simple example for generalized Fibonacci graceful labeling for K_3.



Observation 6.1.3. The edge numbered s_q must have 0 and s_q as the vertex numbers. Any vertex adjacent to a vertex labeled 0 must be labeled with the numbers of the sequence s_n.

Theorem 6.1.4. *K_n is generalized Fibonacci graceful if and only if $n \leq 3$.*

Proof. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of K_n.

In K_n, each v_i is adjacent to all other vertices. Let v_1 be labeled as 0. By the Observation 6.1.3 all the vertices except v_1 must be labeled with the numbers of the sequence s_n. Let v_1 and v_2 be labeled as 0 and s_q respectively. Thus v_3 must be either s_{q-1} or s_{q-2} so that the edge v_2v_3 will get a number s_{q-2} or s_{q-1}. Otherwise the edge v_2v_3 will not get a generalized Fibonacci number.

Without loss of generality, let v_3 be labeled s_{q-2} so that the edge v_2v_3 will get s_{q-1}. If we label v_4 as s_{q-1} then the edge v_1v_4 will get s_{q-1} which is already received by v_2v_3.

If we label v_4 except s_q, s_{q-1}, s_{q-2}, the edge v_2v_4 and v_1v_4 will not get the number of the sequence s_q.

Hence K_n is not generalized Fibonacci graceful if $n > 4$.

□

Theorem 6.1.5. *The complete bipartite graph $K_{m,n}$ is not generalized Fibonacci graceful for all $m, n > 2$.*

Proof. Let $V_1 = \{v_i/1 \leq i \leq m\}$ and $V_2 = \{u_j/1 \leq j \leq n\}$ be the vertices of two partition sets of $K_{m,n}$.

Suppose the vertex v_1 is labeled 0. Then the vertices $u_1, u_2, \ldots, u_n$ are labeled with the generalized Fibonacci numbers. But we can't find a number α to label $v_2, v_3, \ldots, v_m$ such that $|f(u_j) - \alpha|$ is equal to a generalized Fibonacci number for $1 \leq j \leq n$.

Hence $K_{m,n}$ is not a generalized Fibonacci graceful. □

Theorem 6.1.6. *Any path P_n is generalized Fibonacci graceful.*

Proof. Let G be a path P_n.

Let $V(G) = \{v_i : 1 \leq i \leq n\}$ be the vertices of G and

$E(G) = \{v_i, v_{i+1} : 1 \leq i \leq n-1\}$ be the edges of G.

Define an injective function

$f : V(G) \rightarrow [0, s_{n-1}]$ where s_{n-1} is the $(n-1)^{th}$ number of the sequence by

$$
\begin{aligned}
f(v_1) &= 0 \\
f(v_i) &= f(v_{i-1}) + (-1)^i s_{n-(i-1)} \text{ for } 2 \leq i \leq n-2 \\
f(v_{n-1}) &= f(v_{n-2}) + (-1)^{n-2} s_1 \\
f(v_n) &= f(v_{n-1}) + (-1)^{n-1} s_2 \\
\min_{v \in V} f(v) &= \min_{v \in V} \{f(v_1), f(v_i) : 2 \leq i \leq n\} \\
&= \min\{0, f(v_2), f(v_3), \ldots, f(v_n)\} \\
&= 0. \\
\max_{v \in V} f(v) &= \max_{v \in V} \{f(v_1), f(v_i) : 2 \leq i \leq n\} \\
&= \max_{v \in V} \{0, s_{n-1}, s_{n-1} - s_{n-2}, \ldots, \} \\
&= s_{n-1}.
\end{aligned}
$$

Thus f is a function from $V(G)$ into the set $[0, s_{n-1}]$.

Next we claim that f is one-one. When n is odd, we have

$f(v_2) > f(v_4) > \cdots > f(v_{n-1}) > f(v_n) > f(v_{n-2}) > \cdots > f(v_3) > f(v_1)$.

When n is even, we have

$f(v_2) > f(v_4) > \cdots > f(v_n) > f(v_{n-1}) > \cdots > f(v_3) > f(v_1)$.

Thus f is one-one.

New let us find the edge labelings.

$$\begin{aligned}
\text{Let } E &= \{f(v_i v_{i+1}) : 1 \le i \le n-1\} \\
&= \{f(v_1 v_2), f(v_2 v_3), f(v_3, v_4), \dots, f(v_{n-1} v_n)\} \\
&= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|, |f(v_3) - f(v_4)|, \\
&\quad \cdots |f(v_{n-1} - f(v_n)|\} \\
&= \{|f(v_1) - f(v_1) + s_{n-1}|, |f(v_2) - f(v_2) + s_{n-2}|, \\
&\quad |f(v_3) - f(v_3) + s_{n-3}|, \dots \\
&\quad |f(v_{n-1} - f(v_{n-1}) - (-1)^{n-1} s_2|\} \\
&= \{s_{n-1}, s_{n-2}, s_{n-3}, \dots, s_3, s_1, s_2\}.
\end{aligned}$$

Hence P_n is generalized Fibonacci graceful. □

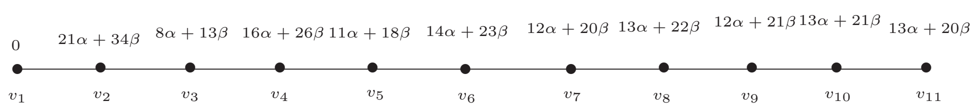


Figure: Generalized Fibonacci Graceful Labeling of P_{11}

Theorem 6.1.7. *P_n^2 is generalized Fibonacci graph, where P_n is a path of n vertices.*

Proof. Let $G = P_n^2$, where P_n^2 is the $2th$ power of P_n. Let $V(G) = \{v_i / 1 \le i \le n\}$ be the vertices of G and let $E(G) = \{v_i v_{i+1} / 1 \le i \le n-1\} \cup \{v_{i-2} v_i / 3 \le i \le n\}$ so that the graph G has n vertices and $2n - 3$ edges.

Define an injection $f : V(G) \to \{0, s_1, s_2, \dots, s_q\}$ where $q = 2n - 3$ by

$f(v_1) = 0$ and $f(v_i) = s_q - s_{q-2} + s_{q-4} \cdots + (-1)^{i-2} s_{q-2(i-2)}$ for $i = 2, 3, \ldots, n$.

From the definition of f,

$$\min_{v \in V} = \min_{v \in V} \{f(v_1), f(v_2), f(v_3), \ldots, f(v_n)\}$$
$$= \min_{v \in V} \{0, s_q, s_q - s_{q-2}, \ldots, s_q - s_{q-2} + s_{q-4} + \cdots + (-1)^{n-2} s_1\}$$
$$= 0$$
$$\max_{v \in V} = \max\{0, s_q, s_q - s_{q-2}, \ldots, s_q - s_{q-2} + s_{q-4} + \cdots + (-1)^{n-2} s_1\}$$
$$= s_q.$$

Thus f is a function from $V(G)$ into $\{0, s_1, s_2, \ldots, s_q\}$.

Next we claim that f is one-one. We see that,

$f(v_2) > f(v_4) > \cdots > f(v_{n-1}) > f(v_n) > f(v_{n-2}) > f(v_{n-4}) > \cdots > f(v_1)$ if n is odd and

$f(v_2) > f(v_4) > \cdots > f(v_n) > f(v_{n-1}) > f(v_{n-3}) > \cdots > f(v_1)$ if n is even.

Thus f is one-one.

Next we claim that the edge labels are distinct.

$$\text{Let } E_1 = \{f(v_i v_{i+1} / 1 \leq i \leq n-1\}$$
$$= \{f(v_1 v_2), f(v_2 v_3), f(v_3 v_4), \ldots, f(v_{n-1} v_n)\}$$
$$= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|,$$
$$|f(v_3) - f(v_4)|, \ldots, |f(v_{n-1}) - f(v_n)|\}$$

$$= \{|0 - s_q|, |s_q - (s_q - s_{q-2})|, \ldots$$

$$|s_q - s_{q-2} + s_{q-4} - \cdots + (-1)^{n-3} s_3$$

$$- (s_q - s_{q-2} + s_{q-4} - \cdots + (-1)^{n-3} s_3 + (-1)^{n-2} s_1\}$$

$$= \{s_q, s_{q-2}, s_{q-4}, \ldots, s_1\}$$

$$\text{Let } E_2 = \{f(v_{i-2}v_i)/3 \leq i \leq n\}$$

$$= \{f(v_1v_3), f(v_2v_4), f(v_3v_5), \ldots, f(v_{n-2}v_n)\}$$

$$= \{|f(v_1) - f(v_3)|, |f(v_2) - f(v_4)|, |f(v_3) - f(v_5)|,$$

$$\ldots |f(v_{n-2}) - f(v_n)|\}$$

$$= \{|0 - (s_q - s_{q-2})|, |s_q - (s_q - (s_q - s_{q-2} + s_{q-4})|, \ldots,$$

$$|s_q - s_{q-2} + s_{q-4} - \cdots + (-1)^{n-4} s_5$$

$$- (s_q - s_{q-2} + s_{q-4} - \cdots + (-1)^{n-4} s_5$$

$$- (s_q - s_{q-2} + s_{q_4} - \cdots + (-1)^{n-4} s_5$$

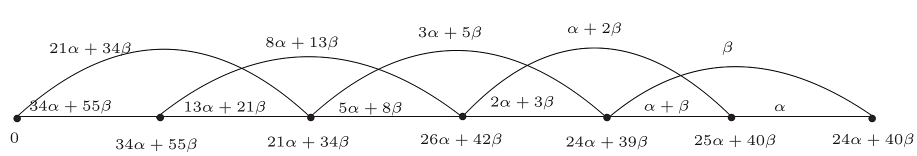
$$+ (-1)^{n-3} s_3 + (-1)^{n-2} s_q)|\}$$

$$= \{s_q - s_{q-2}, s_{q-2} - s_{q-4}, \ldots, s_3 - s_1\}$$

$$= \{s_{q-1}, s_{q-3}, \ldots, s_2\}.$$

Therefore $E_1 \cup E_2 = \{s_1, s_2, s_3, \ldots, s_{q-1} s_q\}$. Hence the edge labels are distinct. Thus P_n^2 is generalized Fibonacci graceful graph. □

Example 6.1.8.



Theorem 6.1.9. *A caterpillar is generalized Fibonacci graceful.*

Proof. Let G be a caterpillar.

Let $\{v_i/1 \leq i \leq m\}$ be the vertices of the path of G. Let $\{v_{ij}/1 \leq i \leq m, 1 \leq j \leq n_i\}$ be the leaves of G. Then $E(G) = \{v_i v_{i+1}/1 \leq i \leq m-1\} \cup \{v_i v_{ij}/1 \leq i \leq m, 1 \leq j \leq n_i\}$. Then $E(G) = m - 1 + n_1 + n_2 + n_3 + \cdots + n_m = n$ (say)

Define $f : E(G) \rightarrow [0, s_n]$, where $[0, s_n]$ is the set of all non-negative integers from 0 to s_n by

$$f(v_1) = 0$$
$$f(v_{1j}) = S_{n-(j-1)} : 1 \leq j \leq n_1$$
$$f(v_2) = f(v_1) + s_{n-n_1}$$
$$f(v_{2j}) = f(v_2) - s_{n-n_1-j} : 1 \leq j \leq n_2$$
$$f(v_3) = f(v_2) - s_{n-(n_1+n_2)-1}$$
$$f(v_{3j}) = f(v_3) + s_{n-(n_1+n_2)-1-j} : 1 \leq j \leq n_3$$
$$f(v_4) = f(v_3) + s_{n-(n_1+n_2+n_3)-2}$$
$$f(v_{4j}) = f(v_4) - s_{n-(n_1+n_2+n_3)-2-j} : 1 \leq j \leq n_4$$

Proceeding like this

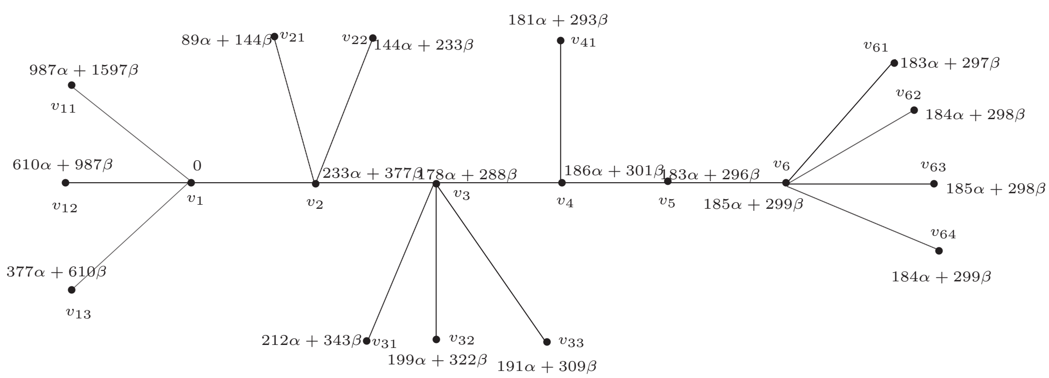
$$f(v_m) = \begin{cases} f(v_{m-1}) - s_{n-(n_1+n_2+n_3+\cdots+n_{m-1})-(m-2)} & \text{if } m \text{ is odd} \\ f(v_{m-1}) + s_{n-(n_1+n_2+\cdots+n_{m-1})-(m-2)} & \text{if } m \text{ is even} \end{cases}$$

$$f(v_{mj}) = \begin{cases} f(v_m) - s_{n-(n_1+n_2+\cdots+n_{m-1})-(m-2)-j} & \text{if } m \text{ is odd, } 1 \leq j \leq n_m \\ f(v_m) + s_{n-(n_1+n_2+\cdots+n_{m-1})-(m-2)-j} & \text{if } m \text{ is even, } 1 \leq j \leq n_m \end{cases}$$

For $j = n_m$

$$
\begin{aligned}
f(v_m n_m) &= f(v_m) \pm S_{n-(n_1+n_2+\cdots+n_{m-1})-(m-2)-n_m} \\
&= f(v_m) \pm S_{n-(n_1+n_2+\cdots+n_m)-m+2} \\
& f(v_m) \pm S_1.
\end{aligned}
$$

Clearly the vertex labels are distinct and resulting edge labels are of the form $\{S_1, S_2, S_3, \ldots, S_n\}$. Hence G is generalized Fibonacci graceful. □



Corollary 6.1.10. *The bistar $B_{m,n}$ is generalized Fibonacci graceful graph.*

Theorem 6.1.11. *Oliver tree O_n are generalized Fibonacci graceful where $n \geq 3$.*

Proof. Let G be the oliver tree O_n having n paths of lengths $1, 2, 3, \ldots, n$ adjoint to one vertex v.

Let $V(G) = \{v\} \cup \{v_{ji} : 1 \leq i \leq n, 1 \leq j \leq n, j \leq i\}$ be the vertices of G and let $E(G) = \{vv_{1i} : 1 \leq i \leq n\} \cup \{v_{ji}v_{j+1i} : 1 \leq j \leq n-1, 2 \leq i \leq n, j \leq i\}$ be the edge set of G.

Now $|E(G)| = \frac{n(n+1)}{2} = q$ (say)

Define an injection $f : V(G) \to \{0, s_1, s_2, s_3, \ldots, s_q\}$ by

$$f(v) = 0$$

$$f(v_{1i}) = s_{(q-(i-1))} : 1 \le i \le n$$

$$f(v_{ji}) = \begin{cases} f(v_{j-1i}) - s_{q+\frac{j(j-5)}{2}} - j(n-1) + i + 1 : \\ 2 \le i \le n, 1 < j \le i, j \text{ is even} \\ f(v_{j-1i}) + s_{q+\frac{(j-1)(j-4)}{2}} - j(n-1) + i - 1 \\ 3 \le i \le n, 1 < j \le i,\ j\ is\ odd \end{cases}$$

From the definition of f,

$$\begin{aligned} \min_{v \in V} f(v) &= \min_{v \in V}\{\{f(v)\} \cup \{f(v_{1i}) : 1 \le i \le n\} \\ &\quad \cup \{f(v_{ji}) : 2 \le i \le n, 1 < j \le i\}\} \\ &= \min_{v \in V}\{0 \cup \{f(v_{1i}) : 1 \le i \le n\} \\ &\quad \cup \{f(v_{ji}) : 2 \le i \le n, 1, < j \le i\}\} \\ &= 0 \\ \max_{v \in V} f(v) &= \max_{v \in V}\{f(v) \cup \{f(v_{1i}) : 1 \le i \le n\} \\ &\quad \cup \{f(v_{ji}) : 2 \le i \le n, 1 < j \le i\}\} \\ &= \max_{v \in V}\{f(v) \cup \{f(v_{11}), f(v_{12}), \ldots, f(v_{1n})\} \\ &\quad \cup \{f(v_{ji}) : 2 \le i \le n, 1 < j \le i\}\} \\ &= f(v_{11}) \\ &= s_q. \end{aligned}$$

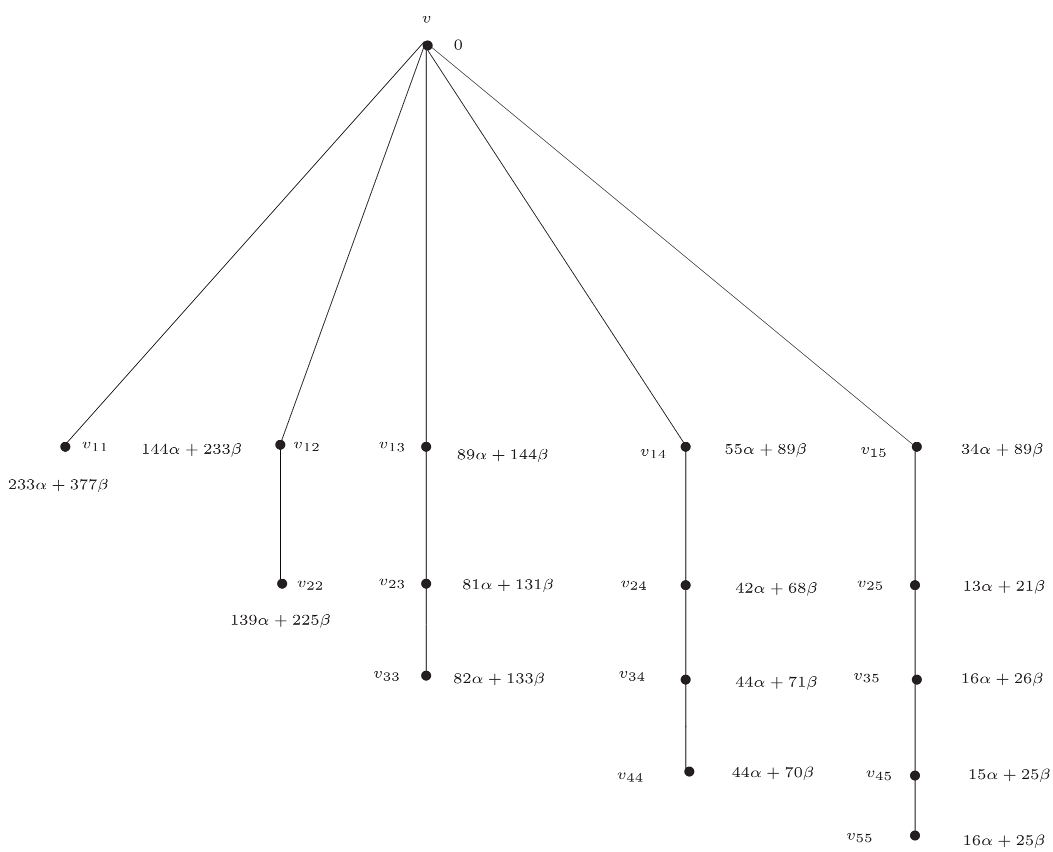
Thus f is a function from $V(G)$ into $\{0, s_1, s_2, \ldots, s_q\}$.

Next we claim that f is one-one. We see that,

$$f(v_{11}) > f(v_{12}) > f(v_{13}) > f(v_{14}) > \cdots > f(v_{1n})$$
$f(v_{22}) > f(v_{23}) > f(v_{24}) > \cdots > f(v_{2n})$
$f(v_{33}) > f(v_{34}) > \cdots > f(v_{3n})$
$\ldots f(v_{n-1n-1}) > f(v_{n-1n}) > f(v_{nn}) > f(v)$.

Also $f(v_{11}) < f(v_{22}) < f(v_{33}) < \cdots < f(v_{nn}) < f(v)$ and $f(v_{1n}) < f(v_{2n}) < f(v_{3n}) < \cdots < f(v_{n-1}v_n) < f(v)$.

Thus f is one-one. □



Theorem 6.1.12. *If G is Eulerian and generalized Fibonacci graceful then $q \equiv 0, 1 (mod\ 3)$.*

Proof. Since G is Eulerian, it can be decomposed into edge disjoint cycles. We have the result [16].

Suppose that integers not necessarily distinct are assigned to the vertices of a graph G and each edge of G is given an edge number equal to the absolute difference of the vertex numbers at its end points. Then the sum of the edge numbers around any circuit of G is even. By the above result, we have $s_1 + s_2 + s_3 + \cdots + s_q$ is even.

Case (i). For any α and β, by the nature of our generalized Fibonacci numbers, this is possible when q is of the form $3k$.

$$\therefore q \equiv 0 (mod\ 3).$$

Case (ii). If α is even and for any β, by the nature of our generalized Fibonacci numbers, this is possible when q is of the form $3k + 1$.

$$\therefore q \equiv 1 (mod\ 3).$$

If G is Eulerian and generalized Fibonacci graceful, then $q \equiv 0, 1 (mod\ 3)$. □

Remark 6.1.13. The above condition is only a necessary condition but not a sufficient condition. We have K_5 is Eulerian, but it is not generalized Fibonacci graceful.

Theorem 6.1.14. *The cycle C_n is generalized Fibonacci graceful if and only if $n \equiv 0, 1 (mod\ 3)$.*

Proof. Let G be a cycle of length n, C_n. Let $V(G) = \{v_i : 1 \leq i \leq n\}$ be the vertices of G and let $E(G) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$ be the edges of G. By the Theorem 6.1.12 if G is Eulerian generalized Fibonacci graceful, then $n \equiv 0, 1(mod\ 3)$.

Case (i). Suppose $n \equiv 0(mod\ 3)$, then $n = 3m$(say).

Let $\{v_1, v_2, v_3, \ldots, v_{3m}\}$ be the vertex set and $\{e_1, e_2, e_3, \ldots, e_{3m}\}$ be the edge set of G when $e_i = v_i v_{i+1}$ for $i = 1, 2, 3, \ldots, (3m-1)$ and $e_{3m} = v_{3m} v_1$. Define an injective function,

$f : V(G) \rightarrow \{0, s_1, s_2, s_3, \ldots, s_{3m}\}$ by

$$f(v_1) = 0$$

$$f(v_i) = \begin{cases} s_{i-1} & \text{if } i \equiv 2(mod\ 3) \\ s_i & \text{if } i \equiv 0(mod\ 3) \\ s_{i+1} & \text{if } i \equiv 1(mod\ 3) \end{cases}$$

We observe that the vertices $f(v_1) = 0$.

$v_2, v_5, v_8 \ldots v_{3m-1}$ receive $s_1, s_4, s_7 \ldots s_{3m-2}$ respectively

$v_3, v_6, v_9 \ldots v_{3m}$ receive $s_3, s_6, s_9 \ldots s_{3m}$ respectively

$v_4, v_7, v_{10} \ldots v_{3m-2}$ receive $s_5, s_8, s_{11} \ldots s_{3m-1}$ respectively.

Therefore $f(v) \in \{0, s_1, s_2, \ldots, s_{3m}\}$. Next we claim that f is one-one. From the definition of f,

$f(v_1) < f(v_2) < f(v_3) < f(v_5) < f(v_4) < f(v_6) < f(v_8) < f(v_7) < f(v_9) < f(v_{11}) < f(v_{10}) < \cdots < f(v_{3m})$.

Therefore f is one-one.

Next we claim that the edge labels are distinct.

Let $E_1 = \{f(v_2v_3), f(v_5v_6), f(v_8v_9), \dots, f(v_{3m-1}v_{3m})\}$

$= \{|s_1 - s_3|, |s_4 - s_6|, |s_7 - s_9|, \dots |s_{3m-2} - s_{3m}|\}$

$= \{s_2, s_5, s_8 \cdots s_{3m-1}\}$

Let $E_2 = \{f(v_3v_4), f(v_6v_7), f(v_9v_{10}) \dots f(v_{3m-3}v_{3m-2})\}$

$= \{|s_3 - s_5|, |s_6 - s_8|, |s_9 - s_{11}|, \dots |s_{3m-3} - s_{3m-1}|\}$

$= \{s_4, s_7, s_{10} \dots s_{3m-2}\}$

Let $E_3 = \{f(v_4v_5), f(v_7v_8), \dots, f(v_{3m-2}v_{3m-1}\}$

$= \{|s_5 - s_4|, |s_8 - s_7|, \dots, |s_{3m-1} - s_{3m-2}|\}$

$= \{s_3, s_6, s_9, \dots, s_{3m-3}\}$

Let $E_4 = \{f(v_1v_2), f(v_{3m}v_1)\}$

$= \{|0 - s_1|, |s_{3m} - 0|\}$

$= \{s_1, s_{3m}\}$.

Therefore $E_1 \cup E_2 \cup E_3 \cup E_4 = \{s_1, s_2, s_3, \dots, s_{3m}\}$. Thus the edge labels are distinct. Therefore G is generalized graceful, if $n \equiv 0(mod\ 3)$.

Case (ii). Suppose $n \equiv 1(mod\ 3)$, then $n \equiv 3m + 1$ (say)

Let $\{v_1, v_2, v_3, \dots, v_{3m+1}\}$ be the vertex set and $\{e_1, e_2, e_3, \dots, e_{3m+1}\}$ be the edge set of G where $e_i = v_iv_{i+1}$ for $i = 1, 2, 3, \dots, 3m$ and $e_{3m+1} = v_{3m+1}v_1$.

Define an injection $f : V(G) \to \{0, s_1, s_2, s_3, \ldots, s_{3m+1}\}$ by

$f(v_1) = 0$

$f(v_3) = s_1$ and

$$f(v_i) = \begin{cases} s_{i+1} & \text{if } i \equiv 2(mod\ 3) \\ s_{i-1} & \text{if } i \equiv 0(mod\ 3), \text{ when } i \neq 3 \\ s_i & \text{if } i \equiv 1(mod\ 3) \text{ and} \end{cases}$$

We observe that the vertices $f(v_1) = 0; f(v_3) = s_1$.

$v_2, v_5, v_8 \ldots v_{3m-1}$ receive $s_3, s_6, s_9 \ldots s_{3m}$ respectively

$v_6, v_9, v_{12} \ldots v_{3m}$ receive $s_5, s_8, s_{11} \ldots s_{3m-1}$ respectively

$v_4, v_7 \ldots v_{3m+1}$ receive $s_4, s_7 \ldots s_{3m+1}$ respectively.

Therefore $f(v) \in \{0, s_1, s_2, s_3, \ldots, s_{3m+1}\}$. Next we claim that f is one-one.

From the definition of f,

$f(v_1) < f(v_3) < f(v_2) < f(v_4) < f(v_6) < f(v_5) < f(v_7) < f(v_9) < f(v_8) < f(v_{10}) < \cdots < f(v_{3m+1})$.

Therefore f is one-one.

Next we claim that the edge labels are distinct.

$$\begin{aligned} \text{Let } E_1 &= \{f(v_6v_7), f(v_9v_{10}) \ldots f(v_{3m}v_{3m+1})\} \\ &= \{|s_5 - s_7|, |s_8 - s_{10}|, \ldots |s_{3m-1} - s_{3m+1}|\} \\ &= \{s_6, s_9, s_{12}, \ldots, s_{3m}\} \end{aligned}$$

$$E_2 = \{f(v_4v_5), f(v_7v_8), f(v_{10}v_{11}) \dots f(v_{3m-2}v_{3m-1})\}$$
$$= \{|s_4 - s_6|, |s_7 - s_9|, |s_{10} - s_{12}|, \dots |s_{3m-2} - s_{3m}|\}$$
$$= \{s_5, s_8, s_{11}, \dots, s_{3m-1}\}$$
$$E_3 = \{f(v_5v_6), f(v_8v_9), \dots, f(v_{3m-1}v_{3m})\}$$
$$= \{|f(v_5) - f(v_6)|, |f(v_8) - f(v_9)|, \dots, |f(v_{3m-1})| - f(v_{3m})|$$
$$= \{|s_6 - s_5|, |s_9 - s_8|, |s_{12} - s_{11}|, \dots, |s_{3m} - s_{3m-1}|\}$$
$$= \{s_4, s_7, s_{10}, \dots, s_{3m-2}\}$$
Let $E_4 = \{f(v_1v_2), f(v_2v_3), f(v_3v_4), f(v_{3m+1}v_1)\}$
$$= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|,$$
$$|f(v_3) - f(v_4)|, |f(v_{3m+1}) - f(v_1)|$$
$$= \{|0 - s_3|, |s_3 - s_1|, |s_1 - s_4|, |s_{3m+1} - 0|\}$$
$= \{s_3, s_2, s_1, s_{3m+1}\}$ where $s_1 - s_4 = 2s_2 = s_1$.

Therefore $E_1 \cup E_2 \cup E_3 \cup E_4 = \{s_1, s_2, s_3, s_4, \dots, s_{3m}, s_{3m+1}\}$. Thus the edge labels are distinct. Therefore G is generalized graceful if $n \equiv 1(mod\ 3)$. □

Example 6.1.15.

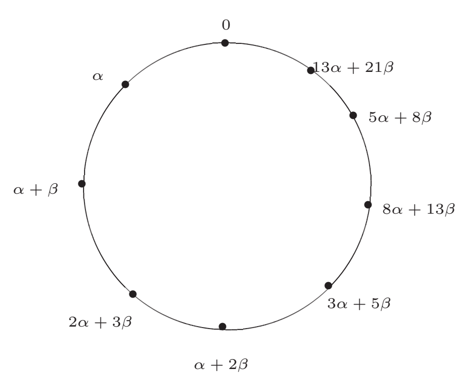


Figure: Generalized Fibonacci Graceful Labeling for C_9

Example 6.1.16.

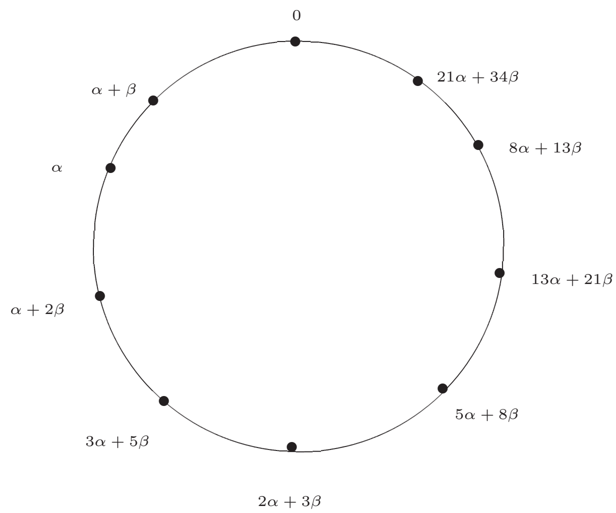


Figure: Generalized Fibonacci Graceful Labeling for C_{10}

Theorem 6.1.17. *Every fan F_n is a generalized Fibonacci graceful graph.*

Proof. Let $G = F_n$ be a fan F_n. Let $V(G) = \{v\} \cup \{v_i/1 \leq i \leq n\}$ be the vertex set. Let $E(G) = \{vv_i/1 \leq i \leq n\} \cup \{v_{i-1}v_i/2 \leq i \leq n\}$ be the edge set, so that $|V(G)| = n + 1$ and $|E(G)| = 2n - 1$.

Define an injection $f : V(G) \to \{0, s_1, s_2, \ldots, s_{2n-1}\}$ by

$f(v) = 0$ and $f(v_i) = s_{2i-1}$ for $i = 1, 2, 3, \ldots, n$.

From the definition of f

$$\begin{aligned}\min_{v \in V} f(v) &= \min_{v \in V}\{f(v), f(v_1), f(v_2), \ldots, f(v_n)\} \\ &= \min_{v \in V}\{0, s_1, s_3, \ldots, s_{2n-1}\} \\ &= 0 \text{ and}\end{aligned}$$

$$\begin{aligned}\max_{v\in V} f(v) &= \max_{v\in V}\{0, f(v_1), f(v_2), \ldots, f(v_n)\}\\ &= \max_{v\in V}\{0, s_1, s_3, s_5, \ldots, s_{2n-1}\}\\ &= s_{2n-1}.\end{aligned}$$

Thus f is a function from $V(G)$ into $\{0, s_1, s_2, \ldots, s_{2n-1}\}$. We claim that f is one-one.

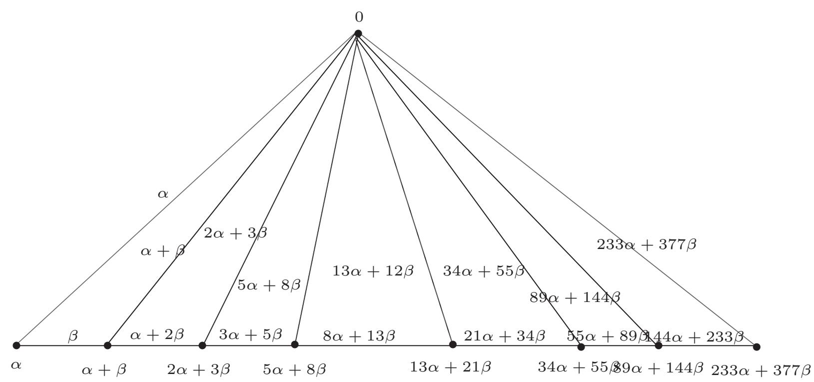
We see that $f(v) < f(v_1) < f(v_2) < \cdots < f(v_n)$. Thus f is one-one.

Next we claim that the edge labels are distinct.

$$\begin{aligned}\text{Let } E_1 &= \{f(vv_i)/1 \leq i \leq n\}\\ &= \{f(vv_1), f(vv_2), f(vv_3), \ldots, f(vv_n)\}\\ &= \{|f(v) - f(v_1)|, |f(v) - f(v_2)|, |f(v) - f(v_3)|,\\ &\quad \ldots |f(v) - f(v_n)|\}\\ &= \{|0 - s_1|, |0 - s_3|, \ldots, |0 - s_{2n-1}|\}\\ &= \{s_1, s_3, \ldots, s_{2n-1}\}\\ \text{Let } E_2 &= \{f(v_{i-1}v_i)/2 \leq i \leq n\}\\ &= \{f(v_1v_2), f(v_2v_3), f(v_3, v_4), \ldots, f(v_{n-1}v_n)\}\\ &= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|,\\ &\quad |f(v_3) - f(v_4)|, \ldots, |f(v_{n-1}) - f(v_n)|\}\\ &= \{|s_1 - s_3|, |s_3 - s_5|, |s_5 - s_7|, \ldots |s_{2n-3} - s_{2n-1}|\}\\ &= \{s_2, s_4, s_6, \ldots, s_{2n-2}\}.\end{aligned}$$

Thus $E_1 \cup E_2 = \{s_1, s_2, s_3, \ldots, s_{2n-1}\}$. Hence the edge labels are distinct. Therefore G is a generalized Fibonacci graceful graph. $\square$

Example 6.1.18.



Theorem 6.1.19. *Any star $K_{1,n}$ is a generalized Fibonacci graceful.*

Proof. Let G be a star $K_{1,n}$.

Let v be the central vertex of G and $\{v_i/1 \leq i \leq n\}$ be the pendent vertices of G and let $\{vv_i/1 \leq i \leq n\}$ be the edges of G.

Define an injection $f : V(G) \rightarrow \{0, s_1, s_2, \ldots, s_n\}$ by $f(v) = 0$ and $f(v_i) = s_i; 1 \leq i \leq n$.

From the definition of f,

$$\begin{aligned}
\min_{v \in V} f(v) &= \min_{v \in V}\{f(v), f(v_i) : 1 \leq i \leq n\} \\
&= \min_{v \in V}\{\{0\} \cup \{s_i : 1 \leq i \leq n\}\} = 0 \\
\max_{v \in V} f(v) &= \max_{v \in V}\{f(v)\} \cup \{f(v_i) : 1 \leq i \leq n\} \\
&= \max_{v \in V}\{0\} \cup \{s_i : 1 \leq i \leq n\} \\
&= s_n.
\end{aligned}$$

Thus f is a function from the set G to the set $\{0, s_1, s_2, \ldots, s_n\}$.

Next we find the vertex label of $f(v_i)$.

For $1 \leq i \leq n$, the vertex labels $f(v_i)$ are in the set $A = \{s_1, s_2, s_3, \ldots, s_n\}$.

For $1 \leq i \leq n$, the edge labels of $f(vv_i)$ are in the set,

$$
\begin{aligned}
B &= \{|f(v) - f(v_1)|, |f(v) - f(v_2)|, \ldots, |f(v) - f(v_n)|\} \\
&= \{|0 - s_1|, |0 - s_2|, \ldots, |0 - s_n|\} \\
&= \{s_1, s_2, s_3, \ldots, s_n\}
\end{aligned}
$$

Hence G is a generalized Fibonacci graceful graph. □

Example 6.1.20.

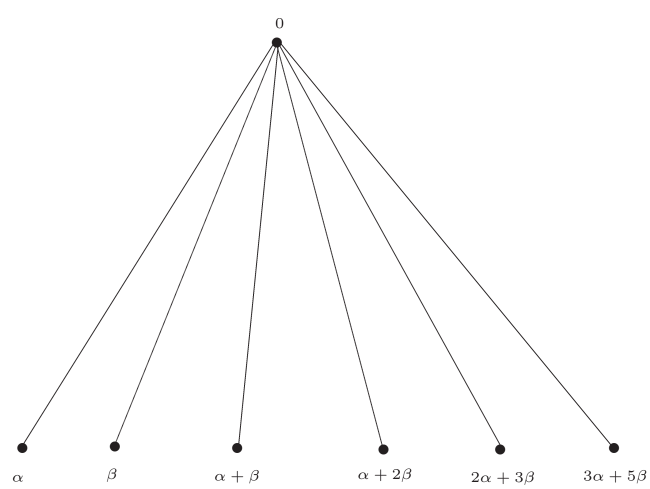


Figure: Generalized Fibonacci Graceful Labeling of $K_{1,6}$

Definition 6.1.21. A triangular snake is obtained from the path $v_1v_2 \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, 3, \ldots, (n-1)$.

Theorem 6.1.22. *A triangular snake is a generalized Fibonacci graceful graph.*

Proof. Let G be a triangular snake graph. Let $V(G) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n-1\}$ be the vertices of a graph G and $E(G) = \{v_i v_{i+1}, v_i u_i, v_{i+1} u_i : 1 \leq i \leq n-1\}$ be the edges of a graph G. Then graph G has $2n-1$ vertices and $3n-3$ edges.

Define $f : V(G) \rightarrow \{0, s_1, s_2, \ldots, s_{3n-3}\}$ as follows:

$$\begin{aligned}
f(v_1) &= 0 \\
f(v_2) &= s_{3n-3} \\
f(v_{i+1}) &= f(v_i) - s_{3n-3i} : 2 \leq i \leq n-1 \\
f(u_1) &= s_{3n-4} \\
f(u_i) &= f(v_i) - s_{3n-1} - 3i : 2 \leq i \leq n-1.
\end{aligned}$$

The edge labels are as follows:

$$\begin{aligned}
f(v_1 v_2) &= s_{3n-3} \\
f(v_i v_{i+1}) &= s_{3n-3i} : 2 \leq i \leq n-1 \\
f(v_1 u_1) &= s_{3n-4} \\
f(v_i u_i) &= s_{3n-1-3i} : 2 \leq i \leq n-1
\end{aligned}$$

$$\begin{aligned}
f(v_2 u_1) &= s_{3n-5} \\
f(v_{i+1} u_i) &= s_{3n-2-3i} : 2 \leq i \leq n-1
\end{aligned}$$

so that the edge labels are in the set

$\{s_{3n-3}, s_{3n-6}, s_{3n-9} \cdots s_3\} \cup$
$\{s_{3n-4}, s_{3n-7}, S_{3n-10} \cdots s_2\} \cup$
$\{s_{3n-5}, s_{3n-8}, s_{3n-11} \cdots s_1\}$.

Obviously $\min_{v \in V(G)} V(G) = 0$ and $\max_{v \in V(G)} V(G) = s_{3n-3}$. Thus f is a function from $V(G)$ to the set $\{0, s_1, s_2, \ldots, s_{3n-3}\}$.

Clearly f is one-one and the edge labels are in the set $E(G) = \{s_1, s_2, s_3, \ldots, s_{3n-3}\}$.

Hence f is generalized Fibonacci graceful. □

Example 6.1.23.

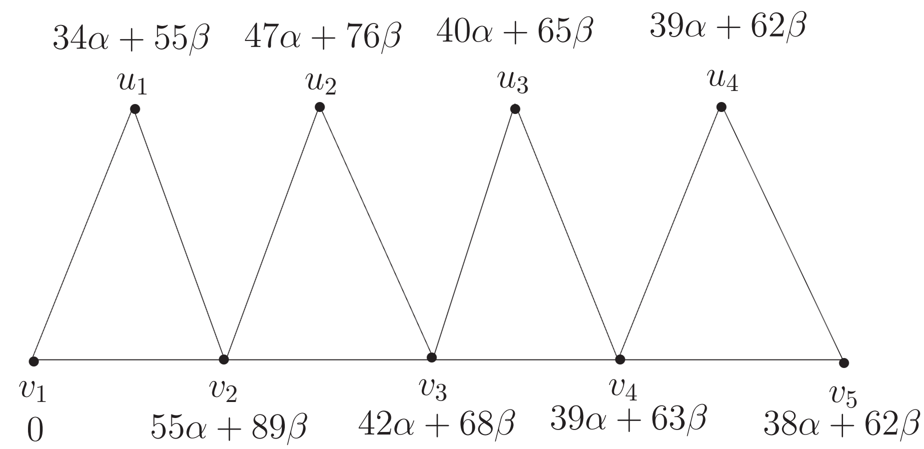


Figure : Generalized Fibonacci Graceful Lableing of a Triangular Snake

Definition 6.1.24. pendent edge extension G^* of a graph $G(p, q)$ is obtained from the graph G by adjoining a pendent edge to each vertex of G. So G^* is a $(2p, p+q)$ graph.

Theorem 6.1.25. *If G is generalized Fibonacci graceful, then its pendent edge extension G^* is generalized Fibonacci graceful.*

Proof. Let $G(p, q)$ be a generalized Fibonacci graceful graph with respect to the labeling f. Let the vertex set of G be $\{v_i : 1 \le i \le p\}$ where $f(v_1) < f(v_2) < f(v_3) < \cdots < f(v_p)$.

Let the vertex set of G^* be $\{v_i, u_i : 1 \le i \le p\}$ where u_i's are pendent vertices joined respectively with v_i's where $1 \le i \le p$.

Define an injection $\phi : V(G^*) \to \{0, s_1, s_2, \ldots, s_{p+q}\}$ by

$$\phi(v_i) = f(v_i)$$

$$\phi(u_i) = f(v_i) + s_{p+q-i} \text{ for } 1 \le i \le p.$$

Already f is a function from $V(G)$ into $\{0, s_1, s_2, \ldots, s_q\}$.

Therefore

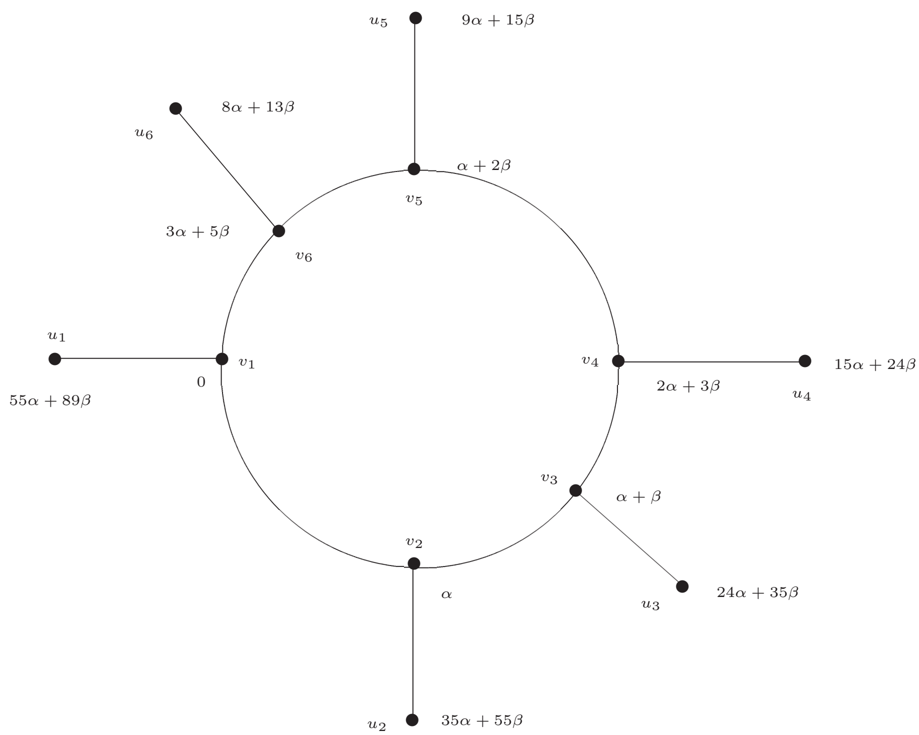
$$\min_{v \in V} \phi(v) = \min_{v \in V} f(v) = 0 \text{ and}$$

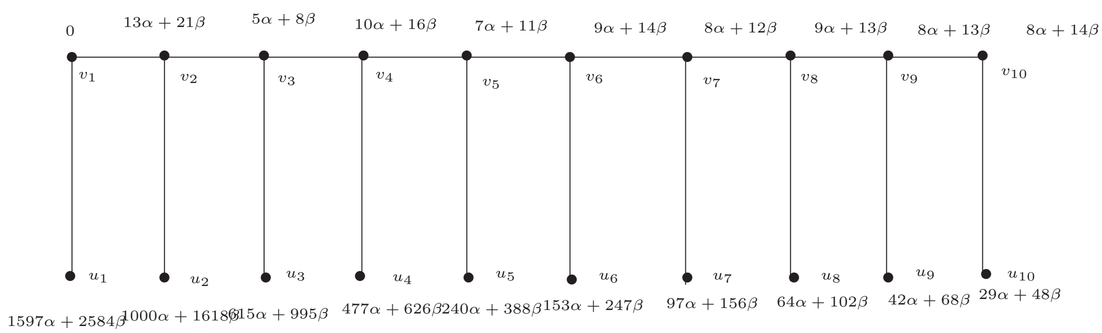
$$\max_{v \in V} \phi(v) = \max_{v \in V} f(v) = s_{p+q}.$$

Thus ϕ is a function from $V(G^*)$ into $\{0, s_1, s_2, \ldots, s_{p+q}\}$. Since f is one-one. ϕ is also one-one. Next, we claim that the edge labels are distinct. Already by the definition of f and ϕ, the edge labels of G are $s_1, s_2, s_3, \ldots, s_q$.

$$\begin{aligned}
\text{Let } E_2^* &= \{\phi(u_i v_i) : 1 \le i \le p\} \\
&= \{|\phi(u_i) - \phi(v_i)| : 1 \le i \le p\} \\
&= \{|f(v_i) + s_{p+q-(i-1)} - f(v_i)| : 1 \le i \le p\} \\
&= \{s_{p+q}, s_{p+q-1}, s_{p+q-2}, \ldots, s_{q+1}\}.
\end{aligned}$$

Thus the edge labels of G^* is $E_1 \cup E_2 = \{s_1, s_2, s_3, \ldots, s_{p+q}\}$. Hence ϕ is a generalized Fibonacci graceful labeling. $\square$





Theorem 6.1.26. *If G_1 and G_2 are generalized Fibonacci graceful, then their union $G_1 \cup G_2$ is generalized Fibonacci graceful.*

Proof. Let $G(p_1, q_1)$ and $G_2(p_2, q_2)$ be two generalized Fibonacci graceful graphs with respect to the labelings f_1 and f_2 respectively.

Let $\{v_i, u_j : 1 \leq i \leq p_1, 1 \leq j \leq p_2\}$ be the vertex set of $G_1 \cup G_2$.

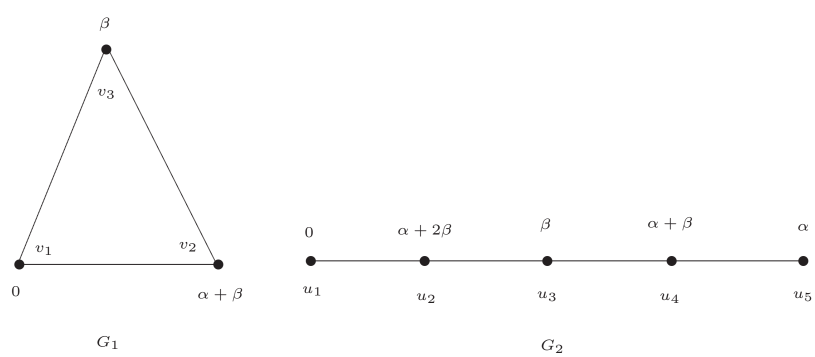
Define $\phi : V(G_1 \cup G_2) \rightarrow \{0, s_1, s_2, \ldots, s_{q_1+q_2}\}$ by

$$\phi(v_i) = f(v_i) + s_1 : 1 \leq i \leq p_1 \text{ and}$$

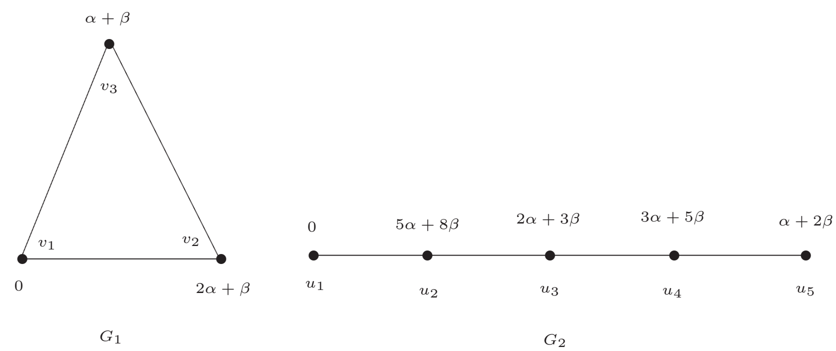
$$\phi(u_i) = \begin{cases} 0 & \text{if } f_2(u_i) = 0 \\ s_{r_i+q_1} & \text{if } f_2(u_i) = s_{r_i} \text{ with } 1 \leq i \leq p_2 \\ & \text{and } r_i \in \{1, 2, 3, \ldots, q_2\} \end{cases}$$

so that the edges of G_1 will get $s_1, s_2, \ldots, s_{q_1}$ and edges of G_2 will get $s_{q_1+1}, s_{q_1+2}, \ldots, s_{q_1+q_2}$. Then ϕ is a generalized Fibonacci graceful labeling. □

Example 6.1.27. Let $G_1 = C_3, G_2 = P_4$ with the labelings as follows:



The generalized Fibonacci graceful labelings for $G_1 \cup G_2$ is as follows:



6.2 COMPARISON OF GRACEFUL AND GENERALIZED FIBONACCI GRACEFUL GRAPHS

In this section, we compare the concept of graceful and generalized Fibonacci graceful graphs with the help of examples.

(i) C_3 is both graceful [30] and generalized Fibonacci graceful (Theorem 6.1.14).

(ii) K_4 is graceful but not generalized Fibonacci graceful (Theorem6.1.4).

(iii) C_6 is generalized Fibonacci graceful (Theorem6.1.14) but not graceful [30].

(iv) $K_n (n \geq 5)$ is neither generalized Fibonacci graceful (Theorem6.1.4) nor graceful [4]

(v) C_7 is generalized Fibonacci graceful but not Fibonacci graceful.

6.3 CONCLUSION AND SCOPE

For further work, we have planned to investigate common properties of various graph labeling schemes with generalized Fibonacci graceful labeling and to classify them.