CHAPTER 1

INTRODUCTION

1.1 MODERN METHODS OF STRUCTURAL ANALYSIS

Structural analysis is concerned with the determination of the forces (stresses) and displacements (strains) experienced by the structure under the loading. The basic requirements to be satisfied by any method of analysis are the equilibrium between the internal stresses and the external loads, the compatibility of displacements at the joints and the satisfaction of the prescribed force or deformation conditions at the supports.

Till 1930s the only means of solving indeterminate structures was to formulate simultaneous equations using slope deflection equation or its variation. Without the relaxation method or computers which were not known till then, the engineer had a formidable task solving large number of these equations and indeterminate structures. The moment distribution method of Hardy Cross in 1930 provided a spurt and the industry saw a great leap. Still many structures were beyond this method. Rigorous methods for solution of multistoried buildings with sway, for example, though within reach involved a formidable task. Air craft structures posed an even more difficult task.

It was only after World War II that computers came into the field. It demanded a fresh look at the more “Exact” methods of computation in favour of iteration method such as moment distribution method. With the development of the digital computer, matrix notations and algebra came to be used in the organization of calculations. By formulating the tools of matrix structural analysis in a mathematically consistent fashion, the analyst achieves a systematic approach that is convenient for automatic computation.
Matrix methods afford the structural analysis of a class of problems to be represented as a series of algebraic operations on matrices representing force and displacement groups, and thus permits programming in terms of these symbols that are to be substituted later by numerical values. Thus by changing numerical data, solutions of structures of different geometry and loading can readily be obtained. Therefore one can develop general program for a given class of structures such as pin jointed frames, plane frames, etc, without any specific geometry of the structure or loading in view. From the practical view point of design office, the power of such computations is immense as general purpose program once written out and tested together with a powerful computational services, it can be used to solve a variety of problems.

Basically there are two different types of matrix methods to analyse structures, namely force and displacement methods. These are frequently called compatibility method and equilibrium method respectively. Also the alternative names, flexibility method and stiffness method are equally popular. The names force and displacement methods are based on the features that certain force or displacement variables finally determine the solution. When forces are used as variable, equilibrium is satisfied first and the requirement of compatibility are used as conditions to determine the unknown force variables. When displacements are chosen as variables, the compatibility requirements are satisfied first and the requirements of equilibrium form the necessary equations to determine the displacement variables.

Any approach of analysis, be it a force or displacement approach, breaks up the total effort in to a number of steps. The analysis proceeds from the part to whole. The analysis procedure essentially consists of two steps: (i) Member analysis, and (ii) Structural analysis. The first step is the study of the member itself as an independent body. Properties such as stiffness or
flexibility, which are required in the second stage are determined. The second step then consists of the analysis of the assembly of such members. In the first step certain displacement or force parameters are taken as unknowns and the member analysis is formulated in terms of these unknowns. The second step determines the set of all such unknowns. This step involves the knowledge of structural connectivity and from the computational viewpoint is a major issue.

As mentioned above both flexibility and stiffness methods lead to a set of simultaneous equations to be solved. The number of unknowns in the flexibility method is generally smaller. However there is some arbitrariness in the force approach in the sense that the analyst has certain freedom to choose the set of force unknowns. The displacement approach, however, is more definite in its steps (without any alternatives) and thus becomes easier from the point of view of developing general purpose programs. Some analysts on the other hand, consider this a disadvantage and prefer the flexibility method since it offers freedom in the selection of the redundants.

In practice one is required to analyse not only skeletal structures but also continuum structures such as plate and shell structures. In continuum structures the material is continuously distributed over the whole structure and not concentrated along discrete lines constituting individual members. In other words in a continuum structure, which may be a surface structure (two dimensional structure) or solid structure (three dimensional structure), there are no physically recognizable members or joints. If the stiffness method is to be adopted for the analysis of such structures it is obviously necessary to imagine that the structure is assembled form discrete elements called finite elements connected at discrete joints called as nodes. It is of course necessary to ensure that the structure assembled from the finite elements satisfies the basic equilibrium and compatibility requirements.
An important feature of the finite element method (FEM), which sets it apart from other approximate numerical methods, is its ability to formulate solutions for each element before putting them together to represent the entire problem. In essence, a complex problem reduces to considering a series of greatly simplified problems. Another advantage of the FEM is the variety of ways in which one can formulate the properties of individual elements. Out of the various approaches available to formulate the element properties, direct approach and variational approach are generally used for structural engineering problems. While direct approach can be used to derive properties of simple (1D) elements i.e. members of framed structures, the variational approach can be used to derive properties of both simple (1D) and sophisticated (2D and 3D) element shapes.

The variational approach relies on the calculus of variations and involves minimization of a function. For problems of solid mechanics, the functional turns out to be the potential energy, the complimentary energy or some derivatives of these, such as the Reissner variational principle. Following are the variational approaches available for deriving the element properties.

The displacement based FEM employs the principle of minimum potential energy analogous to the principle of virtual displacement. A displacement function is assumed for each element. Compatibility of displacement within the element and across the interelement boundaries has to be satisfied.

In the force based FE approach, principle of minimum complimentary energy analogous to the principle of virtual forces is employed. An equilibrium field is assumed within the element in such a way so as to satisfy internal equilibrium and continuity of stresses between the elements.

In the mixed method, which is based on the Hellinger-Reissner variational principle, both displacements and stress fields are assumed separately for
each element and these have to be continuous within the element and between the elements.

In the hybrid method, stress distributions are assumed within the element and displacement distributions are assumed along the boundaries of the element. Both internal equilibrium and compatibility of displacements along inter element boundaries are to be satisfied.

Among the various approaches discussed above, the displacement formulation is the most popular and widely used for structural problems because of its simplicity and easy applicability to a variety of problems.

1.2 FLEXIBILITY IN FORCE BASED METHOD

The fundamental equation in the matrix form for the flexibility method is \( \{D\} = [F]\{A\} \), which states that the displacements can be expressed in terms of actions by the formulation of flexibility matrix \([F]\) that represents displacements due to the unit values of the actions.

For Statically indeterminate structures, the displacements can be obtained by super-imposing those due to loads and those due to redundants for the displacements at the release where the redundants acts, the following equation can be written

\[
\{D_Q\} = \{D_{QL}\} + [F]\{Q\} \quad \ldots (1.1)
\]

where the column matrix \( \{D_Q\} \) represents the actual displacements at the releases that are either prescribed or known, \( \{D_{QL}\} \) represents the displacements corresponding to redundants in the released structure due to the actual loads, and \([F]\) is a square flexibility matrix representing the displacements at the releases in the released structure due to unit values of the redundants \( \{Q\} \).
If a computer is available for use, one can combine system analysis and member analysis by analyzing the system through a member approach. Matrix formulation allows to develop the necessary flexibility matrices by first considering the flexibilities of the individual members and then combining these to analyse the system - including members- as a whole. In the formalized version of the flexibility method use is made of transfer matrices in finding \( \{D_{QL}\} \) and \([F]\) and compatibility equation is expressed as

\[
\{D_Q\} = [A^{TMQ}][F_M][A_{ML}] + [A^{TMQ}][F_M][A_{MQ}]{Q}
\] ...

(1.2)

Where \( A_{ML} \) and \( A_{MQ} \) are known as transfer matrices and these represent respectively member end action due to actual loading and unit redundants, and \( [F_M] \) is known as member flexibility matrix.

There is considerable freedom in the manner one can generate compatibility equations in the formalized approach of flexibility method. Six different approaches which differ mainly in the formulation of \( \{D_{QL}\} \) and \([F]\) are discussed below.

**1.2.1 Conventional Approach**

The approach described by Gere and Weaver [1] is called here as conventional approach. In this approach the compatibility equation is used exactly in the manner as given in Eq. (1.2). After selecting the suitable released structure the displacements are obtained with the help of given equation as

\[
\{D_{QL}\} = [A^{TMQ}][F_M][A_{ML}] \quad \text{and,} \quad [F] = [A^{TMQ}][F_M][A_{MQ}]
\] ...

(1.3)

The determination of transfer matrices \([A_{MQ}]\) and \([A_{ML}]\) requires that the released structure be analysed by statics for unit values of redundants and also for the combined joint loads. For each member the size of \([A_{MQ}]\), \([A_{ML}]\) and \([F_M]\) depends upon the internal actions considered for the member.
These matrices are formed for the whole structure from the submatrices containing the contribution of individual members. The choice of the redundants greatly influence the amount of computational work required for the generation of the transfer matrices and because of the multiple choices in the selection of redundants one can not generate these matrices automatically through the computer. i.e. one has to supply these matrices as input to the computer. Moreover, most of the elements of the flexibility matrix are nonzero in this approach unless the released structure is chosen to ensure localized phenomena.

1.2.2 Connection Matrix Approach

The approach which requires the use of connection matrix in the generation of \( \{D_{QL}\} \) and \([F]\) matrices is called here as connection matrix approach \([2]\). The connection matrix is generated by writing the force equilibrium equation at each joint between the applied external forces and the member forces and discarding all rows corresponding to known displacements and thus introducing the boundary conditions. This matrix relates member forces to given joint forces. Therefore it is known as connection matrix \([C]\). The numbers of excess columns than the numbers in this connection matrix automatically represents the indeterminacy of structure. For the generation of \([C]\) matrix, a translation matrix is required. For the generation of \([A_{MQ}]\) and \([A_{ML}]\) matrices, proper partition is made for these in the connection matrix. Therefore, there is no need to divide the method into two steps i.e. one for the determinate structure and the other for the indeterminate structure. After the selection of redundants, the rearrangement of columns in \([C]\) matrix and \([C_I]\) and \([C_{II}]\) is made. From which \([A_I]\) and \([A_{II}]\) are derived. From \([A_I]\) and \([A_{II}]\) by simply adding zero and identity matrix, one can obtain the \([A_{MQ}]\) and \([A_{ML}]\) matrices.

The proper selection of redundants is important as it effects the stability and accuracy requirements. The redundants are selected such that this matrix becomes sparse, banded and well conditioned. The sparseness can be
achieved, if attention is paid to the fact that a force in the direction of one redundant should not introduce displacements in the direction of all other redundants or its effect must be as localized as possible. The banded property which economizes memory space could be achieved by careful selection and numbering of the redundants. A well conditioned matrix can be obtained if redundants cause larger displacements in their own direction than in the direction of the other redundants. In connection matrix method displacements and member end actions are represented in global co-ordinate system. The method of connection matrix is advantageous only when the inverted form of the $[C]$, or alternatively $[AM_Q]$ and $[AML]$ can be set up directly but this is possible only in very few cases.

1.2.3 Subframe Approach
In the subframe approach, given by Shaw [3], for the generation of $[AM_Q]$ matrix i.e. Self equilibrating force system matrix the structure is broken up into small subframes. Every subframe is cut at suitable places and redundants are applied. Bending moment diagrams are drawn for all redundants and these diagrams are used to write $[AM_Q]$ matrix. This method has advantage that no inversion of connection matrix is required. Small subframe prevents spreading of redundant forces on more members and hence in this approach the effects are localized. For generating $[AML]$ matrix, the bending moment diagram is drawn for applied loads and then $[AML]$ matrix which is also known as static equilibrium matrix is formed by noting member end forces from the BMD. Although the technique of subframe is applicable to all structures weather fixed or hinged footed but it requires too many diagrams to be drawn and hence much work is to be done manually before making use of the computer for the analysis.

1.2.4 Rank Force Approach
The most significant difference in the stiffness and flexibility method is that one has to prepare more input data when adopting the flexibility approach. 

8
The additional input consists of manually generating the basic redundant load systems which initially one has to decide whether or not the structure is redundant and, if so, to what degree. A proper set of redundancies must then be selected. For large size problems, these requirements are very time consuming, laborious and prone to error. Argyris in 1962 [4] suggested a method of reducing this work by considering redundant subsystems but this method is applicable only for specific structural configuration and cannot be used for general analysis. To remove these deficiencies a research was carried out at the Boeing Company and as a result of this work “the rank technique” was developed. This technique furnishes a method for automatic selection of redundancies in the matrix force method and removes the need to generate the basic and redundant load systems.

An automated structural analysis system which adopts the force approach and incorporates the rank technique was developed by Robinson [5] and he called this technique as “The Rank Force Method”. An attractive feature of the method is that the structure is systematically and automatically investigated to determine the basic characteristics such as stability and indeterminacy. A consistent set of redundants is automatically isolated and moment diagram are not required unlike subframe approach to generate \([A_{MQ}]\) and \([A_{ML}]\) matrices. However, the method is not simple and because of the complete generalization it involves large size matrices compared to the other methods and that is why it has not become that popular.

1.2.5 Modified Force Approach

A modified force approach was developed by Patel and Patodi [6] for the analysis of rigid jointed framed structures. The approach partly follows the subframe approach to retain its advantage of localized phenomena. In this approach a determinate structure is formed in a special manner to generate \([A_{ML}]\) and use is made of small sub frames for the generation of \([A_{MQ}]\) matrix. This approach differs from the subframe approach in the generation of \([A_{MQ}]\)
matrix. The generation of $[B_F]$ matrix is carried out from the coordinates of forward ends of the members. Thus the labour involved in selecting the released structure and drawing BMD for each redundant is completely saved. The approach does not require any selection of redundants. These are automatically selected as in the rank force approach of Robinson. These new ideas for the generation of transfer matrices ($A_{ML}$ and $A_{MQ}$) were, however restricted to rigid jointed frames with encastré ends only. To accommodate hinged, roller and guided support conditions, a new matrix called as “Hinge Matrix $H_M$” is required and a modified form of compatibility equation is needed for the calculation of primary unknowns. The main drawback of this approach is that it requires a different treatment for pin jointed structures and hinged, roller and guided support conditions.

1.2.6 Integrated Force Method

In 1973, a new version of the force method, was developed by Patnaik [7] and named it “Integrated Force Method (IFM)”. It was shown in a comparison with early force methods that the IFM makes the automation as convenient as stiffness method and yet retains the known potential for superior stress field accuracy of finite element models that is associated with force method solution techniques. He also pointed out that with the further development IFM can become robust and versatile formulation to deal with variety of problems.

1.3 SCOPE AND OBJECTIVES OF THE PRESENT WORK

It is clear from the literature review, presented in the next chapter, that there are two alternative formulations available for the calculation of stresses directly i.e. hybrid stress method and the force method. In hybrid method, the inversion of the flexibility matrix is necessary in order to generate the element stiffness matrix, which may become a huge computational burden, especially if higher order stress field is strongly required. On the other hand, the advantageous and attractive part of the force method is that it allows the
forces in the element to be considered directly as unknowns, which is very appealing feature to the design engineer as the properties of the members of structures most often depend upon the member forces rather than joint displacements.

In case of framed structures subjected to static loading; only problems of beam, plane truss and plane frame have been attempted using Integrated Force Method. No work has been reported in the literature for the analysis of space truss and space frame structures. One of the objectives of the present work, therefore, is to systematize and generalize the analytical work including the development of program for each type of skeletal structure with pre- and post - processing facilities using VB as programming platform. Also, the aim is to include the effect of secondary stresses caused by initial strains, temperature variation and support disturbances in the compatibility condition through vector of initial deformation.

In the field of dynamic analysis, except for one problem of propped cantilever beam subjected to a lumped mass at the centre, no work has been reported in the literature for the force vibration analysis of different types of framed structures using IFM. Hence in the present work it is planned to extend the IFM to deal with other types of framed structures for force based eigen value analysis considering both Lumped and Consistent mass matrices including development of the program for the same.

It is also clear from the literature survey that Patnaik and his team have developed IFM based formulation for the analysis of isotropic plane stress and plate bending problems. The formulation is based on selection of displacement functions using Langragian/Hermite or generalized polynomial, and proper selection of stress polynomial. One of the objectives of the present work is to extend the IFM to handle two dimensional problems of continuum structures by appropriate selection of stress and displacement fields and to demonstrate its applicability to a variety of plane stress, plane
strain and isotropic plate bending problems. It is also proposed to develop the formulation for different types of orthotropic plate problems using suitable displacement and stress polynomial functions.

Further, the objective of present work is to develop a rectangular element formulation with nine force and 12 displacement degrees of freedom to deal with plate bending problems under different inertial properties using both lumped and consistent mass matrices to obtain the natural frequency, internal forces and mode shapes for each frequency value.

The objective of the present work is also to develop a modified integrated force method known as Dual Integrated Force Method (DIFM) for variety of problems of framed and continuum structures and to compare, where possible, with the known solutions to validate the proposed formulation.

Also, one of the important objectives of the present work is to develop the IFM based formulation in the polar coordinates to handle different types of circular and annular plate bending problems and to validate the formulation by comparing the results with those available in the literature.

It is also one of the objectives of the present work to carry out the buckling analysis of different types of beams, plane trusses and plane frames using Integrated Force Method. It is also planned to develop the necessary formulation to handle plate buckling problems.

1.4 ORGANIZATION OF THE THESIS

Chapter 1 starts with the classification of structures and brief account of various methods of analysis available for the same with due emphasis on flexibility in force based methods. After highlighting the scope and objectives of the present work, it gives brief account of the organization of the thesis.
Chapter 2 is devoted to literature survey. Some of the major developments taken place in the analysis of structure in last eight decades are highlighted with specific reference to matrix methods, finite element method and integrated force method.

In Chapter 3 concepts, methodology and variational functionals adopted in IFM are discussed in detail. Different issues related to compatibility conditions are also discussed with the emphasis on force compatibility conditions. Derivation of Equilibrium Equations (EE), Compatibility Conditions (CC) and Deformation Displacement Relations (DDR) are discussed. How concatenation of compatibility matrix [C] with basic equilibrium matrix [S] is also explained.

A new matrix based method is developed with minor modifications in the IFM based formulation in Chapter 4 and is named as Dual Integrated Force Method (DIFM). Different types of elements are formulated to facilitate analysis of different types of framed structures.

Different stress functions which are required for solving different types of 2D in-plane problems and out of plane bending problems are discussed in Chapter 5. Firstly, a rectangular element (RECT_5F_8D) having 8 displacement and 5 force degrees of freedom is discussed. Next, development of higher order elements by using Airy Stress Functions is explained. Also, force and displacement polynomials for triangular and curved elements are discussed. For plate bending problems, the use of force polynomial with different terms and approximating the displacements using Hermitian formula is discussed.

After highlighting some of the important features of selected environments for the development of the programs in Chapter 6, necessary matrices are derived in Chapter 7 for solving different types of framed structure problems
 thru IFM. Computer program for plane and space structures are prepared using VB.NET. Results obtained are depicted in terms of internal moments and nodal displacements for different types of framed structures and are compared with the available solutions.

**Chapter 8** is devoted to dynamic analysis of various types of framed structures. Matrices required for dynamic analysis are formulated and a program is developed in VB.NET to find natural frequencies and thus the internal forces are calculated in addition to nodal displacements. IFM results are compared with the available solutions.

In **Chapter 9** different types of plane stress and plane strain problems are solved. Plate bending problems are also attempted using RECT_9F_12D element under different types of loading and boundary conditions. Linear Independent Unknown Technique is employed to calculate compatibility matrix \( [C] \) from the basic equilibrium matrix \( [B] \) using Matlab software. Computer program is also developed in VB.NET with input and output modules to handle different types of 2D problems.

**Chapter 10** is devoted to free vibration analysis of plate bending problems using IFM formulation. First four natural frequencies are calculated and internal moments and nodal displacements are worked out. The values calculated here for natural frequencies are validated using the standard available solutions in the literature.

After discussing the different types of orthotropy in **Chapter 11** modifications required in the formulation to take care of material orthotropy are presented and a variety of problems of GFRP (Glass Fiber Reinforced Plastic) plate problems are solved under different boundary conditions. Problem of bending of Reinforced Cement Concrete Slab with double sided stiffened beam in only one direction is also attempted.
Dynamic analysis of orthotropic plates is attempted in Chapter 12. Values of natural frequencies are calculated using the standard Eigen operators given in Matlab. For the first four frequencies, internal moments and nodal displacements are calculated using the secondary equations.

Various matrices required in IFM are formulated in polar coordinates to deal with axisymmetric plate problems in Chapter 13. A new element (CIRC_2F_4D) is developed with two force and four displacement degrees of freedom. A variety of circular and annular plate problems are solved and results are validated by comparing with the known solutions.

Chapter 14 is aimed at development of IFM for the buckling problems. After deriving the geometric stiffness matrix, different types of problems are solved. It includes beam, truss and frame problems in addition to examples of buckling of rectangular plates. Comparison is made with the known solutions to validate the formulation and computer implementation.

Finally, Chapter 15 highlights the conclusions and contributions of the present work followed by the scope for the future work.