CHAPTER 2

LITERATURE REVIEW

2.1 HISTORICAL OUTLINE OF MATRIX METHODS

The concept of framework analysis emerged during 1850-1875, due to the large amount of collaborative effort of Maxwell, Castigliano, Mohr and others. The concepts developed by them truly represent the corner stone of methodology of Matrix Structural Analysis (MSA), which did not take proper form and shape for about 80 years. Even overall progress in the development of theory plus analytical techniques was very slow from 1875 to 1920. This was due to grass-root limitation of solving a large number of algebraic equations required to get a large number of unknowns. The structures of primary interest in that period were basic pin-jointed and rigid-jointed structures which were mainly studied using truly force approach with force parameter in a member as unknown.

In 1915, Maney in United States presented the method of Slope Deflection expressing the moments in terms of deflections and slopes at the rigid joints of framed structures. A similar idea was put forth by Ostenfield in Denmark. These ideas are considered as forerunners of MSA. Handling large size problems, was considered difficult till 1930 in both the approaches i.e., force and displacement based methods. Structural analysis work geared up, when Hardy cross in 1930 [8] introduced a method of Moment Distribution. This method made feasible the solution of different types of problems with various complexities in very easy manner, which were considered otherwise quite difficult using other methods. Thus moment distribution method become a strong staple in structural analysis for next 25 years.

In 1943, Courant [9] addressed the topic of theoretical and applied mathematics. The equivalence between boundary value problems of partial differential equations on the one hand and problems of the calculus of
Variations on the other hand was the main theme of the discussion in the paper. Variational formulations are fundamentally simpler to use for approximating extremum problems for different practical applications. Thus, methods emerged which could not fail to attract engineers and physicists; after all, the minimum principles of mechanics are more suggestive than the differential equations. Great successes achieved in applications were soon followed by further progress in the understanding of the theoretical background.

In the year 1945, Southwell [10] explained how the relaxation method helps in converting the higher order mathematically complex problems to lower degree by different numerical techniques. He also suggested a new approach for stress calculation in frameworks by the method of relaxation.

Later on digital computer appeared first in early 1950’s, but there real significance to both theory and practice did not become widely apparent immediately. Some of the researchers attempted codification of well established frame work analysis procedures in a format suited to the digital computer, which is known as today’s matrix format. Later on Argyris, Kelsey, Turner, Clough and Martin combined the concept of frame work analysis and continuum analysis, which resulted in the complete procedure in matrix format.

The use of the so-called matrix methods is nothing but reformulation of existing methods and principles in matrix form, at the same time introducing a generalization of such methods because of the capabilities of computer techniques. The flexibility method is a generalization of the consistent deformation method that enables the engineer not only to include all possible types and directions of loads but also to calculate at the same time all the desired internal forces, reactions, and displacements. Displacements that are needed in this method are usually calculated by the unit load method. The stiffness method is a generalization and extension of the well-
known slope deflection method that makes it possible to account not only for the effect of bending moment but for all types of deformations, such as those caused by shear, axial forces, twisting moments, and so forth. In addition, this method allows the determination of all member end actions and reactions at the same time.

Jenen and Dill in 1944 [11], Kempner in 1945 [12], Benscoster in 1946 [13], Plunkett in 1949 [14] and Falkenheiner in 1953 [15] were the first to develop the basic principles of the matrix force method, with particular reference to standard orthogonal stringer sheet assemblies under loads. Falkenheiner’s paper included the clear statement on the importance of self equilibrating load systems as redundancies and also the derivation of the flexibility matrix for the assembled structure. The principle of least work was used in the derivation of the governing equations. Falkenheiner’s original contribution was nevertheless, more advanced than much of the subsequent work on the force method.

The first person to apply the force method to the swept wing aircraft structure appears to have been Levy [16]. In 1947 Levy idealized the swept wing into an assemblage of simple structural units to which the force method could be applied. The idealization was more realistic than those previously used. Subsequently, Langefors in 1952 [17] gave another independent account of the force method based on minimum strain energy approach. Wehle and Lansing in 1952 [18] in their paper described clearly the basic force method and the flexibility matrix of an assembled structure. The discussion of redundancies was rather vague with no mention of self equilibrating load systems. They tabulated the flexibilities of various components and suggested to derive matrices by writing down and solving formally all the equilibrium equations with some of the internal forces taken as redundancies. Lang and Bisplinghoff in 1951 [19] obtained a matrix formulation of a strain energy analysis for a sweptback wing given previously
by Levy [16]. Langefors in 1955 [20] discussed very briefly the relative merits of the force and displacement methods. He used the matrix formulation of the unit load therein but in a more restricted form, and considered it as following from strain energy or Kron’s work [21]. Also, he proposed to solve highly redundant systems by splitting them into a number of subsystems, each of which is to be analysed separately.

A recent modern exposition of the elements of the force method was given by Denke in 1954 [22], who effectively derived the matrix formulation of the unit load theorem, which he quoted to arise from Maxwell-Mohr approach. He selected as redundancies forces at hypothetical cuts by inversion of equilibrium conditions. Denke argued that since the strains in the Maxwell-Mohr relations may be due to any cause, the incorporation of thermal strains or other initial strains is straightforward. Denke also showed how the procedure could be applied to certain non-linear cases [23].

The literature on the displacement method is not so extensive. For aeronautical structures the first semi-matrix scheme was proposed by Levy 1953 [24]. He considered as components the sparse and ribs for bending and the cells for torsion, the stiffness of which were obtained by inversion of the flexibilities.

In 1956, Livesley [25] gave the concept of automated structural design of structural frames in which he considered the problem of finding the lightest structural frame of given geometrical form. Following the development of geometrical analogue and an iterative method of solution, an analytic technique was presented which gave an exact solution to the problem discussed in the paper. A brief description of program developed was included and it was shown that with slight modification in the developed program it can be used to determine the collapse load of a given frame.
These initial contributions were followed by the work of Argyris in 1954 [26] and Argyris and Kelsey in 1957 [27]. Argyris and Kelsey subsequently [28] in their book gave a very general formulation of matrix structural theory based on the fundamental energy principles of elasticity. Although the work emphasized the force method, it also showed that the displacements, rather than internal forces, could be chosen as the primary unknowns of the structural problems. This alternative choice of unknown than brought the stiffness matrix into the formulation. In this early work the stiffness matrix was not calculated directly as in present days work, but rather was obtained by inverting the more familiar flexibility matrix at that time. Hence the stiffness method was considered as an alternative choice for solving the equations generated from the force method. Thus, Argyris and Kelsey presented a format unification of force and displacement methods using dual energy theorems. Although practical application of the duality proved ephemeral, this work systematized the concept of assembly of structural system equations form elemental components.

A mixed force displacement method was proposed by Klein in 1957 [29-30] who presented the force displacement relations for all elements. The two groups of equations were solved jointly for the forces and displacements. The matrix formulation appeared only in the final equations. Since both forces and displacement were considered as unknowns, the number of unknowns were inevitably very high. Kron in 1959 [31] developed a theory to solve the complex structures by dividing the structure into number of subsystems which could be analysed separately. These subsystems when joined by links represented the static redundancies of assembly of subsystems.

The above publications carried the traditional force and displacement methods to an advanced stage of development. Although the method thereby became applicable to wide classes of structural problems, it took several years for this work to become widely known outside the aircraft industry.
In 1966, Bazant [32] gave complete account of theoretical and historical development in analysis of framed structures. He also demonstrated that matrix analysis of framed structures consists of two major methods in which stiffness method is an extension of slope deflection method and flexibility method is an extension of consistent deformation approach.

In 1968, Prezemieniecki [33] explained in detail the background required for deriving the stiffness and flexibility properties of all types of structural elements. He suggested various procedures for deriving the above properties using theorems based on virtual work and complementary work principles. He included various examples in his book to validate the above structural properties. The concept of substructure technique was also given in a systematic way for stiffness as well as flexibility matrix based approaches. The book also contains sufficient details regarding the dynamic analysis that is applicable to framed structures using various mass inertial properties. Using stiffness and flexibilities properties, different types of problems of vibration analysis are attempted. Concept of critical damping is also explained in detail accompanied with solution of problems based on damped structural system. Different aspects of nonlinear structural analysis are explained with reference to bar and beam elements using geometric stiffness matrix. Various important aspects of large deflection analysis using matrix force method are also given in his book in a systematic manner.

2.2 FINITE ELEMENT METHODIZATION

The finite element method had its beginning in the area of structural analysis. The first developments were in the aircraft industry, where researchers were striving to model the thin membranes of the fuselage and wing of a jet liner. Membrane elements were used in conjunction with the already established beam and frame elements. A classic paper by Turner, Clough, Martin and Topp appeared in the Journal of Aeronautical Science in 1956 [34]. This marked the beginning of the analysis of large complex
structural systems. In 1960 Clough [35] coined the term “Finite Element Method” which was developed as an extension to established structural analysis techniques. This development is often noted as the beginning of modern finite element analysis.

Two earlier classical papers in mid 1950, by Argyris & Kelsey [26] and Turner et al. [34] merged the initial concepts of discretized frame analysis and continuum analysis and kicked off the explosive developments in the finite element methods. Following the linear state formulation for each finite element, extension for practical applications have continued to include various field problems.

Additional papers concerned with the basic theory appeared during the 1960s. For example, Melosh’s doctoral work in 1963 led to a paper [36] placing the finite element method on the principle of minimum potential energy. Also in 1963, Besseling [37] presented the analogy between the matrix equation of structural analysis and continuous field equations of elasticity. The question of upper and lower bound was discussed by Venbeke [38] in a basic paper that introduced the alternative possibility of defining stress or equilibrium elements based on the principle of minimum complementary energy.

Other papers further demonstrated the rich theoretical base offered by the variational principles for defining finite elements. Jones in 1964 [39] pointed out the advantages that could be secured by using Reissner’s general variational principle. This led to the development of mixed-element, which depends on assumed displacement and stresses. However, the conditions to be satisfied by these assumed functions are considerably less stringent than those required when seeking a displacement i.e. compatible or stress i.e. equilibrium element. Hence this approach is quite useful when certain complex elements are to be derived.
A variant of Jones theory was also published in 1964 by Pian [40]. In this also, element specifications are defined in terms of both displacements and stresses. However, the variation formulation was in terms of both the minimum potential and complementary energy principles. Again the added flexibility led to advantages of particular value for certain complex cases. This approach has led to what is now known as the hybrid element.

Establishing the finite element on the variational principles led to advances that would have otherwise been impossible. This also permitted questions concerned with boundness and the convergence to be discussed with rigor. In addition, the variational formulation permitted a unified approach to be used for determining generalized nodal forces for surface tractions, body forces, thermal gradients, inertia forces and so on. Hence, it became possible to express fully the general elasticity problem in finite element terms. From a conceptual point of view the new theoretical foundations permitted the physical element to be replaced by its mathematical equivalent. The element thus could now be visualized as a small region of space within which the unknown function was to be prescribed in a simple manner. Moreover, the conditions to be met in choosing the function could be stated with certainty. This immediately lifted the method outside the borders of solid mechanics.

These ideas and the underlying theory became widely known with the publication of the text by Zienkiewicz and Cheung [41]. This was the first comprehensive treatment of the subject, and the text had a far-reaching influence on subsequent developments.

In structural dynamics, three developments are of particular interest. The first was by Archer [42], who in 1963 showed how to correctly determine the mass matrix for distributed mass systems, when using the finite element method. This made the mass matrix consistent with the determination of the stiffness matrix and generalized nodal forces in the stiffness equation. Two years later Guyan [43] pointed out how the mass matrix could be reduced in
a manner consistent with the reduction of the assemblage stiffness matrix. Such reductions are invariably used in dynamics calculation since they permit the eigen value problem to be condensed to a numerically suitable size. Then Hurty in 1965 [44] presented a technique for undertaking the dynamic analysis of higher order structural systems. This procedure is based on using natural modes of vibration of structural subsystems, calculated in terms of displacement mode functions for the various components.

The period from 1965 may be considered as golden age of finite element developments. Any attempt at cataloging this vast literature would require many pages. A brief account of some of the summary papers along with papers which are related to present work is given below.

In 1966, Burton [45] handled a two-dimensional problem of equilibrium of a perfectly elastic body with coupling stresses criteria. The general form of solution of the problem in the case of an orthotropic medium indicated that there exists a close analogy between the equations governing the behavior of a plane rectangular lattice composed of rigidly interconnected elastic beams, and the general set of equations of the two-dimensional couple-stress theory for certain orthotropic bodies.

In 1968, Fellipa and Clough [46] formulated a fully compatible general quadrilateral plate bending element. The element was assembled from four partially constrained linear curvature compatible triangles, arranged such that no mid-side nodes occur on the external edges of the quadrilateral; thus, the resulting element had only 12 DOF. Also they described the modifications required for incorporating shear distortion. Results were presented for static analyses with and without shear distortion, plate vibration and plate buckling studies; all performed with this quadrilateral element. They claimed it to be the most efficient bending element.
Gallagher in 1969 presented detailed account of contributions to two
important subject areas. One treats plates and shells [47] while the other
discusses the finite element method’s contributions to problems of stability
analysis [48].

In 1972, Neale, et. al. [49] formulated triangular and rectangular elements,
both based on the hybrid formulation for the analysis of plate bending
problems. Static and dynamic analyses were performed for which results
were presented to prove superiority of the hybrid elements over the
equivalent displacement elements. Results were also compared with those
obtained with elements having strain degrees of freedom.

In 1975, Kikuchi [50], studied convergence rates of the ACM non-conforming
scheme for thin plate bending elements with the shape of the domain as
rectangular and the exact solution being sufficiently smooth. Error bounds
of moments and deflection and error of eigen value were found of the order of
square of the maximum mesh size. This result was also confirmed by
numerical experimentation.

Also in 1975, Anderson [51] formulated a Mixed isoparametric element for
the Saint-Venant torsion problem of laminated and anisotropic bars. Both
triangular and quadrilateral elements were considered for the analysis
purpose. The “generalized” element stiffness matrix was obtained by using a
modified form of the Hellinger - Reissner mixed variational principle. The
working of the mixed iso-parametric elements was demonstrated by means
of numerical examples.

In 1978, Gupta and Rao [52] developed the stiffness and mass matrices for a
twisted beam element having linearly varying breadth and depth. The angle
of twist was assumed to vary linearly along the length of the beam. The
effects of shear deformation and rotary inertia were considered in deriving
the elemental matrices. The first four natural frequencies and mode shapes
were calculated for cantilever beams of various depth and breadth taper
ratios at different angles of twist. The results were compared with those available in literature.

In 1985, Gallagher and Ding [53] attempted diversified application of force method in optimization process, based on the size variable-independence of the equilibrium equations in matrix format. A rational reduced basis reanalysis and approximate reanalysis methods were presented. Because the static redundancy of pin-jointed structures is often low, and the decomposed coefficient matrix is known, these techniques can be advantageous in the structural optimization process. Several truss structures were studied for the purpose of analysis and validation of results.

In 1993, Argyris and Tenek [54] formulated a three-nodes layered triangular element constrained to comply with an assumed linear direct strain distribution across its thickness and including transverse shear deformation for bending analysis of plates. The concepts of the natural mode and matrix displacement methods together with decomposition and lumping ideas based on appropriate kinematic idealizations were combined for stiffness matrix derivation.

In 1993, Wilson [55], summarized the evolution of computational and numerical methods for the static and dynamic analysis of finite element systems. The majority of the material discussed was based on the reflection of the personal experience of the author’s working for the past 35 years. It was the opinion of the author, that at the present time, the finite element method is far from being completely automated for complex structures.

In 1993, Kosmatka and Friedman [56] developed an improved two noded beam element with appropriate stiffness, mass, and consistent matrices. Timoshenko beam element was developed based upon Hamilton's principle. Cubic and quadratic polynomials were used for the transverse and rotational displacements respectively, where the polynomials were made interdependent by requiring them to satisfy the two homogeneous differential
equations associated with Timoshenko's beam theory. Numerical results were presented to show that for short beam subjected to complex distributed loading the proposed element predicts shear and moment resultants and natural frequencies better than the existing Timoshenko beam elements.

In 1996, Gupta and Meek [57] presented summary of the works of several eminent authors and research workers associated with the invention of the analysis technique now referred to as the Finite Element Method. It is believed to be an accurate record of the historical sequence of published papers on FEM in the international literature. The complete development of the ideas, which led to the method of analysis in which the field equations of mathematical physics are approximated over simple regions (triangles, quadrilaterals, tetrahedrons, etc.), and then assembled together so that equilibrium or continuity is satisfied at the interconnecting nodal points, are presented in interesting manner by the authors.

In 1998, Fellipa and Park [58] presented variational framework for the development of partitioned solution algorithms in structural mechanics. This framework was obtained by decomposing the discrete virtual work of unassembled structure into that of partitioned substructures. New aspects of the formulation were an explicit use of substructural rigid-body mode amplitudes as independent variables and direct construction of rank-sufficient interface compatibility conditions. The resulting discrete variational functional was shown to be variationally complete, thus yielding a full rank solution matrix. Four specializations of the suggested framework were also discussed in detail.

In the year 2000, Fellipa [59] gave again a historical background of major contribution made by Collar, Duncan, Argyris and Turner, who really worked hard and shaped the Matrix Based Structural Analysis. He divided the complete development into following three parts: (1) Formative period in which methodological concept were developed, (2) The period in which Matrix
methods assumed bewildering complexity in response to conflicting demand and restrictions, and (3) The period in which development of Direct Stiffness Method took place, through which MBSA completely morphed into the present implementation of structural mechanics.

In year 2005, Kikuchi [60] presented theory and examples of partial approximation as a modification of the displacement based FE analysis. This method needs various types of shape functions for different terms in the potential energy expression to curtail the processes in the standard displacement method. The theory is explained with simple example.

In year 2006, Samuelsson and Zinenkiewicz [61] presented a brief history of the development of stiffness method tracing the evolution of the complete method for discrete type of structures i.e. trusses and frames composed of two noded members. Brief description is also given in the same paper for the application of the same method for continuum type of problems, which are modeled by finite difference and finite element methods.

In year 2006, Oztorun [62] developed a FE based formulation for the static and dynamic analysis of linear-elastic space structures. He suggested that the finite element method can be efficiently used for the analysis of linear-elastic structures with shear walls. The element considered for the study had six degrees of freedom at each node and an in-plane rotational degree of freedom, which makes it compatible with 3D beam-type finite element model. The rigidity coefficients of the element were determined analytically. The element can be used with facility for the modeling of a shear wall with the connections of slab components. Convergence studies were carried out on actual structures by using several models to check the performance of the rectangular finite elements. Acceptable results were obtained with coarse mesh and reasonable convergence was observed on the models tested.

Kuoa et al. [63] in the year 2006, proposed a simple method for deriving the geometric stiffness matrix (GSM) of a three-noded triangular plate element.
(TPE). It was found that when the GSM of the element is combined into the global matrix, the structural stiffness matrix becomes symmetric and satisfies both the rigid body requirements and incremental force and moment equilibrium (IFE) conditions. In addition, the advantage is that the derivation of the matrix needs only simple matrix operations without cumbersome non-linear virtual strain energy derivations and tedious numerical integrations.

In 2007, Kaveh et al. [64] developed force based methods which were found highly efficient for the generation of sparse and banded null bases and flexibility matrices. However, these methods require special considerations when the support conditions are indeterminate. These considerations with special methods were presented in the paper, which lead to efficient utilization of force method for indeterminate support conditions with no substantial decrease in sparsity.

In 2011, Lang and Yan [65] found that finding deformation and internal force of rectangular plate under complex boundary and free corner supported condition is comparatively complicated. Based on the elastic thin plate flexure theory, the rectangular plate with complex boundary was firstly divided into several simple rectangular elements. Introducing the concept of generalized supporting edge in the common boundary, the unknown coefficients of the plate bending functions were determined by satisfying the boundary conditions and thus the geometric compatibility conditions, and the analytic solution of the original plate was finally obtained. It provides a valuable reference for the inner force analysis and engineering design of rectangular plates under various edge and loading conditions.

### 2.3 Evolution of Integrated Force Method

A novel formulation was developed in 1973 by Patnaik [7] for the analysis of discrete structures by considering member forces as primary unknowns.
instead of conventional way of treating the redundants as prime unknowns. He named the method as Integrated Force Method (IFM). Illustrative examples for the determination of forces and displacements were presented for pin-jointed and rigid connected frame structures for various load conditions.

In 1977, Patnaik and Yadagiri [66] developed IFM for frequency analysis of spring systems, beams and trusses. In the paper, this concept of eigen analyses was explored considering force mode shape as primary variable. After calculating the frequency, internal forces and moments were calculated using the homogenous equation by substituting each frequency value. Utilizing the force mode shape concept, direct design of structures was attempted by keeping the frequency constrains.

In 1984, Patnaik and Joseph [67] developed two schemes for the generation of compatibility conditions for discrete structures which were used in IFM. The first scheme was based on displacement deformation relation and was recommended for the basic analysis of structures. The second scheme was developed using Linear Programming (LP), in which Equilibrium Equations (EEs) were used as constraints and linearized internal strain energy of structure as objective function. Linear programming had advantage of sparsity of coefficient matrix. Both the schemes were included in IFM and thus using the same different problems were attempted.

In the year 1986, Patnaik [68] compared the IFM with the Standard Force Method (SFM). Standard Force Method was considered as solution technique as a part of IFM for static analysis. IFM bypasses the popular concept of selection of redundant. IFM was applied to twenty bay truss and cantilever plate problems. It has been observed in the paper that the numerical stability of IFM is much better and superior than the SFM.
In the year 1989, Patnaik [69] extended the application of Integrated Force Method to structural optimization problems, in which prior knowledge of design variable was necessary. In the paper two methods were presented; first was of Linear Programming and other was based on fully utilized design concept. The Nonlinear Mathematical Programming (NLP) was elegantly removed without disturbing the main aspect which resulted in considerable reduction in computational time.

Nagabhushanam and Patnaik [70] in the year 1989 explained the basic methodology of IFM, in which development of equilibrium equation and compatibility conditions were explained in detail. The generation of EEs was very simple, and straightforward while the development of CCs was intricate. They considered both the conditions i.e. field and boundary compatibility conditions with applications to FE based problems. The key feature in the development of compatibility conditions was based on the concept of node determinacy, which was used to reduce the complexity of the displacement deformation relation. It also helped to reduce the mathematical complexity by eliminating the node at intermediate level.

In the year 1990, Patnaik et al. [71] applied IFM to plane stress and plate bending problems of deep cantilever beam and clamped plate subjected central point loading. Software named as GIFT was developed based on IFM and displacement based modified approach which was named as IFMD. The same problems were solved by commercially available software “MSc-NASTRAN”, which was based on displacement based FE approach and another software “NHOST”, which was based on mixed method. IMF with simultaneous emphasis on stress equilibrium and strain compatibility gave acceptable solutions even for coarser discretization.

In the year 1990, Patnaik and Satish [72] used the Boundary Compatibility Conditions (BCCs) that geared the force method and brought up IFM upto
the level of stiffness method. Based on the new concept and theoretical development of BCCs novel formulation, various problems of static and dynamic analysis of framed and continuum structures were attempted which provided quite accurate results.

In the year 1996, Kaljevic et al. [73] explored the possibility of finding secondary stresses in beams and circular annulus by using IFM. The initial strain was directly incorporated into the compatibility conditions using vector of initial deformation. Two problems were successfully attempted i.e. support settlement in two span continuous beam and thermal strain effect in circular annulus.

In the year 1996, Kaljevic et al. [74] developed triangular and quadrilateral elements with necessary displacements and stress fields. Elements were developed by discretizing the potential and complimentary strain energies. The displacement field was approximated using the Lagrange’s, Hermitian’s and generalized coordinate systems. The stress field was approximated by using the complete polynomial of correct order. Airy’s stress theory was explored for its constant, linear, quadratic and cubic stress fields which were used in IFM. The resulting matrices were also checked for equations of equilibrium and conditions of compatibility. The elements were insensitive to the orientation from local to global axis. Elements having large number of unknowns in the stress field increased the size of matrices and finally to work out the solution became cumbersome. Thus, in the same paper different schemes were discussed to reduce the number of unknowns without disturbing the accuracy. For comparison purpose, quadratic, linear and constant fileds were assumed. Various examples were considered to validate the developed library of 2D elements. The results were checked and verified with the available solutions.
A two dimensional beam element having eight displacement degrees of freedom and five force degrees of freedom was used in the year 1997, by Patnaik et al. [75] for frequency analysis by using IFM and IFMD. Two lumping mass were considered at extreme nodes for which force mode shape based frequency analysis was carried out. From the frequency vectors, internal forces and mode shapes were worked out. Solutions for frequencies and force mode shapes were compared with the standard displacement based eigen value analysis and analytical solutions.

In the year 2004, Patnaik et al. [76] gave an account of theoretical development with various numerical examples of framed and continuum structures. It was just a summary of the work done by Patnaik and his team on IFM till that year.

In the year 2005, Dhananjaya et al. [77] presented a formulation of a 2D element based on Mindlin - Reissner theory with element having 12 displacement degrees of freedom and 9 force degrees of freedom using IFM. The performance of the element was checked by solving different benchmark problems. Results for deflection and internal moments were checked by the exact solution and displacement based FE method; a good agreement was indicated.

Also, in 2005, Dhananjaya et al. [78] extended IFM to laminated composite plate bending problems. A quadrilateral element having total 20 displacement degrees of freedom and 16 force degrees of freedom was developed including shear deformation theory. Simply supported plates under central point load and uniformly distributed loading were considered for the analysis purpose. Different discretization schemes were used for checking the accuracy and convergence with reference to an exact solution.
In the year 2005, Dhananjaya et al. [79] also developed a general purpose program for automatic generation of the equilibrium and flexibility matrices which were based on Kirchoff’s and Mindlin-Reissner’s plate bending theory. Using the equilibrium and flexibility matrices for triangular, rectangular and quadrilateral elements, different types of thin and thick plate bending problems were attempted by using the IFM formulation. The performance of the developed element was evaluated for its accuracy and convergence and it was found satisfactory.

Further, in the year 2005, Dhananjaya et al. [80] developed closed form equilibrium and flexibility matrices for four noded and eight noded plate bending elements. The necessary matrices were developed by discretizing the different strain energies required. Numerous standard plate bending problems were solved to get results for deflections and internal moments; a good agreement was indicated, with the available solutions.

In the year 2006, Patnaik and Pai [81] formulated the boundary compatibility conditions in the incomplete form of Beltarmi’s Michell’s formulation. It is now recognized as “Completed Beltrami’s Michell’s Formulation (CBMF)”. Using this concept different problems of circular boundary with mixed boundary conditions were efficiently attempted.

Recently, in the year 2010, Dhananjaya et al. [82] developed closed form solutions for equilibrium and flexibility matrices by using Mindlin-Reissner’s theory of plate bending based on IFM. The rectangular element with 12 displacement degrees of freedom and 9 force degrees of freedom was developed. Large scale plate bending problems were attempted using IFM and results for deflection and moments were calculated and compared with the available small deflection theory.