CHAPTER 8

DYNAMIC ANALYSIS OF FRAMED STRUCTURES

8.1 SCREEN SHOTS OF COMPUTER IMPLEMENTATION

A preprocessor is developed in VB6 to facilitate data input through a number of forms developed for the dynamic analysis of different types of framed structures. Using menu editor and multiple display interface one can link different forms by hiding and un-hiding operations depending upon logic of the program. After invoking the preprocessor the first step is selection of type of structure out of the group of framed structures i.e. Beam, Plane Truss, Plane Frame, Grid, Space Truss and Space Frame. Once the structure is selected by the user, input data is supplied using Labels, Textboxes which are linked to different forms. The geometrical and material properties are supplied using various text boxes. Support conditions are selected by assigning property through the next form. Visual feedback is provided for each user action. The next form highlights the key diagram which enables the complete geometry of the structure.

By clicking the command button “Development of Matrices”, various necessary properties are transferred from Form 1 to Form 3 and Form 2 to Form 3. This command button develops basic equilibrium matrix [B] and necessary global flexibility matrix [G] depending upon the basic unknowns ‘m’ and ‘n’, which are free nodal displacements and internal unknowns nodal moments respectively. Coding is written for development of text file which enables us to supply both the matrices named as IFM_FRAME.txt and Gmatrix.txt. The lumped mass matrix [M] is directly written in the main processor. The connection between VB 6 and Matlab is done through COM approach by linking an automation server in the given program.

Once the major matrices are developed the mathematical and matrix based operations are carried out in the main processor part of the software. Matlab
facilitates a temporary command window which is developed by using interface procedure. The basic matrices \([B]\) and \([G]\) are activated by calling a text file. The lumped Mass matrix \([M]\) is directly typed using command window. The Matlab in-built editor enables various types matrix based operations at the command prompt i.e. matrix addition, multiplication, transpose, inverse and eigen operator.

The step-by-step procedure followed for the dynamic analysis is illustrated here with the help of an example of a propped cantilever beam. A mild steel propped cantilever beam having two elements is separated by lumped mass \(M_o = 10\ \text{kN-Sec}^2/\text{m}\). Each element is of 1m length having flexural rigidity \(EI\) as 1666.67 kN-m\(^2\) as shown in Fig. 8.1.

![Fig. 8.1 Propped Cantilever Beam](image)

**Step 1:** From MDI form 1 main menu the type of structure is selected as beam as depicted in Fig. 8.2.

![Fig. 8.2 Selection of Type of Structure](image)

**Step 2:** From beam option which displays different types of beams by using check box button propped cantilever case is selected. Length and flexural rigidity of each span are entered as shown in Fig. 8.3.
Fig. 8.3 Geometrical and Flexural Properties of Beam

**Step 3:** Once the entered data is accepted it depicts on screen the next form related to boundary conditions. The desired support condition is selected as depicted in **Fig. 8.4.** Selection of plot button draws the beam on the screen and clicking next button lead to the next form.

Fig. 8.4 Beam with Boundary Conditions

**Step 4:** Form 3 shows the button for drawing key diagram of the beam in addition to button for the calculation of the different matrices such as Basic
Equilibrium Matrix [B], Global Flexibility Matrix [G] and Mass Matrix [M] as shown in Fig. 8.5. In the same form clicking next button connects VB6 to Matlab using COM procedure as shown in Fig. 8.6.

Fig.8.5 Key Diagram with Lumped Mass

Fig.8.6 Connection of Visual Basic with Matlab

**Step 5:** Once Matlab command window is depicted the main processor starts and matrices are called using text file named as IFM_FRAME.txt, Gmatrix_Frame.txt and Mass.txt respectively as can be seen in Fig. 8.7.
**Step 6:** In the command window by changing path to folder named as “CCPROG” which is available at desktop, connection is established between a .m file which is Matlab based program for developing compatibility condition using LIUT technique. Thus by typing `z = mtechexamplemod(B)` in command window, one can enter number of compatibility conditions required. The dependent unknown codedepB and independent unknown codeindepB is auto selected for further calculation. The complete problem having three moment unknowns ‘n’ and having two free displacements at intermediate joint ‘m’ requires one compatibility condition which is named as z.cMatrix in matlab code. The screen shot of development of compatibility conditions is depicted in Fig. 8.8.
Step 7: Once ‘z.cMatrix’ is available it is checked for its null property by multiplying z.cMatrix with [B] matrix. Thus by typing ‘z.cTransposeB’ the null property check can be satisfied as depicted in the Fig. 8.9. Secondly, normalized global compatibility condition is also calculated by multiplying z.cMatrix with Gmatrix. The global equilibrium matrix [Smatrix] is developed by concatenation of [B] with the ‘CCmatrix’. The inverse can be calculated by typing inv(Smatrix) in command window, which is named as ‘Sinv’. The Jmatrix which is m rows of [Smatrix⁻¹]T is also worked out using the same. The complete procedure related to each operation is depicted in Fig. 8.9.
Fig. 8.9 Different Matrix Operations

**Step 8:** Next, eigen value analysis is carried out using the inbuilt module. Product of [Mmatrix], [Jmatrix] and [Gmatrix], named as [M.IGmatrix], is calculated directly. By typing directly on command window [M_modes, Omega] = eig(Smatrix, M*JG) the necessary operation is carried out. Here Omega is a diagonal matrix of size (n x n), having possible values of frequencies along diagonal terms. Next, by substituting each value of ‘ω’ in matrix, the possible modes are calculated which is known as modal moment vector. It is named as M_modes corresponding to each value of frequency and calculated directly using Matlab software. **Figure 8.10** depicts the natural frequency value (ω = 47.8091 rad/sec) calculated in last line which is found to match with the exact solution 47.8091 rad/sec based on
standard dynamics approach. The first column of M_mode matrix represents internal moments corresponding to value of $M_1 = 1$, for which $M_2$ and $M_3$ are worked out.

![MATLAB Command Window](image)

**Fig. 8.10 First Frequency and Moment Modal Values**

**Step 9**: The first modal deflection values corresponding to first modal moments are calculated using standard IFM based formula i.e. $[\delta] = \text{[Jmatrix]}*[\text{Gmatrix}]*[\text{M_modes}].$ Next, after normalizing the deflection value to unity the rotational value corresponding to that is calculated as depicted in **Fig. 8.11**.

![MATLAB Command Window](image)

**Fig. 8.11 First Modal Displacement Vector**
8.2 EXAMPLES OF DYNAMICS ANALYSIS OF BEAMS

8.2.1 Simply Supported Beam Example

A mild steel beam which is simply supported at extreme ends has two segments of 1m each on either side of lumped mass Mo equal to 10 kN-Sec²/m at centre as shown in Fig. 8.12, with EI = 1666.67 kN-m².

![Fig. 8.12 Simply Supported Beam Example](image)

**Step 0 – Solution strategy:** The beam has two segments on either side of the lumped mass \((M_0)\). It has four internal moments \((M_1, M_2, M_3, M_4)\) out of which two extreme moments are zeros. The given problem has internal unknown moments as two (Fig. 8.13) and nodal displacements at intermediate joint as two. Thus the problem becomes statically determinate.

![Fig. 8.13 Free Body Diagram](image)

**Steps 1 – Formulate the equilibrium equations:** The EEs are written in terms of unknown moments at the intermediate joint i.e. at lumped mass along vertical displacement \(\delta_{20}\) and rotational displacement \(\Theta_{20}\) follows.

\[
\begin{bmatrix}
-1 & -1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
M_2 \\
M_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

or \([B][M] = [P]\) \hspace{1cm} \text{... (8.1)}

**Step 2 – Derive the deformation displacement relations:** The DDR is obtained in the form \(\{\beta\} = [B]^T\{\delta\}\) and is written as
\[
\begin{pmatrix}
\beta_2 \\
\beta_3
\end{pmatrix} =
\begin{bmatrix}
-1 & 1 \\
-1 & -1
\end{bmatrix}
\begin{pmatrix}
\delta_{23} \\
\Theta_{23}
\end{pmatrix}
\]

\ldots

(8.2)

**Step 3**  **Formulate force deformation relations:** Force deformation relation for a beam problem is already derived in the previous illustrative example. The FDR in this case will be in terms of \([G]\) matrix which is of size \(n \times n\). For end moments \(M_i\) and \(M_j\), the FDR can be written as

\[
\beta_i = \frac{\ell}{6EI} (2M_i + M_j) \quad \text{and} \quad \beta_j = \frac{\ell}{6EI} (M_i + 2M_j)
\]

\ldots (8.3)

Formulating above equations for two elements, one has

\[
\beta_2 = \frac{1}{6EI} (2M_2) \quad \beta_3 = \frac{1}{6EI} (2M_3)
\]

\ldots (8.4)

In matrix form, the same can be expressed as

\[
\begin{pmatrix}
\beta_2 \\
\beta_3
\end{pmatrix} = \frac{2}{6EI} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{pmatrix}
M_2 \\
M_3
\end{pmatrix} = [G]\{M\}
\]

\ldots (8.5)

**Step 4 – Derive the lumped mass matrix:** The Lumped mass matrix is directly written as given below.

\[
[M_{\text{mass}}] = \begin{bmatrix}
10 & 0 \\
0 & 0
\end{bmatrix}
\]

\ldots (8.6)

**Step 5 – Calculate natural frequency:** IFM based frequency equation is as follows:

\[
[S]\{F\} = \lambda[J][G][M]\{F\}
\]

\ldots (8.7)

In case of beam bending problem the \(\{F\}\) vector represents internal moments \(\{M\}\), while the same for pin jointed structures represents internal forces \(\{F\}\). \([S]\) is the global equilibrium matrix which is equal to \([B]\) in this problem, \([G]\) is the flexibility matrix, \([M]\) is the mass matrix and \(\lambda = \omega^2\) is known as eigen operator for frequency analysis.
Substituting all the above matrices in Eq. (8.7) and solving directly using Matlab eigen operator, the value of $\omega$ is found as $31.6227$ radians/sec, which is found to be matching with the standard dynamics formula of $\sqrt{\frac{48EI}{ML^3}} = 31.6228$ radians/sec.

**Step 6 – Calculate modal moments:** Substituting the value of frequency obtained in Eq. (8.7) and calculating the modal moments, one gets

$$\begin{bmatrix} M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{... (8.8)}$$

**Step 7 – Calculating nodal displacements:** The nodal displacements can be calculated by substituting the values of moments into formula given below.

$$\begin{bmatrix} \delta_{23} \\ \theta_{32} \end{bmatrix} = [J][G] [\text{matrix}] [M] = \begin{bmatrix} -0.2 \times 10^{-3} \\ 0.000 \end{bmatrix}$$

$$\text{... (8.9)}$$

It shows that the deflection value is maximum where slope equals to zero.

**8.2.2 Fixed Beam Example:**

A fixed mild steel beam as per **Fig. 8.14** has two segments of 1m each on either side of the lumped mass $M_0$ equal to 10 kN-Sec$^2$/m at centre. The beam has $EI = 1666.67$ kN-m$^2$.

![Fixed Beam Example](image)

**Fig. 8.14 Fixed Beam Example**

**Step 0 – Solution strategy:** The given problem has four internal unknown moments ($M_1$, $M_2$, $M_3$ and $M_4$) and two nodal displacements at intermediate joint. Thus the problem is statically indeterminate (**Fig. 8.15**).
**Fig. 8.15 Free Body Diagram**

**Step 1 – Formulate the equilibrium equations:** The EEs are written in terms of unknown moments at the intermediate joint i.e. at lumped mass along vertical displacement \( \delta_{23} \) and rotational displacement \( \Theta_{23} \) as

\[
\begin{bmatrix}
1 & -1 & -1 & 1 \\
0 & 1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
[B][M] = [P] 
\]

... (8.10)

**Step 2 – Derive the deformation displacement relations:** The DDR is obtained as \( \{\beta\} = [B]^T \{\delta\} \).

\[
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix}
= \begin{bmatrix}
-1 & 0 \\
-1 & 1 \\
-1 & -1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
\delta_{23} \\
\Theta_{23}
\end{bmatrix}
\]

... (8.11)

**Step 3 – Generate the compatibility conditions:** The two compatibility conditions can be expressed using Matlab code as

\[
\begin{bmatrix}
16 & 9 & 9 & 2 \\
12 & 9 & 9 & 6
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

Or \([C][\beta] = [0]\)

... (8.12)

**Step 3 – Formulate force deformation relations:** For end moments \( M_i \) and \( M_j \), the FDR can be written as

\[
\beta_i = \frac{\ell}{6EI} \left(2M_i + M_j\right) \quad \text{and} \quad \beta_j = \frac{\ell}{6EI} \left(M_i + 2M_j\right)
\]

... (8.13)

Writing above equations for the two members leads to
\[
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{pmatrix} = \frac{2}{6EI} \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix} \begin{pmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4
\end{pmatrix} = [G][M] \quad \ldots (8.14)
\]

**Step 4 – Derive the lumped mass matrix:** The Lumped mass matrix can be directly written as given below.

\[
[M_{\text{mass}}] = \begin{pmatrix}
10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \quad \ldots (8.15)
\]

**Steps 5 – Calculate natural frequency:** Calculate global normalized compatibility conditions by multiplying [C] matrix and Global Flexibility matrix [Gmatrix]. By concatenating into basic equilibrium matrix [B], a global equilibrium matrix [S] is developed. The basic IFM based equation now can be written by substituting all the related matrices into Eq. (8.7) as

\[
\begin{pmatrix}
1 & -1 & 1 \\
0 & 1 & -1 \\
0.41 & 0.34 & 0.2 & 0.13 \\
0.33 & 0.3 & 0.24 & 0.21
\end{pmatrix} \begin{pmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4
\end{pmatrix} = \frac{\omega^2}{1666.67} \begin{pmatrix}
10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0.25 & -0.25 & -0.25 & 0.25 \\
-0.25 & 0.5 & -0.5 & 0.25 \\
5.55 & -0.6944 & 0.6944 & -6.9944 \\
-4.6296 & 1.6204 & 1.6204 & 7.8704
\end{pmatrix} \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix} \quad \ldots (8.16)
\]

Solving above equations one can get the solution as \( \omega = 63.2455 \text{ rad/Sec} \), which is found to be matching with the exact solution of 63.2456 rad/seconds.

**Step 6 – Calculate modal moments:** Substituting the values of frequency calculated in Eqn.(8.16) the internal nodal moments can be calculated which is as

\[
\begin{pmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4
\end{pmatrix} = \begin{pmatrix}
1.0 \\
-1.0 \\
-1.0 \\
1.0
\end{pmatrix} \quad \ldots (8.17)
\]

**Steps 7 – Calculate Modal Moments (Internal Moments)**
Substituting the values of nodal moments into Eq. (8.8) the nodal moments are calculated which is as below

\[
\begin{pmatrix}
\delta_{23} \\
\theta_{22}
\end{pmatrix} = \begin{pmatrix}
1.0 \times 10^{-4} \\
0.000
\end{pmatrix}
\]

... (8.18)

### 8.2.3 Continuous Beam Example

A two span continuous beam having two extreme ends as fixed is shown in Fig. 8.16. It is to be analysed considering four segments of 1m each on both sides of two lumped mass ‘Mo’. Consider EI as 666.67 kN-m².

![Fig. 8.16 Two Span Continuous Beam Example](image)

Following three possibilities are considered here: (i) Direct Lumping of Constant Mass at given nodes (ii) Lumped Mass calculated based on basis of mass per unit length and number of members meeting at a joint [LMcase] and (iii) Consistent Mass which is again based on mass per unit length and number of members meeting at a joint [CMcase]. Solutions obtained based on different mass criteria are verified here by Stiffness based eigen value analysis.

#### (i) DNLM Case

**Step 0 – Solution strategy:** The continuous beam is analysed by considering four segments with lumping mass of Mo equals 10kN at the centre of each span. Each beam segment has internal moments \((M_1, M_2)\). Thus, the complete problem has total eight internal unknown moments \((n)\) and total five possible free joint displacements \((m)\) as shown in Fig. 8.17. Thus the problem has 3 degrees of static indeterminacy.

![Fig. 8.17](image)
Fig. 8.17 Free Body Diagram

**Step 1 – Formulate the equilibrium equations:** The EEs are written in terms of unknown moments at the intermediate joints i.e. at lumped masses and at intermediate junctions corresponding to vertical displacements $\delta_{23}$, $\delta_{67}$ and rotational displacements $\Theta_{23}$, $\Theta_{45}$, and $\Theta_{67}$. In matrix form

$$
\begin{bmatrix}
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
M_5 \\
M_6 \\
M_7 \\
M_8 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
$$

Which can be seen as $[B][M] = [P]$ or $[S][M] = [P]$.

**Step 2 – Derive the deformation displacement relations:** The DDR is obtained as $\{\beta\} = [B]^T \{\delta\}$.

$$
\begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_8 \\
\end{bmatrix}
\begin{bmatrix}
\delta_{23} \\
\Theta_{23} \\
\delta_{45} \\
\Theta_{45} \\
\delta_{67} \\
\Theta_{67} \\
\end{bmatrix}
= [B]^T
$$

**Step 3 – Generate the compatibility conditions:** The three compatibility conditions for the beam problem can be worked out by using .m file of Matlab which is named as mtechexamplemod.

$$
\begin{bmatrix}
-8 & 2 & 2 & 12 & 12 & 9 & 9 & 6 \\
1 & 1 & 1 & 1 & 3 & 3 & 3 & 5 \\
2 & 9 & 9 & 9 & 16 & 9 & 9 & 2 \\
\end{bmatrix}
\{\beta\} = \{0\}
$$

Which can be written in the form $[C]\{\beta\} = \{0\}$

**Step 4 – Formulate force deformation relations:** For end moments $M_i$ and $M_j$, the FDR can be written as
\[
\beta_i = \frac{\ell}{6EI} \left( 2M_i + M_j \right) \\
\beta_j = \frac{\ell}{6EI} \left( M_i + 2M_j \right)
\]

Formulating above equations for all the three members and arranging in a matrix form one can write,

\[
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7 \\
\beta_8
\end{bmatrix} = \frac{1}{6EI} \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 1
\end{bmatrix} \begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
M_5 \\
M_6 \\
M_7 \\
M_8
\end{bmatrix} = [G][M]
\]

**Step 5 – Derive the mass matrix for DNLM Case:** The lumped mass matrix can be is directly written as

\[
[M_{DNLM}] = \begin{bmatrix}
10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

**Step 6 – Calculate natural frequency:** The global normalized compatibility conditions are obtained by multiplying [C] matrix by Global Flexibility matrix [Gmatrix] and then concatenating into basic equilibrium matrix [B], a global equilibrium matrix [S] is developed.

The governing IFM based equations are developed by substituting all the required matrices in Eq. (8.7). After solving the equations using matlab based eigen module ‘eig(a,b)’. The calculated frequencies based on DNLM case are as follows;

\[
\begin{bmatrix}
\omega_1 \\
\omega_2
\end{bmatrix} = \begin{bmatrix}47.8092 \\
63.2456\end{bmatrix} \text{ rad/sec}
\]
In which first natural frequency is matching with value of propped cantilever case while second one is matching with the fixed case. As the mass is on both the side, a mechanism is developed at intermediate joint such that it works as a propped cantilever beam where moment equals to zero, with possible rotational value. On the other hand, whenever both the masses during motion are on one side of the beam the same joint having zero rotation with reacting moment values is considered as second mode of vibration.

**Step 7 – Calculate modal moments:** Substituting the above values of frequencies into basic equation of IFM, the internal nodal moments can be calculated. For the first natural frequency, the moments are

\[
\begin{pmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
M_5 \\
M_6 \\
M_7 \\
M_8
\end{pmatrix} = \begin{pmatrix}
-1.00 \\
0.833 \\
0.833 \\
0.00 \\
0.00 \\
-0.833 \\
-0.833 \\
1.00
\end{pmatrix}
\]

**Step 8 – Calculate nodal displacements:** Substituting the values of nodal moments calculated above into Eq. (8.9), the nodal displacements for first mode moments are found as

\[
\begin{pmatrix}
\delta_{23} \\
\theta_{23} \\
\theta_{45} \\
\delta_{67} \\
\theta_{67}
\end{pmatrix} = \begin{pmatrix}
-0.1106 \\
-0.05 \\
0.2 \\
0.1166 \\
-0.05
\end{pmatrix} \times 10^{-3}
\]

(i) **Considering Lumping Mass Criteria (LM Case)**

In this case, the contributory mass is considered at each node as per the number of members meeting at a junction corresponding to lateral deflection direction only. Considering mass per unit length of each member as 1kN/m, the lumped mass matrix for each member written as
\[
[M_1] - [M_2] - [M_3] - [M_4] = \begin{bmatrix}
0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

After assembly, the global lumped mass matrix corresponding to vertical displacement based contributory criteria at joints (2-3) and (6-7) can be written as

\[
[M_L] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(ii) **Considering Consistent Mass (CM Case)**

In this case, actual contributory mass is calculated at each node as per number of members meeting at junction, where contribution corresponding to rotation is also considered in addition to that corresponding to lateral displacement. Considering mass per unit length of each member as 1kN/m, Thus for all the beam members of 1m length the consistent mass is found as

\[
[M_1] = [M_2] = [M_3] = [M_4] = 2.389 \times 10^{-3} \begin{bmatrix}
156 & \text{Symm} \\
22 & 4 \\
54 & 13 & 156 \\
-13 & -3 & -22 & 4 \\
\end{bmatrix}
\]

After assembly, the global consistent mass matrix is found as

\[
[M_c] = \begin{bmatrix}
0.7428 & 0 & 0.1904 & \text{Symm} \\
-0.0309 & -0.00714 & 0.1904 & 0 \\
0 & 0 & 0.0309 & 0.7424 \\
0 & 0 & -0.00714 & 0 & 0.1904 \\
\end{bmatrix}
\]

By following the steps given above, the frequencies are worked out based on LM and CM cases. By substituting the frequencies, the internal forces and nodal displacements are calculated for the first frequency. The frequency values are checked using Standard stiffness based eigen value analysis. The
comparison is presented in Table 8.1. The corresponding moments for the first natural frequency value are given in Table 8.2.

**Table 8.1 Natural Frequencies for Continuous Beam**

<table>
<thead>
<tr>
<th>Natural Frequency (ω) Radians/Second</th>
<th>Lumped Mass</th>
<th>Consistent Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>IFM Stiffness Method</td>
<td>IFM Stiffness Method [93]</td>
<td></td>
</tr>
<tr>
<td>151.188</td>
<td>151.187</td>
<td>158.881</td>
</tr>
<tr>
<td>200.001</td>
<td>200.00</td>
<td>232.11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>596.11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>836.66</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1558.33</td>
</tr>
</tbody>
</table>

**Table 8.2 Moments and Nodal Displacements**

<table>
<thead>
<tr>
<th>Based on Lumped Mass</th>
<th>Based on Consistent Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal Moments</td>
<td>Nodal Displacements (δ) x 10⁻³</td>
</tr>
<tr>
<td>M₁ = -1.0</td>
<td>δ₂₃ = 1.00</td>
</tr>
<tr>
<td>M₂ = 0.83</td>
<td>θ₂₃ = 0.00</td>
</tr>
<tr>
<td>M₃ = 0.83</td>
<td>θ₄₅ = 0.00</td>
</tr>
<tr>
<td>M₄ = 0.00</td>
<td>δ₆₇ = 1.00</td>
</tr>
<tr>
<td>M₅ = 0.00</td>
<td>θ₆₇ = 0.00</td>
</tr>
<tr>
<td>M₆ = -0.83</td>
<td></td>
</tr>
<tr>
<td>M₇ = -0.83</td>
<td></td>
</tr>
<tr>
<td>M₈ = 1.00</td>
<td></td>
</tr>
</tbody>
</table>

### 8.3 EXAMPLES OF DYNAMIC ANALYSIS OF PLANE TRUSS

Two examples of plane truss are solved here using IFM based formulation by considering (1) Direct Nodal Lumped Mass of 10kN at nodes under consideration (2), Lumped mass criteria as per members meeting at joints and (3) Consistent mass criteria. The solution obtained is compared with the stiffness based Eigen values analysis [93].

#### 8.3.1 Three Member Truss Example
A three member truss shown in Fig. 8.18 has three joints out of which two are loaded with direct nodal lumping of mass (Mo) equals to 10 kN-Sec²/m. The members have axial rigidity as 2.01 x 10⁶ kN. The truss is to be analysed under different mass criteria by considering unit weight as 1 kN/m.

![Fig. 8.18 Three Member Plane Truss Example](image)

The complete methodology being same, the necessary changes in the derivation of mass matrices for different cases are discussed here.

(i) **Direct Nodal Lump of Mass (DNLM Case)**

Three possible nodal displacements in the truss are δ₃H, δ₂H and δ₂V as shown in Fig. 8.18. Hence there are three possible frequencies corresponding to each nodal displacement with three internal force unknowns. Thus, the problem is having basic equilibrium matrix [B] which directly become a square global equilibrium matrix [S] as it does not require compatibility conditions. The nodal lumping mass matrix for Mₒ can be written as

\[
[M_{DNLM}] = \begin{bmatrix}
2_n & 2_v & 3_n \\
10 & 0 & 0 \\
0 & 10 & 0 \\
0 & 0 & 10
\end{bmatrix}
\]

After deriving all the necessary matrices and substituting in Eq. (8.16), the values of natural frequencies are calculated and are compared with the
stiffness method based eigen values analysis in Table 8.3. After substituting each IFM based frequency value in same Eq. (8.16) relative internal force in each member is calculated by considering $F_1$ equals to 1, using Matlab software. By substituting the relative internal forces of each member the corresponding nodal displacements are calculated. The relative internal forces as well as nodal displacements obtained for first natural frequency are reported in Table 8.4.

<table>
<thead>
<tr>
<th>Table 8.3 Natural Frequencies (DNLM Case)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural Frequency ($\omega$) rad/sec</td>
</tr>
<tr>
<td>IFM</td>
</tr>
<tr>
<td>194.855</td>
</tr>
<tr>
<td>448.33</td>
</tr>
<tr>
<td>613.36</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 8.4 Internal Forces and Nodal Displacements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal Forces (F)</td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td>$F_1 = -1.0$</td>
</tr>
<tr>
<td>$F_2 = 0.232$</td>
</tr>
<tr>
<td>$F_3 = 0.232$</td>
</tr>
</tbody>
</table>

(ii) **Considering Lumping Mass Criteria (LM Case)**

In this case, actual contributory mass is calculated at each node as per the number of members meeting at the joint. The mass matrix for each member can be written as

$$[M_1] = [M_2] = [M_3] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}$$

The transformation from local to global direction is carried out by

$$[M_L] = [T]^T[M_1][T]$$
Where \([T]\) is the transformation matrix for a plane truss member and is given by

\[
[T] = \begin{bmatrix}
C_x & C_y & 0 & 0 \\
-C_y & C_x & 0 & 0 \\
0 & 0 & C_x & C_y \\
0 & 0 & -C_y & C_x
\end{bmatrix}
\]

Next, the global lumped mass matrix is calculated, which is as follows

\[
[M_L] = \begin{bmatrix}
1.2069 & \text{Symm} \\
0 & 1.2069 \\
0 & 0 & 1.2069
\end{bmatrix}
\]

After deriving all the necessary matrices and substituting in Eq. (8.16), the values of natural frequencies are calculated and are compared with those obtained using stiffness based eigen value analysis (Table 8.5). Relative internal force in each member is also calculated by considering \(F_1\) equals to 1. Finally, the corresponding nodal displacements are calculated. Results obtained for the the relative internal forces as well as nodal displacements for first natural frequency are presented in Table 8.6.

### Table 8.5 Natural Frequencies for Plane Truss

<table>
<thead>
<tr>
<th>IFM</th>
<th>Stiffness Method [93]</th>
</tr>
</thead>
<tbody>
<tr>
<td>560.85</td>
<td>560.83</td>
</tr>
<tr>
<td>1290.5</td>
<td>1290.4</td>
</tr>
<tr>
<td>1765.4</td>
<td>1765.3</td>
</tr>
</tbody>
</table>

### Table 8.6 Internal Forces and Nodal Displacements

<table>
<thead>
<tr>
<th>Internal Force (F)</th>
<th>Nodal Displacements (\delta) x 10^{-5}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8111</td>
<td>0.1879</td>
</tr>
<tr>
<td>0.8111</td>
<td>0.0434</td>
</tr>
<tr>
<td>1.000</td>
<td>0.0434</td>
</tr>
</tbody>
</table>

**(iii) Considering Consistent Mass Criteria (CM Case)**
In this case, actual contributory mass is calculated at each node as per the number of members meeting at junction. Thus for all the members of the plane truss the lumped mass matrix is as under.

\[
[M_1] = \frac{1 \times 1}{6} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = [M_2] \quad \text{and,} \quad [M_3] = \frac{1 \times \sqrt{2}}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}
\]

The global consistent mass matrix is as follows

\[
[M_L] = \begin{bmatrix} 2_h & 2_v & 3_h \\ 0.8064 & 0.00 & 0.2356 \\ 0.00 & 0.8046 & 0.00 \\ 0.2356 & 0.00 & 0.8046 \end{bmatrix} \text{ Symm}
\]

After deriving all the necessary matrices and substituting in Eq. (8.16), values of natural frequencies are calculated and are verified by stiffness based eigen analysis as depicted in Table 8.7 whereas results obtained for the relative internal forces as well as nodal displacements for the first natural frequency value are depicted in Table 8.8.

**Table 8.7 Natural Frequencies for Plane Truss**

<table>
<thead>
<tr>
<th>IFM</th>
<th>Natural Frequency ((\omega)) rad/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>204.03</td>
<td>204.0</td>
</tr>
<tr>
<td>507.71</td>
<td>507.69</td>
</tr>
<tr>
<td>749.47</td>
<td>749.413</td>
</tr>
</tbody>
</table>

**Table 8.8 Internal Forces and Nodal Displacements**

<table>
<thead>
<tr>
<th>Internal Force (F)</th>
<th>Nodal Displacements ((\delta) \times 10^{-5})</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.7868</td>
<td>0.1812</td>
</tr>
<tr>
<td>-1.00</td>
<td>0.0391</td>
</tr>
<tr>
<td>0.9273</td>
<td>0.0498</td>
</tr>
</tbody>
</table>

**8.3.2 Eleven Member Truss Example**
A 11-member truss having four joints loaded with direct nodal lumping of mass \((M_o)\) equals to 10 kN-Sec\(^2\)/m. The axial rigidity is constant for all the members and is equal to 2.01 x 10\(^6\) kN. The frequency analysis is to be carried out for lumped and consistent mass criteria by considering weight as 1 kN/m.

![Diagram of a 11-member plane truss with labeled joints and forces](image)

**Fig. 8.19 Eleven Member Plane Truss Example**

(i) **Considering Direct Nodal Lumping Mass (DNLM Case)**

There are total eight possible nodal displacements in the truss. Hence there are eight possible frequencies corresponding to each displacement while there are eleven internal force unknowns. The basic equilibrium matrix \([B]\) is of size 8 x 11. Thus, the problem is three degree statically indeterminate as per IFM procedure.

The mass matrix \(M_o\) can be written by referring Eq. (8.6) as

\[
[M_{DNLM}] = \begin{bmatrix}
2_h & 2_v & 3_h & 3_v & 4_h & 4_v & 5_h & 5_v \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Sym & 10 & 0 & 0 & 10 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 10 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 10
\end{bmatrix}
\]
After deriving all the necessary matrices and substituting in Eq. (8.16), values of natural frequencies are calculated and are verified by stiffness based eigen value analysis (Table 8.9). After substituting each IFM based frequency value in same Eq (8.16), the relative internal forces in each member are calculated by considering \( F_2 \) equals to 1. In this case as member one is constrained at support in both the direction, there is no internal force developed in it. By substituting the relative internal force of each member into Eq. (8.8) and using Matlab, corresponding nodal displacements are calculated. The relative internal forces as well as nodal displacements for first natural frequency are depicted in Table 8.10.

### Table 8.9 Natural Frequencies

<table>
<thead>
<tr>
<th>IFM</th>
<th>Stiffness Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>104.665</td>
<td>101.04</td>
</tr>
<tr>
<td>312.9164</td>
<td>308.33</td>
</tr>
<tr>
<td>350.038</td>
<td>325.44</td>
</tr>
<tr>
<td>611.7482</td>
<td>589.33</td>
</tr>
<tr>
<td>643.8708</td>
<td>643.08</td>
</tr>
<tr>
<td>770.7261</td>
<td>748.19</td>
</tr>
<tr>
<td>789.6878</td>
<td>774.65</td>
</tr>
<tr>
<td>910.3956</td>
<td>874.34</td>
</tr>
</tbody>
</table>

### Table 8.10 Internal Forces

<table>
<thead>
<tr>
<th>Internal Forces ((F))</th>
<th>Nodal Displacements ((\delta) \times 10^6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 = 0.00 )</td>
<td>( \delta_{2H} = -0.8718 )</td>
</tr>
<tr>
<td>( F_2 = 1.00 )</td>
<td>( \delta_{2V} = -0.8718 )</td>
</tr>
<tr>
<td>( F_3 = 0.5049 )</td>
<td>( \delta_{3H} = 1.00 )</td>
</tr>
<tr>
<td>( F_4 = -0.5049 )</td>
<td>( \delta_{3V} = 1.00 )</td>
</tr>
<tr>
<td>( F_5 = -1.00 )</td>
<td>( \delta_{4H} = -0.8718 )</td>
</tr>
<tr>
<td>( F_6 = 0.00 )</td>
<td>( \delta_{4V} = -0.8718 )</td>
</tr>
<tr>
<td>( F_7 = 0.3305 )</td>
<td>( \delta_{5H} = 1.00 )</td>
</tr>
<tr>
<td>( F_8 = 0.00 )</td>
<td>( \delta_{5V} = 1.00 )</td>
</tr>
<tr>
<td>( F_9 = -0.3650 )</td>
<td></td>
</tr>
</tbody>
</table>
\[ F_{10} = 0.3650 \]
\[ F_{11} = -0.3305 \]

(i) Considering Lumping Mass Criteria (LM Case)

The global is found as

\[
[M_L] = \begin{bmatrix}
2.9142 & 0 & 0 & 0 & 0 & 0 & 0 \\
2.9142 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.7071 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.7071 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.7071 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.7071 & 0 & 0 & 0 & 0 & 0 & 0 \\
2.9142 & 0 & 0 & 0 & 0 & 0 & 0 \\
2.9142 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Following the procedure used in previous example, results obtained for natural frequencies, internal forces and nodal displacements are reported here in Table 8.11 and Table 8.12.

**Table 8.11 Natural Frequencies for Plane Truss**

<table>
<thead>
<tr>
<th>IFM</th>
<th>Natural Frequency (ω) rad/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>240.44</td>
<td>231.68</td>
</tr>
<tr>
<td>688.911</td>
<td>650.068</td>
</tr>
<tr>
<td>703.41</td>
<td>678.89</td>
</tr>
<tr>
<td>1357.11</td>
<td>1238.9</td>
</tr>
<tr>
<td>1368.44</td>
<td>1339.4</td>
</tr>
<tr>
<td>1550.11</td>
<td>1553.83</td>
</tr>
<tr>
<td>1550.11</td>
<td>1553.83</td>
</tr>
<tr>
<td>1944.44</td>
<td>1864.55</td>
</tr>
</tbody>
</table>

**Table 8.12 Internal Forces and Nodal Displacements**

<table>
<thead>
<tr>
<th>Internal Force (F)</th>
<th>Nodal Displacements (δ) x 10^{-5}</th>
</tr>
</thead>
<tbody>
<tr>
<td>F_1 = 0.00</td>
<td>δ_{2H} = -0.04977</td>
</tr>
<tr>
<td>F_2 = 1.00</td>
<td>δ_{2V} = 0.0944</td>
</tr>
<tr>
<td>F_3 = 0.5513</td>
<td>δ_{3H} = -0.0644</td>
</tr>
<tr>
<td>F_4 = -0.5513</td>
<td>δ_{3V} = 0.2344</td>
</tr>
<tr>
<td>F_5 = -1.000</td>
<td>δ_{4H} = 0.0644</td>
</tr>
<tr>
<td>F_6 = 0.00</td>
<td>δ_{4V} = 0.2345</td>
</tr>
<tr>
<td>F_7 = 0.2950</td>
<td>δ_{5H} = 0.0497</td>
</tr>
</tbody>
</table>
\[ F_8 = 0.000 \quad \delta_{5Y} = 0.0944 \]
\[ F_9 = -0.3273 \]
\[ F_{10} = -0.3273 \]
\[ F_{11} = -0.2950 \]

(ii) **Considering Consistent Mass Criteria (CM Case)**

Based on consistent mass criteria, the global consistent mass matrix is found as follows

\[
[M_c] = \begin{bmatrix}
2_n & 2_y & 3_n & 3_y & 4_n & 4_y & 5_n & 5_y \\
1.9423 & 0 & 0.1667 & 0 & 0.2356 & 0 & 0.1667 & 0 \\
1.9423 & 0 & 0.1667 & 0 & 0.2356 & 0 & 0.1667 & 0 \\
0.333 & 0 & 0.1667 & 0 & 0 & 0 & 0 & 0 \\
0.333 & 0 & 0.1667 & 0 & 0 & 0 & 0 & 0 \\
1.1378 & 0 & 0.1667 & 0 & 0 & 0 & 0 & 0 \\
\text{Symm} & 1.1378 & 0 & 0.1667 & 0 & 0 & 0 & 0 \\
1.9423 & 0 & 0 & 0 & 0 & 0 & 0 & 1.9423 \\
1.9423 & 0 & 0 & 0 & 0 & 0 & 1.9423 & 0 \\
\end{bmatrix}
\]

Natural frequency results are presented in **Table 8.13** along with the comparison with solution obtained based on stiffness method of analysis whereas the results obtained for internal forces and nodal displacements are reported here in **Table 8.14**.

### Table 8.13 Natural Frequencies for Plane Truss

<table>
<thead>
<tr>
<th>IFM</th>
<th>Eigen Analysis (Stiffness Based)</th>
</tr>
</thead>
<tbody>
<tr>
<td>257.994</td>
<td>247.4</td>
</tr>
<tr>
<td>720.41</td>
<td>709.01</td>
</tr>
<tr>
<td>916.041</td>
<td>850.7</td>
</tr>
<tr>
<td>1723..11</td>
<td>1648.44</td>
</tr>
<tr>
<td>1368.44</td>
<td>1690.01</td>
</tr>
<tr>
<td>1799.11</td>
<td>1846.88</td>
</tr>
<tr>
<td>2073.50</td>
<td>2057.01</td>
</tr>
<tr>
<td>2602.22</td>
<td>2494.88</td>
</tr>
</tbody>
</table>

### Table 8.14 Internal Forces and Nodal Displacements

<table>
<thead>
<tr>
<th>Internal Forces (F)</th>
<th>Nodal Displacements (δ) x 10^{-5}</th>
</tr>
</thead>
<tbody>
<tr>
<td>F_1 = 0.00</td>
<td>F_9 = 0.000</td>
</tr>
<tr>
<td>F_2 = 1.00</td>
<td>F_9 = 0.3310</td>
</tr>
<tr>
<td>F_3 = 0.6188</td>
<td>F_{10} = -0.3310</td>
</tr>
</tbody>
</table>
8.4 EXAMPLE OF DYNAMIC ANALYSIS OF PLANE FRAME

A fixed footed portal frame is considered here with direct nodal lumping of mass \((M_0)\) equals to 10 kN Sec\(^2\)/m at the centre of beam as depicted in Fig. 8.20. The members have EI equals to 1666.67 kN – m\(^2\). The frequency analysis is carried out for lumped and consistent mass criteria by considering unit weight of each member as 1kN/m.

![Fig. 8.20 Plane Frame Example with Free Body Diagram](image)

1. Considering Direct Nodal Lumping Mass (DNLM Case)

The problem is having following five possible nodal displacements: (1) Horizontal displacement (Lateral sway) of beam member \(\delta_{23H}\). (2) Rotational displacement at junction of members 1-2 \(\theta_{12}\),(3) Rotational displacement at junction of member 2-3 \(\theta_{23}\),(4) Rotational displacement at joint 3-4 \(\theta_{34}\) and (5) Vertical displacement at lumped mass \(\delta_{23V}\). Hence there are five possible frequencies one corresponding to each displacement. Eight internal moments are as shown in Fig. 8.20. The problem has basic equilibrium
matrix $[B]$ of size $5 \times 8$ and hence the problem is three degree statically indeterminate as per IFM.

The nodal lumping mass matrix can be written by referring Eq. (8.27) as

$$
[M_{DNLM}] = \begin{bmatrix}
\delta_{23H} & \delta_{23V} & \delta_{23} & \delta_{34} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 \\
\text{Sym} & 0 & 0 & 0
\end{bmatrix}
$$

Following the procedure used in the previous example, IFM and stiffness method based frequencies are worked out and are given in Table 8.15. After substituting IFM based first natural frequency value in basic equation of IFM formulation all the internal moments are worked out which are depicted in Table 8.16.

<table>
<thead>
<tr>
<th>Natural Frequency (ω) rad/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>IFM</td>
</tr>
<tr>
<td>-----------------------------</td>
</tr>
<tr>
<td>52.9151</td>
</tr>
<tr>
<td>126.4912</td>
</tr>
</tbody>
</table>

**Table 8.16 Internal Moments and Nodal Displacements**

<table>
<thead>
<tr>
<th>Internal Moments (F)</th>
<th>Nodal Displacements (δ) × 10^{-3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>M_1 = 1.00</td>
<td>δ_{23H} = 0.125</td>
</tr>
<tr>
<td>M_2 = -0.75</td>
<td>Θ_{12} = -0.075</td>
</tr>
<tr>
<td>M_3 = -0.75</td>
<td>δ_{33V} = 0.0375</td>
</tr>
<tr>
<td>M_4 = 0.00</td>
<td>Θ_{12} = 0.00</td>
</tr>
<tr>
<td>M_5 = 0.00</td>
<td>Θ_{34} = -0.075</td>
</tr>
<tr>
<td>M_6 = 0.75</td>
<td></td>
</tr>
<tr>
<td>M_7 = 0.75</td>
<td></td>
</tr>
<tr>
<td>M_8 = -1.00</td>
<td></td>
</tr>
</tbody>
</table>

(ii) **Considering Lumping Mass Criteria (LM Case)**
Considering mass per unit length of each member as 1kN/m and after suitable transformation the global lumped mass matrix is obtained as

\[
[M_L] = \begin{bmatrix}
\delta_{33}\delta_{33} & \delta_{33} \theta_{33} & \delta_{33} \theta_{33} & \theta_{33} & \theta_{33} \\
1.0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Results obtained for natural frequencies, internal force and nodal displacements are reported in Table 8.17 and 8.18.

**Table 8.17 Natural Frequencies (LM Case)**

<table>
<thead>
<tr>
<th>Natural Frequency (ω) rad/sec</th>
<th>IFM</th>
<th>Stiffness Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>159.2038</td>
<td>159.2038</td>
<td></td>
</tr>
<tr>
<td>221.5624</td>
<td>221.5624</td>
<td></td>
</tr>
</tbody>
</table>

**Table 8.18 Internal Forces and Nodal Displacements**

<table>
<thead>
<tr>
<th>Internal Moments (M)</th>
<th>Nodal Displacements (δ) x 10^{-3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>M₁ = 1.00</td>
<td>δ₂₃H = 0.1323</td>
</tr>
<tr>
<td>M₂ = -0.6788</td>
<td>θ₁₂ = -0.0969</td>
</tr>
<tr>
<td>M₃ = -0.6788</td>
<td>δ₂₃V = 0.0924</td>
</tr>
<tr>
<td>M₄ = -0.5833</td>
<td>θ₁₂ = 0.00</td>
</tr>
<tr>
<td>M₅ = -0.5833</td>
<td>θ₃₄ = -0.0969</td>
</tr>
<tr>
<td>M₆ = 0.6768</td>
<td></td>
</tr>
<tr>
<td>M₇ = 0.6768</td>
<td></td>
</tr>
<tr>
<td>M₈ = -1.00</td>
<td></td>
</tr>
</tbody>
</table>

(iii) **Considering Consistent Mass Criteria (CM Case)**

The global consistent mass matrix by following the procedure outlined earlier is found as

\[
\begin{bmatrix}
\delta_{23H} & \theta_{12} & \delta_{23V} & \theta_{23} & \theta_{34}
\end{bmatrix}
\]
\[
[M_c] = \begin{bmatrix}
0.7428 & -0.0523 & 0 & 0 & -0.0261 \\
0.0119 & 0.01547 & -0.00178 & 0 & \\
0.7428 & 0 & -0.01547 & \\
Symm & 0.00476 & -0.00178 & \\
& & & & 0.00476 \\
\end{bmatrix}
\]

Results obtained considering consistent mass criteria are reported here in Tables 8.19 and 8.20.

**Table 7.19 Natural Frequencies for Plane Frame**

<table>
<thead>
<tr>
<th>Natural Frequency (ω) rad/sec</th>
<th>IFM</th>
<th>Stiffness Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>152.88</td>
<td>152.00</td>
<td></td>
</tr>
<tr>
<td>208.11</td>
<td>208.1</td>
<td></td>
</tr>
<tr>
<td>1152.13</td>
<td>1152.0</td>
<td></td>
</tr>
<tr>
<td>2420.01</td>
<td>2420.0</td>
<td></td>
</tr>
<tr>
<td>11100.54</td>
<td>11100.0</td>
<td></td>
</tr>
</tbody>
</table>

**Table 7.20 Internal Moments and Nodal Displacements**

<table>
<thead>
<tr>
<th>Internal Moments (M)</th>
<th>Nodal Displacements (δ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>M_1 = -0.5159</td>
<td>( \delta_{23H} = -0.0001 )</td>
</tr>
<tr>
<td>M_2 = 0.2457</td>
<td>( \Theta_{12} = 0.0001 )</td>
</tr>
<tr>
<td>M_3 = 0.3218</td>
<td>( \delta_{23V} = -0.0001 )</td>
</tr>
<tr>
<td>M_4 = 1.00</td>
<td>( \Theta_{12} = 0.000 )</td>
</tr>
<tr>
<td>M_5 = -1.00</td>
<td>( \Theta_{34} = 0.0001 )</td>
</tr>
<tr>
<td>M_6 = -0.3218</td>
<td></td>
</tr>
<tr>
<td>M_7 = -0.2457</td>
<td></td>
</tr>
<tr>
<td>M_8 = 0.5081</td>
<td></td>
</tr>
</tbody>
</table>

**8.5 DYNAMIC ANALYSIS OF A GRID STRUCTURE**

A grid structure with four members orthogonal to each other is shown in Fig. 8.21. Each member has flexural rigidity (EI) and torsional rigidity (GJ) equals to 1666.67 and 758.89 kN-m² respectively. It also has mass polar moment of inertia (Im) and polar moment of inertia (J) about centroidal axis as 1250 kN-m² and 9.8174 x 10⁻⁶ m⁴ respectively. The frequency analysis is to be carried out for direct nodal lumping of mass (Mo) equals to 10 kN-
Sec²/m. It is also analysed for lumped and consistent mass criteria by considering unit weight of each member as 1kN/m.

![Diagram of a grid structure](image)

**Fig. 8.21 Example of a Grid Structure**

\[ V_1 = (M_2 - M_1)/1 \]
\[ V_2 = -V_1 \]

![Free body diagram](image)

**Fig. 8.22 Free Body Diagram**

(i) **Considering Direct Nodal Lumping Mass (DNLM Case)**

The grid problem has total three possible nodal displacements at meeting point of all members i.e. Vertical displacement (δv), Bending rotation (θM) and Torsional rotation (θT). As there are total sixteen internal unknowns in terms of bending and torsional moments as shown in **Fig. 8.22**, the
problem will have equilibrium matrix \([B]\) of size 3 x16. Thus, the problem becomes thirteen degree statically indeterminate as per IFM.

The nodal lumping mass matrix can be written by referring Eq. (8.27) as

\[
[M_{DNL}] = \begin{bmatrix}
10 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix}
\]

Based on the IFM the natural frequency of grid structure is found as 59.076 rad/sec, whereas using stiffness method the natural frequency is found as 59.073 rad/sec. Results obtained based on IFM for internal moments and nodal displacement are reported in Table 8.21.

<table>
<thead>
<tr>
<th>Flexural Moments (M)</th>
<th>Torsional Moments (T)</th>
<th>Nodal Displacements (δ) x 10^-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>M₁ = 1.00</td>
<td>T₁ = 0.00</td>
<td>δᵥ = 2.72</td>
</tr>
<tr>
<td>M₂ = -0.7106</td>
<td>T₂ = -0.0659</td>
<td>θₘ = 0.052</td>
</tr>
<tr>
<td>M₃ = 0.7106</td>
<td>T₃ = 0.0659</td>
<td>θᵣ = 0.054</td>
</tr>
<tr>
<td>M₄ = -1.00</td>
<td>T₄ = 0.00</td>
<td></td>
</tr>
<tr>
<td>M₅ = -0.6118</td>
<td>T₅ = 0.0329</td>
<td></td>
</tr>
<tr>
<td>M₆ = 0.4671</td>
<td>T₆ = 0.00</td>
<td></td>
</tr>
<tr>
<td>M₇ = 0.6118</td>
<td>T₇ = -0.0329</td>
<td></td>
</tr>
<tr>
<td>M₈ = -0.4671</td>
<td>T₈ = 0.00</td>
<td></td>
</tr>
</tbody>
</table>

(ii) Considering Lumping Mass Criteria (LM Case)

Considering mass per unit length of each member as 1kN/m, based on lumping mass criteria the global lumped mass matrix can be calculated as

\[
[M_L] = \begin{bmatrix}
3.00 & 0 & 0 \\
0 & 1875.0 & 0 \\
0 & 0 & 1875.0 
\end{bmatrix}
\]

After deriving all the necessary matrices and after substituting in various equations the frequency values are found as reported in Table 8.22. After
substituting each IFM based frequency value in eigen equation, the relative internal forces in each member are calculated using Matlab software. Once the internal moment in each member are calculated the relative nodal displacements are obtained as reported in Table 8.23.

### Table 8.22 Natural Frequencies for Grid Structure

<table>
<thead>
<tr>
<th>Natural Frequency (ω) rad/sec</th>
<th>IFM</th>
<th>Stiffness Method [93]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1463</td>
<td>2.1412</td>
<td></td>
</tr>
<tr>
<td>2.4363</td>
<td>2.4343</td>
<td></td>
</tr>
<tr>
<td>122.4798</td>
<td>122.4768</td>
<td></td>
</tr>
</tbody>
</table>

### Table 8.23 Internal Moments and Nodal Displacements

<table>
<thead>
<tr>
<th>Flexural Moments (M)</th>
<th>Torsional Moments (T)</th>
<th>Nodal Displacements (δ) x 10^4</th>
</tr>
</thead>
<tbody>
<tr>
<td>M_1 = 1.00</td>
<td>T_1 = 0.00</td>
<td>δ_y = 1.00</td>
</tr>
<tr>
<td>M_2 = -1.00</td>
<td>T_2 = -0.00</td>
<td>θ_M = -0.0003</td>
</tr>
<tr>
<td>M_3 = 1.00</td>
<td>T_3 = 0.00</td>
<td>θ_T = 0.0003</td>
</tr>
<tr>
<td>M_4 = -1.00</td>
<td>T_4 = 0.00</td>
<td></td>
</tr>
<tr>
<td>M_5 = 0.234</td>
<td>T_5 = 0.00</td>
<td></td>
</tr>
<tr>
<td>M_6 = -0.234</td>
<td>T_6 = 0.00</td>
<td></td>
</tr>
<tr>
<td>M_7 = 0.234</td>
<td>T_7 = -0.00</td>
<td></td>
</tr>
<tr>
<td>M_8 = -0.234</td>
<td>T_8 = 0.00</td>
<td></td>
</tr>
</tbody>
</table>

(iii) **Considering Consistent Mass Criteria (CM Case)**

Based on consistent mass concept, the global consistent mass matrix is as follows

\[
[M_L] = \begin{bmatrix}
2.2286 & 0 & 0 \\
0 & 1250.0 & 0 \\
0 & 0 & 1250.0
\end{bmatrix}
\]

Frequency analysis provides results as per Table 8.24. Moments are calculated using Matlab based module and are reported in Table 8.25.

### Table 8.24 Natural Frequencies for Grid
<table>
<thead>
<tr>
<th>Natural Frequency (ω) rad/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>IFM</td>
</tr>
<tr>
<td>2.6288</td>
</tr>
<tr>
<td>2.949</td>
</tr>
<tr>
<td>142.11</td>
</tr>
</tbody>
</table>

**Table 8.25 Internal Moments and Nodal Displacements**

<table>
<thead>
<tr>
<th>Flexural Moments (M)</th>
<th>Torsional Moments (T)</th>
<th>Nodal Displacements (δ) x 10⁻⁴</th>
</tr>
</thead>
<tbody>
<tr>
<td>M₁ = 1.00</td>
<td>T₁ = 0.00</td>
<td>δᵥ = 0.9977</td>
</tr>
<tr>
<td>M₂ = -1.00</td>
<td>T₂ = -0.00</td>
<td>Θₘ = -0.0004</td>
</tr>
<tr>
<td>M₃ = 1.00</td>
<td>T₃ = 0.00</td>
<td>Θₜ = 0.0004</td>
</tr>
<tr>
<td>M₄ = -1.00</td>
<td>T₄ = 0.00</td>
<td></td>
</tr>
<tr>
<td>M₅ = 0.234</td>
<td>T₅ = 0.00</td>
<td></td>
</tr>
<tr>
<td>M₆ = -0.234</td>
<td>T₆ = 0.00</td>
<td></td>
</tr>
<tr>
<td>M₇ = 0.234</td>
<td>T₇ = -0.00</td>
<td></td>
</tr>
<tr>
<td>M₈ = -0.234</td>
<td>T₈ = 0.00</td>
<td></td>
</tr>
</tbody>
</table>

**8.6 DISCUSSION OF RESULTS**

Using IFM for frequency analysis and getting force mode shapes is one of the unexplored area in vibration theory of structural mechanics. After substituting each frequency value the relative internal forces are worked out and then the relative displacements are worked out by substituting the corresponding internal forces into IFM based equations. Thus, by utilizing the concept of force mode shape a direct design is feasible for the structural members which preferably vibrate under controlled frequency constraints. Some of the important observations of this chapter are as follows:

- Eigen value analysis is carried out by attempting various types of beam problems with varying boundary condition. In all the cases fixed central point mass is considered to enable comparison with the standard cases. By keeping same span, flexural rigidity and vibrating mass for all the cases the overall behavior in terms of maxima/minima frequency values,
relative moments and nodal displacements are studied. For the same relative internal moment values the nodal displacements in simply supported beam are found to be higher compared to fixed one under first modal pattern. A propped cantilever beam which exhibits behavior between the two cases indicates approximately average values in moments and nodal displacement with non zero value of slope at centre of beam. A continuous beam with two equal span length having extreme ends as simply supported is also studied under same criteria. Using IFM based formulation two frequency values are worked out. In the first case, whenever the vibrating mass is on one side of the beam base line, the frequency value of propped cantilever beam and first modal frequency is found to be matching. Thus, a good agreement is found with respect to structural vibration theory. During the second vibratory motion when the both masses are on the opposite side of beam base line, the frequency value of simply supported beam case and the second modal frequency values are found equal. The values for internal moments and nodal displacements are also found to match with the standard theory of vibration.

- All the skeletal framed structures are studied under various types of mass calculations keeping all other properties identical such as: (1) Direct Nodal Lumping Mass (DNLM) where constant intensity of 10kN is lumped at necessary joints, (2) Lumped Mass (LM) is calculated as per its mass per unit length as 1kN/m for all the structural members and (3) Consistent Mass (CM) is calculated as per same mass per unit length. The frequency values for lumped mass criteria are higher than the frequency values calculated using consistent mass criteria. Relative values of internal moments and nodal displacement are also worked out for the first frequency values under LM and CM cases.

- Two pin jointed structures are solved for frequency analysis. One is having three members triangulated panel and other is of eleven
members having two rectangular panels. Both the structures are analysed for eigen values under all the three categories of mass. IFM based analysis for triangulated type panel truss structure is found to give frequency values higher than the conventional stiffness based eigen value approach. Second structure with rectangular panel behaves very rigidly in its own plane and hence requires large amount of inertial force to yield higher natural frequencies.

- For fixed footed portal frame two natural frequency values are found to be matching with the standard stiffness based eigen values corresponding to two lateral displacements for the DNLM and LM Cases. While considering consistent mass criteria, total five values of natural frequencies are calculated by including mass corresponding to rotation in the mass matrix. The relative values of moments and nodal displacements, for all the cases, are also verified for approximate shape of deflection pattern as per the frame vibration criteria as per sequential number of frequencies.

- In case of frequency analysis of unsymmetrical grid structure, all the values corresponding to three displacements are found matching with the stiffness based solutions.