CHAPTER TWO

SOME BITOPOLOGICAL SEPARATION AXIOMS USING SEMI-OPEN, v-OPEN AND g-OPEN SETS

This chapter is divided into three sections. In section one, by using semi-open sets, we introduce and study some new separation axioms in topological and bitopological spaces. Section two dealt with the application of v-open sets in bitopological spaces and section three dealt with some application of g-open sets in topological and bitopological spaces.

(1) Bitopological separation axioms using semi-open sets

In 1963, Levine [Lev1] introduced the concept of semi-open sets. Since then, a considerable number of papers discussing separation axioms, essentially by replacing open sets by semi-open sets, have appeared in the literature. For instance, Maheshwari and Prasad [MP1, MP2, MP3] introduced semi-T₀, semi-T₁, semi-T₂, s-regularity and s-regularity as a generalization of T₀, T₁, T₂, regularity and normality axioms respectively, and investigated their properties. Sharma [Shar] generalized the separation axiom P₁ of Malghan and Banchalli to P₅-axiom and P₁S-axiom using the concept of semi-closure.

The notion of semi-open sets was used by Maheshwari and Prasad [MP1, MP2, MP3] to introduce pairwise semi-T₀, pairwise semi-T₁, pairwise semi-T₂, pairwise s-regular and pairwise s-normal spaces. Bhamini [Bh] defined the notion of pairwise
irresolutely normal spaces which is a bitopological analogue of irresolutely normal spaces.

This section is dealt with the applications of semi-open sets to define some new separation axioms. In this section we introduce the bitopological analogue of $P_S$ and $P_{1S}$-spaces of Sharma [Shar] namely, pairwise $P_S$ and pairwise $P_{1S}$-spaces. Also replacing open sets by semi-open sets in $H_i$ axioms ($i = 0, 1, 2$), and $U_i$ axioms ($i = 0, 1$), of Csaszar [Csa], semi-$H_i$ spaces, ($i = 0, 1, 2$) and semi-$U_i$ spaces, ($i = 0, 1$) are introduced and studied and the bitopological analogues of these spaces are introduced.

DEFINITION 1.1 [Shan]: A space $X$ is known as $R_0$-space if $x \not\in \text{cl} \{ y \}$ implies that $y \not\in \text{cl} \{ x \}$.

DEFINITION 1.2 [MB]: A space $X$ is known as $P_1$-space if $x \not\in \delta\text{cl} \{ y \}$ implies $y \not\in \text{cl} \{ x \}$.

DEFINITION 1.3 [Yan]: A space $X$ is said to be $R_1$-space if for every pair of distinct points $x, y$ of $X$, with $\text{cl} \{ x \} \neq \text{cl} \{ y \}$ there exists an open set $U$ and an open set $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

DEFINITION 1.5 [Do]: A space $X$ is said to be strongly $s$-regular if for each semi-closed subset $A$ of $X$ and $x \not\in A$, there exist disjoint semi-open sets $U$ and $V$ such that $x \in U$ and $A \subseteq V$.

DEFINITION 1.6 [Bh]: A space $X$ is said to be irresolutely normal if for any two disjoint semi-closed sets $A$ and $B$, there exist disjoint semi-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

DEFINITION 1.7: A space $X$ is said to be a $P_S$-space [Shar] if $x \not\in \text{cl} \{ y \}$ implies that
y \not\in \text{scl} \{x\}.

**DEFINITION 1.8:** A space X is said to be a **$\text{P}_{1S}$-space** [Shar] if $x \not\in \delta\text{cl} \{y\}$ implies that $y \not\in \text{scl} \{x\}$.

**DEFINITION 1.9:** A space $(X, T_1, T_2)$ is said to be a **pairwise semi-$R_0$-space** [Nou] if for every $T_i$-semi-open set $G$, $x \in G$ implies that $T_j\text{-scl} \{x\} \subseteq G$; $i \neq j$, $i, j = 1, 2$.

**DEFINITION 1.10:** A space $(X, T_1, T_2)$ is said to be a **pairwise semi-$R_1$-space** [Nou] if for any two distinct points $x, y$ of $X$ such that $x \not\in T_i\text{-scl} \{y\}$, there exists a $T_i$-semi open set $U$ and a $T_j$-semi open set $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$; $i \neq j$, $i, j = 1, 2$.

1. **Pairwise $P_S$-Spaces:**

   Here we introduce the bitopological analogue of $P_S$-spaces.

   **DEFINITION 1.1.1:** A bitopological space $(X, T_1, T_2)$ is said to be a **pairwise $P_S$-space** if $x \not\in T_i\text{-cl} \{y\} \Rightarrow y \not\in T_j\text{-scl} \{x\}$; $i \neq j$, $i, j = 1, 2$. Clearly, every pairwise $R_0$-space is pairwise $P_S$-space but not conversely.

   **EXAMPLE 1.1.2:** Let $X = \{a, b, c\}$, $T_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$,

   $T_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. Then $(X, T_1, T_2)$ is a pairwise $P_S$-space.

   **THEOREM 1.1.3:** A space $X$ is a pairwise $P_S$-space if and only if for each $T_i$-open set $S$ and each $x \in S$, $T_j\text{-scl} \{x\} \subseteq S$. $i \neq j$, $i, j = 1, 2$.

   **PROOF:** Let $S$ is a $T_i$-open set containing $x$ and let $y \not\in S$. Then $x \not\in T_i\text{-cl} \{y\}$. Since $X$ is a pairwise $P_S$-space, we get $y \not\in T_j\text{-scl} \{x\}$. Hence $T_j\text{-scl} \{x\} \subseteq S$.

   Conversely, Let $x \not\in T_i\text{-cl}\{y\}$. So there is a $T_i$-open set $S$ (say) containing $x$ which has empty intersection with $\{y\}$, i.e. $y \not\in S$. By hypothesis, $T_j\text{-scl} \{x\} \subseteq S$ and so $y \not\in T_j\text{-scl} \{x\}$. Hence $X$ is a pairwise $P_S$-space.
THEOREM 1.1.4: For a bitopological space $X$ the following are equivalent.

(a) $X$ is a pairwise $P_S$-space.

(b) For each $x \in X$, $T_j$-$scl \{x\} \subseteq T_i$-$ker \{x\}$.

(c) If $F$ is a $T_i$-closed set in $X$, then $F$ is the intersection of all the $T_j$-semi-open sets containing $F$.

(d) If $S$ is a $T_i$-open set in $X$, then $S$ the union of all $T_j$-semi-closed sets in $X$ contained in $S$.

(e) For $A \neq \emptyset$, and a $T_i$-open set $S$ in $X$ such that $S \cap A \neq \emptyset$, there exists a $T_j$-semi-closed set $F \subseteq S$ such that $F \cap A \neq \emptyset$.

(f) For any $T_i$-closed set $F$ in $X$ and $x \notin F$, $T_j$-$scl \{x\} \cap F = \emptyset$.

PROOF (a) → (b): Let $y \in T_j$-$scl \{x\}$ and $S$ be a $T_i$-open set containing $x$. Since $X$ is a pairwise $P_S$-space. Therefore by theorem 1.1.3, $T_j$-$scl \{x\} \subseteq S$ and thus $y \in S$. Therefore, $x \in T_i$-$cl \{y\}$, i.e. $y \in T_i$-$ker \{x\}$. Hence $T_j$-$scl \{x\} \subseteq T_i$-$ker \{x\}$.

(b) → (c): Let $F$ be a $T_i$-closed set. Let $x \notin F$. Then $X - F$ is a $T_i$-open set containing $x$. If $y \in T_j$-$scl \{x\}$, then from (b), $y \in T_i$-$ker \{x\}$. Therefore, $x \in T_i$-$cl \{y\}$. So $y \in X - F$. Hence $T_j$-$scl \{x\} \subseteq X - F$, which implies, $F \subseteq X - T_j$-$scl \{x\}$. Therefore, $X - T_j$-$scl \{x\}$ is a $T_j$-semi-open set that does not contain $x$. Thus $x$ does not belong to the intersection of all the $T_j$-semi-open sets, which contain $F$. Hence (c) holds.

(c) → (d): By taking complements of (c), we get (d).

(d) → (e): Since $S \cap A \neq \emptyset$, let $x \in S \cap A$. Then $x \in T_i$-open set $S$. Therefore, from (d), $S$ is the union of all the $T_j$-semi-closed sets contained in $S$. Hence there exists a $T_j$-semi-closed set $F$ (say) such that $x \in F \subseteq S$, which implies that $F \cap A \neq \emptyset$. Thus (e) holds.
(e) $\rightarrow$ (f): Let $F$ be a $T_i$-closed set in $X$ and $x \notin F$. Then $X - F$ is a $T_i$-open set in $X$ such that $(X - F) \cap \{x\} \neq \emptyset$. Therefore, from (e), there is a $T_j$-semi-closed set $K$ such that $K \subseteq X - F$ and $K \cap \{x\} \neq \emptyset$. So $T_j\text{-scl}\{x\} \subseteq X - F$. Hence $T_j\text{-scl}\{x\} \cap F = \emptyset$. Thus (f) is true.

(f) $\rightarrow$ (a): Let $S$ be a $T_i$-open set containing $x$. Then, from (f), we have $(X - S) \cap T_j\text{-scl}\{x\} = \emptyset$ and hence $T_j\text{-scl}\{x\} \subseteq S$. Thus by theorem 1.1.3, $X$ is a pairwise $P_S$-space.

THEOREM 1.1.5: A pairwise $P_S$-space $X$ is pairwise semi-$T_1$ if it is pairwise $T_0$-space.

PROOF: Let $x \neq y \in$ pairwise $T_0$-space. Then there exists a $T_i$-open set $G$ containing $x$ but not $y$. Since $X$ is a pairwise $P_S$-space by theorem 1.1.3, $T_j\text{-scl}\{x\} \subseteq G$. Also $y \notin T_j\text{-scl}\{x\}$. Take $H = X - T_j\text{-scl}\{x\}$, which is a semi-open set containing $y$ but not $x$. Also every open set is semi-open. Thus semi-open sets $G$ and $H$ satisfy the requirement of pairwise semi-$T_1$.

2. Pairwise $P_{1S}$-Spaces:

Here we introduced the bitopological analogue of $P_{1S}$-spaces.

DEFINITION 1.2.1: A bitopological space $(X, T_1, T_2)$ is said to be a pairwise $P_{1S}$-space if $x \notin T_i\text{-}\delta\text{cl}\{y\} \Rightarrow y \notin T_j\text{-scl}\{x\}; i \neq j, i, j = 1, 2$.

EXAMPLE 1.2.2: Let $X = \{a, b, c\}$, $T_1 = \{\phi, \{a\}, X\}$, $T_2 = \{\phi, \{b\}, X\}$. Then the space $(X, T_1, T_2)$ is a pairwise $P_{1S}$-space but not a pairwise $P_S$-space.

THEOREM 1.2.3: A bitopological space $(X, T_1, T_2)$ is a pairwise $P_{1S}$-space if and only if for each $T_i$-regularly open set $S$ and each $x \in T_i\text{-scl}\{x\} \subseteq S$.

PROOF: Let $x \in T_i$-regularly open set $S$ in $X$ and $y \notin S$. Then $x \notin T_i\text{-}\delta\text{cl}\{y\}$. Since $X$ is a pairwise $P_{1S}$-space, therefore $y \notin T_j\text{-scl}\{x\}$. Hence $T_j\text{-scl}\{x\} \subseteq S$. 

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Conversely, let \( x \notin T_i\text{-}cl \{y\} \). So there is a \( T_i\)-regularly open set \( S \) (say) containing \( x \) but not \( y \). By hypothesis, \( T_j\text{-}scl \{x\} \subseteq \overline{S} \), therefore, \( y \notin T_j\text{-}scl \{x\} \).

Hence \( X \) is a pairwise \( P_{1S} \)-space.

THEOREM 1.2.4: For a space \( X \), the following are equivalent:

(a) \( X \) is a pairwise \( P_{1S} \)-space.

(b) For each \( x \in X \), \( T_j\text{-}scl \{x\} \subseteq T_i\text{-}δker \{x\} ; i \neq j, i, j = 1, 2. \)

(c) If \( F \) is a \( T_i\)-regularly closed set in \( X \), then \( F \) is the intersection of all the \( T_j\)-semi-open sets containing \( F \); \( i \neq j, i, j = 1, 2. \)

(d) If \( S \) is a \( T_i\)-regularly open set in \( X \), then \( S \) is the union of all the \( T_j\)-semi-closed sets contained in \( S \); \( i \neq j, i, j = 1, 2. \)

(e) For \( A \neq \emptyset \), and a \( T_i\)-regularly open set \( S \) in \( X \) such that \( S \cap A \neq \emptyset \), there exists a \( T_j\)-semi closed set \( F \subseteq S \) such that \( F \cap A \neq \emptyset ; i \neq j, i, j = 1, 2. \)

(f) For any \( T_i\)-regularly closed set \( F \) in \( X \) and \( x \notin F \), \( T_j\text{-}scl \{x\} \cap F = \emptyset ; i \neq j, i, j = 1, 2. \)

PROOF (a) \( \rightarrow \) (b): Let \( y \in T_j\text{-}scl \{x\} \) and \( S \) be a \( T_i\)-regularly open set containing \( x \).

Since \( X \) is a pairwise \( P_{1S} \)-space, therefore by theorem 1.2.3, \( T_j\text{-}scl \{x\} \subseteq S \) and thus \( y \in S \). Therefore \( x \in T_i\text{-}δcl \{y\} \), i.e., \( y \in T_i\text{-}δker \{x\} \). Hence, \( T_j\text{-}scl \{x\} \subseteq T_i\text{-}δker \{x\} \).

(b) \( \rightarrow \) (c): Let \( F \) be a \( T_i\)-regularly closed set. Let \( x \notin F \). Then \( X - F \) is a \( T_i\)-regularly open set containing \( x \). If \( y \in T_j\text{-}scl \{x\} \), then from (b), \( y \in T_i\text{-}δ\text{-}ker \{x\} \) and therefore \( x \in T_i\text{-}δcl \{y\} \). So \( y \in X - F \). Hence, \( T_j\text{-}scl \{x\} \subseteq X - F \), i.e., \( F \subseteq (X - T_j\text{-}scl \{x\}) \).

Since \( X - T_j\text{-}scl \{x\} \) is a \( T_j\)-semi-open set to which \( x \) does not belong. Therefore \( x \) does not belong to the intersection of all the \( T_j\)-semi-open sets containing \( F \). Thus,
intersection of all the $T_j$-semi-open sets is a subset of $F$. Hence, $F$ is the intersection of all the $T_j$-semi-open sets. Hence, (c) holds.

(c) $\rightarrow$ (d): By taking complements of (c), we get (d).

(d) $\rightarrow$ (e): Since $S \cap A \neq \emptyset$, let $x \in S \cap A$. Also, from (d), $S$ is the union of all the $T_j$-semi-closed sets contained in $S$. Thus there is a $T_j$-semi closed set $F$ (say) such that $x \in F \subseteq S$, which implies that $F \cap S \neq \emptyset$. Thus, (e) holds.

(e) $\rightarrow$ (f): Let $F$ be a $T_i$-regularly closed set in $X$ and $x \not\in F$. Then $X - F$ is a $T_i$-regularly open set in $X$ such that $(X - F) \cap \{x\} \neq \emptyset$. Therefore, from (e), there is a $T_j$-semi-closed set $K$ such that $K \subseteq X - F$ and $K \cap \{x\} \neq \emptyset$. But $T_j$-scl $\{x\}$ is the smallest $T_j$-semi-closed set in $X$. So $T_j$-scl $\{x\} \subseteq X - F$. Hence $T_j$-scl $\{x\} \cap F = \emptyset$. Thus (f) is true.

(f) $\rightarrow$ (a): Let $S$ be a $T_i$-regularly open set containing $x$. Then $x \not\in T_i$-regularly closed set $X - S$. So from (f), we have $(X - S) \cap T_j$-scl $\{x\} = \emptyset$ and hence $T_j$-scl $\{x\} \subseteq S$. Thus by theorem 1.2.3, $X$ is a pairwise $P_{1S}$-space.

THEOREM 1.2.5: A pairwise $P_{1S}$-space $X$ is pairwise semi-$T_1$ if it is a pairwise $rT_0$-space.

PROOF: Let $x \neq y \in$ pairwise $rT_0$-space. Then without loss of generality, assume that there is a $T_i$-regularly open set $G$ containing $x$ but not $y$. Since $X$ is pairwise $P_{1S}$, therefore by theorem 1.2.3, $T_j$-scl $\{x\} \subseteq G$. Hence $y \not\in T_j$-scl $\{x\}$, then there exists a $T_j$-semi-open set $H$ (say) containing $y$ but not $x$. Thus $G$ and $H$ are the semi-open sets satisfying the requirements of semi-$T_1$-axiom. Hence $X$ is a pairwise semi-$T_1$-space.

3. Semi-$H_i$-Spaces ($i = 0, 1, 2$) and Semi-$U_i$-Spaces ($i = 0, 1$):
Replacing open sets by semi-open sets and ‘cl’ by ‘scl’ in \( H_i \)-spaces, \((i = 0, 1, 2)\) and \( U_i \)-Spaces \((i = 0, 1)\) of Csaszar [Csa], we introduce semi-\( H_i \)-spaces, \((i = 0, 1, 2)\) and semi-\( U_i \)-Spaces \((i = 0, 1)\).

**DEFINITION 1.3.1:** A space \( X \) is said to be **semi-\( H_0 \)** if for every pair of points \( x \) and \( y \) such that \( x \notin \text{scl} \{y\} \) there exists a semi-open set \( U \) and a semi-open set \( V \) such that \( x \in U, y \in V, U \cap V = \phi \).

**DEFINITION 1.3.2:** A space \( X \) is said to be **semi-\( H_1 \)** if for every pair of points \( x \) and \( y \) such that \( \text{scl} \{x\} \cap \text{scl} \{y\} = \phi \), there exists a semi-open set \( U \) and a semi-open set \( V \) such that \( x \in U, y \in V, U \cap V = \phi \).

**DEFINITION 1.3.3:** A space \( X \) is said to be **semi-\( H_2 \)** if for every semi-closed set \( A \) and a point \( x \) such that \( \text{scl} \{x\} \cap A = \phi \), there exists a semi-open set \( U \) and a semi-open set \( V \) such that \( x \in U, A \subseteq V, U \cap V = \phi \).

**DEFINITION 1.3.4:** A space \( X \) is said to be **semi-\( U_0 \)** if for every pair of points \( x \) and \( y \) such that \( x \notin \text{scl} \{y\} \), there exists a semi-open set \( U \) and a semi-open set \( V \) such that \( x \in U, y \in V, \text{scl} U \cap \text{scl} V = \phi \).

**DEFINITION 1.3.5:** A space \( X \) is said to be **semi-\( U_1 \)** if for every pair of points \( x \) and \( y \) such that \( \text{scl} \{x\} \cap \text{scl} \{y\} = \phi \), there exists a semi-open set \( U \) and a semi-open set \( V \) such that \( x \in V, y \in U, \text{scl} U \cap \text{scl} V = \phi \).

**THEOREM 1.3.6:** Every irresolutely normal space is semi-\( H_2 \).

**PROOF:** Let \( X \) is irresolutely normal space. Let \( x \in X \) and let \( A \) be a semi-closed set such that \( \text{scl} \{x\} \cap A = \phi \). By irresolutely-normality of \( X \), there exists a semi-open set \( U \) and a semi-open set \( V \) such that \( \text{scl} \{x\} \subseteq V, A \subseteq U, U \cap V = \phi \). Hence \( x \in V, A \subseteq U, U \cap V = \phi \). Hence, \( X \) is semi-\( H_2 \).
THEOREM 1.3.7: Every semi-H₂ space is semi-H₁.

PROOF: Let X is semi-H₂ space. Let x and y be two distinct points of X such that scl \( \{x\} \cap \text{scl} \{y\} = \phi \). Since X is semi-H₂, therefore there exists a semi-open set U and a semi-open set V such that \( x \in V, \text{scl} \{y\} \subseteq U, U \cap V = \phi \). Thus \( x \in V, y \in U, U \cap V = \phi \). Hence, X is semi-H₁.

THEOREM 1.3.8: Every semi-H₀ space is semi-H₁.

PROOF: Let X is semi-H₀ space. Let x and y be two distinct points of X such that scl \( \{x\} \cap \text{scl} \{y\} = \phi \). Hence \( x \not\in \text{scl} \{y\} \). Since X is semi-H₂ therefore there exists a semi-open set U and a semi-open set V such that \( x \in V, y \in U, U \cap V = \phi \). Hence, X is semi-H₁.

THEOREM 1.3.9: Every semi-R₁ space is semi-H₀.

PROOF: Let X is semi-R₁ space. Let x \( \not\in \text{scl} \{y\} \). Then \( \text{scl} \{x\} \neq \text{scl} \{y\} \). Thus there exists a semi-open set U and a semi-open set V such that \( x \in U, y \in V, U \cap V = \phi \). Hence, X is semi-H₀.

THEOREM 1.3.10: Every semi-R₁ space is semi-H₁.

PROOF: Follows in view of theorem 1.3.8 and 1.3.9.

THEOREM 1.3.11: Every semi-H₀ space is semi-R₀.

PROOF: Let X is semi-H₀ space. Let x \( \in G \in \text{SO}(T) \) and let y \( \in X - G \). Then x \( \not\in \text{scl} \{y\} \). Since X is semi-H₀, there exists a semi-open set U and a semi-open set V such that \( x \in U, y \in V, U \cap V = \phi \). Thus \( \{x\} \cap V = \phi \) so that y \( \not\in \text{scl} \{x\} \). Hence X \( - G \subseteq X - \text{scl} \{x\} \) or \( \text{scl} \{x\} \subseteq G \). Thus X is semi-R₀.

THEOREM 1.3.12: Every strongly s-regular space is semi-H₂.

PROOF: Let X is strongly s-regular space. Let x \( \in X \) and let A be a semi-closed
subset of $X$ such that $\text{scl} \{x\} \cap A = \emptyset$. Then $x \not\in A$. By strongly s-regularity of $X$, there exists a semi-open set $U$ and a semi-open set $V$ such that $x \in V$, $A \subseteq U$, $U \cap V = \emptyset$. Hence, $X$ is semi-$H_2$.

**THEOREM 1.3.13**: A space is semi-$T_2$ if and only if it is semi-$T_0$ and semi-$H_0$.

**PROOF**: Let $X$ is semi-$T_2$ space. Clearly, $X$ is semi-$T_0$. Let $x, y \in X$ such that $x \not\in \text{scl} \{y\}$. Then $x \neq y$, since $X$ is semi-$T_2$, there exists a semi-open set $U$ and a semi-open set $V$, such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Hence $X$ is semi-$H_0$.

Conversely, let $X$ is semi-$T_0$ and semi-$H_0$. Let $x, y$ be two distinct points of $X$. Since $X$ is semi-$T_0$, there exists a semi-open set $U$ or a semi-open set $V$ such that $x \in U$, $y \not\in V$ or $x \not\in U$, $y \in V$. Thus $x \not\in \text{scl} \{y\}$ or $y \not\in \text{scl} \{x\}$. Let $x \not\in \text{scl} \{y\}$. Since the space is semi-$H_0$, there exists a semi-open set $P$ and a semi-open set $Q$ such that $x \in P$, $y \in Q$, $P \cap Q = \emptyset$. The result follows similarly in case $y \not\in \text{scl} \{x\}$. Hence $X$ is semi-$T_2$.

**THEOREM 1.3.14**: A space is semi-$T_2$ if and only if it is semi-$T_1$ and semi-$H_1$.

**PROOF**: Let $X$ is semi-$T_2$ space. Clearly, $X$ is semi-$T_1$. Let $x, y \in X$ such that $\text{scl} \{x\} \cap \text{scl} \{y\} = \emptyset$. Then $x, y$ are distinct points of $X$ so that there exists a semi-open set $U$ and a semi-open set $V$, such that $x \in V$, $y \in U$, $U \cap V = \emptyset$. Hence, $X$ is semi-$H_1$.

Conversely, let $X$ is semi-$T_1$ and semi-$H_1$. Let $x, y$ be two distinct points of $X$. Since $X$ is semi-$T_1$, therefore $\{x\}$ and $\{y\}$ are semi-closed sets. Hence $\text{scl} \{x\} \cap \text{scl} \{y\} = \emptyset$. Since $X$ is semi-$H_1$, there exists a semi-open set $U$ and a semi-open set $V$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Hence $X$ is semi-$T_2$.

**THEOREM 1.3.15**: Every strongly s-regular space is semi-$U_0$. 

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PROOF: Let X is strongly s-regular space. Let x, y ∈ X such that x ∉ scl {y}. Since the space is strongly s-regular, there exists a semi-open set U and a semi-open set V such that x ∈ U, scl {y} ⊆ V, scl U ∩ scl V = φ. Hence x ∈ U, y ∈ V, scl U ∩ scl V = φ and thus the space is semi-U₀.

THEOREM 1.3.16: Every semi-U₀ space is semi-H₀.

PROOF: Let X is semi-U₀ space. Let x, y ∈ X such that x ∉ scl {y}. Since X is semi-U₀, there exists a semi-open set U and a semi-open set V such that x ∈ U, y ∈ V, scl U ∩ scl V = φ. Hence x ∈ U, y ∈ V, U ∩ V = φ and thus X is semi-H₀.

THEOREM 1.3.17: Every semi-U₁ space is semi-H₁.

PROOF: Let X is semi-U₁ space. Let x, y ∈ X such that scl {x} ∩ scl {y} = φ. Since X is semi-U₁, there exists a semi-open set U and a semi-open set V such that x ∈ U, y ∈ V, scl U ∩ scl V = φ. Hence x ∈ V, y ∈ U, U ∩ V = φ and thus X is semi-H₁.

THEOREM 1.3.18: Every irresolutely normal space is semi-U₁.

PROOF: Let X is irresolutely normal space. Let x, y ∈ X such that scl {x} ∩ scl {y} = φ. Since X is irresolutely normal, there exists a semi-open set U and a semi-open set V such that scl {x} ⊆ V, scl {y} ⊆ U, scl U ∩ scl V = φ. Hence x ∈ V, y ∈ U, scl U ∩ scl V = φ. Hence, X is semi-U₁.

The results of this can be summarized in the following diagram:

```
semi-U₁
  ↓
irresolutely-normal
  ↓
semi-H₂
  ↓
strongly s-regular
  ↓
semi-H₁
```

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semi-U_0 \rightarrow \text{semi-H}_0 \rightarrow \text{semi-R}_0 \\
\text{semi-R}_1

semi-T_2 = semi-T_0 + semi-H_0
semi-T_2 = semi-T_1 + semi-H_1

4. **Pairwise semi-H_i-Spaces (i = 0, 1, 2) and Pairwise semi-U_i-Spaces (i = 0, 1):**

Here we introduce the bitopological analogues of semi-H_i-spaces (i = 0, 1, 2) and semi-U_i-spaces (i = 0, 1).

**DEFINITION 1.4.1:** A bitopological space \((X, T_1, T_2)\) is said to be **pairwise semi-H_0**
if for every pair of points \(x\) and \(y\) such that \(x \not\in T_i\text{-scl}\{y\}\) there exists a \(T_j\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\) such that \(x \in U, y \in V, U \cap V = \phi; i \neq j, i, j = 1, 2.\)

**DEFINITION 1.4.2:** A bitopological space \((X, T_1, T_2)\) is said to be **pairwise semi-H_1**
if for every pair of points \(x\) and \(y\) such that \(T_i\text{-scl}\{x\} \cap T_j\text{-scl}\{y\} = \phi\), there exists a \(T_i\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\) such that \(x \in U, y \in V, U \cap V = \phi; i \neq j, i, j = 1, 2.\)

**DEFINITION 1.4.3:** A bitopological space \((X, T_1, T_2)\) is said to be **pairwise semi-H_2**
if for every \(T_j\)-semi-closed set \(A\) and a point \(x\) such that \(T_i\text{-scl}\{x\} \cap A = \phi\), there exists a \(T_i\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\) such that \(x \in U, A \subseteq V, U \cap V = \phi; i \neq j, i, j = 1, 2.\)

**DEFINITION 1.4.4:** A bitopological space \((X, T_1, T_2)\) is said to be **pairwise semi-U_0**
if for every pair of points \(x\) and \(y\) such that \(x \not\in T_i\text{-scl}\{y\}\), there exists a \(T_i\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\) such that \(x \in U, y \in V, T_j\text{-scl}\ U \cap T_i\text{-scl}\ V = \phi; i \neq j, i, j = 1, 2.\)
DEFINITION 1.4.5: A bitopological space \((X, T_1, T_2)\) is said to be **pairwise semi-U_1** if for every pair of points \(x\) and \(y\) such that \(T_1\)-scl \(\{x\}\) \(\cap\) \(T_2\)-scl \(\{y\}\) = \(\emptyset\), there exists a \(T_1\)-semi-open set \(U\) and a \(T_2\)-semi-open set \(V\) such that \(x \in V\), \(y \in U\), \(T_1\)-scl \(U \cap T_1\)-scl \(V = \emptyset\).

THEOREM 1.4.6: Every pairwise irresolutely-normal space is pairwise semi-H_2.

PROOF: Let \((X, T_1, T_2)\) is pairwise irresolutely-normal space. Let \(x \in X\) and let A be a \(T_1\)-semi-closed set such that \(T_1\)-scl \(\{x\}\) \(\cap\) A = \(\emptyset\). By pairwise irresolutely-normality of \((X, T_1, T_2)\), there exists a \(T_1\)-semi-open set \(U\) and a \(T_1\)-semi-open set \(V\) such that \(T_1\)-scl \(\{x\}\) \(\subseteq\) \(V\), A \(\subseteq\) \(U\), \(U \cap V = \emptyset\). Therefore \(x \in V\), A \(\subseteq\) \(U\), \(U \cap V = \emptyset\). Hence \(X, T_1, T_2\) is pairwise semi-H_2.

THEOREM 1.4.7: Every pairwise semi-H_2 space is pairwise semi-H_1.

PROOF: Let \((X, T_1, T_2)\) is pairwise semi-H_2 space. Let \(x\) and \(y\) be two distinct points of \(X\) such that \(T_1\)-scl \(\{x\}\) \(\cap\) \(T_1\)-scl \(\{y\}\) = \(\emptyset\). Since \(X\) is pairwise semi-H_2, therefore, there exists a \(T_1\)-semi-open set \(U\) and a \(T_1\)-semi-open set \(V\) such that \(x \in V\), \(T_1\)-scl \(\{y\}\) \(\subseteq\) \(U\), \(U \cap V = \emptyset\). Thus \(x \in V\), \(y \in U\), \(U \cap V = \emptyset\). Hence, \((X, T_1, T_2)\) is pairwise semi-H_1.

THEOREM 1.4.8: Every pairwise semi-H_0 space is pairwise semi-H_1.

PROOF: Let \((X, T_1, T_2)\) is pairwise semi-H_0 space. Let \(x\) and \(y\) be two distinct points of \(X\) such that \(T_1\)-scl \(\{x\}\) \(\cap\) \(T_1\)-scl \(\{y\}\) = \(\emptyset\). Hence \(x \notin T_1\)-scl \(\{y\}\). Since \(X\) is pairwise semi-H_0, therefore there exists a \(T_1\)-semi-open set \(U\) and a \(T_1\)-semi-open set \(V\) such that \(x \in V\), \(y \in U\), \(U \cap V = \emptyset\). Hence, \((X, T_1, T_2)\) is pairwise semi-H_1.

THEOREM 1.4.9: Every pairwise semi-R_1 space is pairwise semi-H_0.
PROOF: Let \((X, T_1, T_2)\) is pairwise semi-R_1 space. Let \(x \notin T_1\text{-scl }\{y\}\). Then \(T_j\text{-scl}\{x\} \neq T_i\text{-scl}\{y\}\). Thus there exists a \(T_i\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\) such that \(x \in U, y \in V, U \cap V = \emptyset\). Hence, \((X, T_1, T_2)\) is pairwise semi-H_0.

THEOREM 1.4.10: Every pairwise semi-R_1 space is pairwise semi-H_1.
PROOF: Follows in view of theorem 1.4.8 and 1.4.9.

THEOREM 1.4.11: Every pairwise semi-H_0 space is pairwise semi-R_0.
PROOF: Let \((X, T_1, T_2)\) is pairwise semi-H_0 space. Let \(x \in G \in SO (T_1)\) and let \(y \in X \setminus G\). Then \(x \notin T_i\text{-scl }\{y\}\). Since \(X\) is pairwise semi-H_0, there exists a \(T_i\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\) such that \(x \in U, y \in V, U \cap V = \emptyset\). Thus \(\{x\} \cap V = \emptyset\) so that \(y \notin T_j\text{-scl }\{x\}\). Hence, \(X \setminus G \subseteq X \setminus T_j\text{-scl }\{x\}\) or \(T_j\text{-scl }\{x\} \subseteq G\). Thus \(X\) is pairwise semi-R_0.

THEOREM 1.4.12: Every pairwise strongly s-regular space is pairwise semi-H_2.
PROOF: Let \((X, T_1, T_2)\) is pairwise strongly s-regular space. Let \(x \in X\) and let \(A\) be a \(T_j\)-semi-closed subset of \(X\) such that \(T_i\text{-scl }\{x\} \cap A = \emptyset\). Then \(x \notin A\). By pairwise strongly s-regularity of \((X, T_1, T_2)\), there exists a \(T_i\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\) such that \(x \in V, A \subseteq U, U \cap V = \emptyset\). Hence \((X, T_1, T_2)\) is pairwise semi-H_2.

THEOREM 1.4.13: A space \((X, T_1, T_2)\) is pairwise semi-T_2 if and only if it is pairwise semi-T_0 and pairwise semi-H_0.

PROOF: Let \((X, T_1, T_2)\) is pairwise semi-T_2 space. Clearly \(X\) is pairwise semi-T_0. Let \(x, y \in X\) such that \(x \notin T_i\text{-scl }\{y\}\). Then \(x \neq y\), since \(X\) is pairwise semi-T_2, there exists a \(T_i\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\), such that \(x \in U, y \in V, U \cap V = \emptyset\). Hence \(X\) is pairwise semi-H_0.
Conversely, let \((X, T_1, T_2)\) is pairwise semi-\(T_0\) and pairwise semi-H_0. Let \(x, y\) be two distinct points of \(X\). Since \(X\) is pairwise semi-\(T_0\), there exists a \(T_i\)-semi-open set \(U\) or a \(T_j\)-semi-open set \(V\) such that \(x \in U\), \(y \notin V\) or \(x \notin U\), \(y \in V\). Thus \(x \notin T_i\)-scl \(\{y\}\) or \(y \notin T_j\)-scl \(\{x\}\). Since the space is pairwise semi-H_0, there exists a \(T_i\)-semi-open set \(P\) and a \(T_j\)-semi-open set \(Q\) such that \(x \in P\), \(y \in Q\), \(P \cap Q = \emptyset\). The result follows similarly in case \(y \notin T_j\)-scl \(\{x\}\). Hence \(X\) is pairwise semi-\(T_2\).

**THEOREM 1.4.15:** A space \((X, T_1, T_2)\) is pairwise semi-\(T_2\) if and only if it is pairwise semi-\(T_1\) pairwise semi-H_1.

**PROOF:** Let \((X, T_1, T_2)\) is pairwise semi-\(T_2\) space. Clearly \(X\) is pairwise semi-\(T_1\). Let \(x, y \in X\) such that \(T_i\)-scl \(\{x\}\) \(\cap\) \(T_j\)-scl \(\{y\}\) = \(\emptyset\). Then \(x, y\) are distinct points of \(X\) so that there exists a \(T_i\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\), such that \(x \in V\), \(y \in U\), \(U \cap V = \emptyset\). Hence \(X\) is pairwise semi-H_1.

Conversely, let \((X, T_1, T_2)\) is pairwise semi-\(T_1\) and pairwise semi-H_1. Let \(x, y\) be two distinct points of \(X\). Since \(X\) is pairwise semi-\(T_1\), therefore \(\{x\}\) and \(\{y\}\) are bi-semi-closed sets. Hence \(T_j\)-scl \(\{x\}\) \(\cap\) \(T_i\)-scl \(\{y\}\) = \(\emptyset\). Since \(X\) is pairwise semi-H_1, there exists a \(T_i\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\) such that \(x \in U\), \(y \in V\), \(U \cap V = \emptyset\). Hence \(X\) is pairwise semi-\(T_2\).

**THEOREM 1.4.16:** Every pairwise strongly s-regular space is pairwise semi-U_0.

**PROOF:** Let \((X, T_1, T_2)\) is pairwise strongly s-regular space. Let \(x, y \in X\) such that \(x \notin T_i\)-scl \(\{y\}\). Since the space is pairwise strongly s-regular, there exists a \(T_i\)-semi-open set \(U\) and a \(T_j\)-semi-open set \(V\) such that \(x \in U\), \(T_i\)-scl \(\{y\}\) \(\subseteq\) \(V\), \(T_j\)-scl \(U \cap T_i\)-scl \(V\) =
Hence $x \in U$, $y \in V$, $T_j\text{-scl } U \cap T_i\text{-scl } V = \emptyset$ and thus the space is pairwise semi-$U_0$.

**THEOREM 1.4.17:** Every pairwise semi-$U_0$ space is pairwise semi-$H_0$.

**PROOF:** Let $(X, T_1, T_2)$ is pairwise semi-$U_0$ space. Let $x, y \in X$ such that $x \notin T_i\text{-scl } \{y\}$. Since $X$ is pairwise semi-$U_0$, there exists a $T_i$-semi-open set $U$ and a $T_j$-semi-open set $V$ such that $x \in U$, $y \in V$, $T_j\text{-scl } U \cap T_i\text{-scl } V = \emptyset$. Hence $x \in U$, $y \in V$, $U \cap V = \emptyset$ and thus $X$ is pairwise semi-$H_0$.

**THEOREM 1.4.18:** Every pairwise semi-$U_1$ space is pairwise semi-$H_1$.

**PROOF:** Let $(X, T_1, T_2)$ is pairwise semi-$U_1$ space. Let $x, y \in X$ such that $T_i\text{-scl } \{x\} \cap T_j\text{-scl } \{y\} = \emptyset$. Since $X$ is pairwise semi-$U_1$, there exists a $T_i$-semi-open set $U$ and a $T_j$-semi-open set $V$ such that $x \in V$, $y \in U$, $T_j\text{-scl } U \cap T_i\text{-scl } V = \emptyset$. Hence $x \in V$, $y \in U$, $U \cap V = \emptyset$ and thus $X$ is pairwise semi-$H_1$.

**THEOREM 1.4.19:** Every pairwise irresolutely-normal space is pairwise semi-$U_1$.

**PROOF:** Let $(X, T_1, T_2)$ is pairwise irresolutely-normal space. Let $x, y \in X$ such that $T_i\text{-scl } \{x\} \cap T_j\text{-scl } \{y\} = \emptyset$. Since $X$ is pairwise irresolutely-normal, there exists a $T_i$-semi-open set $U$ and a $T_j$-semi-open set $V$ such that $T_i\text{-scl } \{x\} \subseteq V$, $T_j\text{-scl } \{y\} \subseteq U$, $T_j\text{-scl } U \cap T_i\text{-scl } V = \emptyset$. Hence $x \in V$, $y \in U$, $T_j\text{-scl } U \cap T_i\text{-scl } V = \emptyset$. Thus $(X, T_1, T_2)$ is pairwise semi-$U_1$.

The relations between pairwise semi-$H_i$-axioms, $i \in \{0, 1, 2\}$ and pairwise semi-$U_i$-axioms, $i \in \{0, 1\}$ can be summarized in the following diagram:

$$P \text{ semi-}T_2 = P \text{ semi-}T_0 + P \text{ semi-}H_0$$

$$P \text{ semi-}T_2 = P \text{ semi-}T_1 + P \text{ semi-}H_1$$
P irresolutely-normal

P strongly s-regular

In the above diagram P denotes Pairwise.

2. Bitopological separation axioms using v-open sets

In this section we generalize \( \nu T_i \)-axioms, \( i \in \{0, 1, 2\} \), \( P_V \) and \( P_{1V} \)-axioms of Sharma [Shar] in bitopological spaces by the name of pairwise \( \nu T_i \)-axioms \( i \in \{0, 1, 2\} \), pairwise \( P_V \) and pairwise \( P_{1V} \)-axioms. Using v-open sets we also generalize the concept of pairwise \( R_i \)-spaces, \( i \in \{0, 1\} \) of Murdeshwar and Naimpally and introduce pairwise \( vR_i \)-spaces, \( i \in \{0, 1\} \).

DEFINITION 2.1: A space \((X, T_1, T_2)\) is said to be \textbf{pairwise \( rT_0 \)} [Nou] if for any pair of distinct points of \( X \), there exists a \( T_i \)-regularly open set (\( i = 1 \) or \( 2 \)) containing one of the points but not the other or equivalently there exists a \( T_i\)-\( \delta \)-open set (\( i = 1 \) or \( 2 \)) containing one of the points but not the other.

DEFINITION 2.2: A space \((X, T_1, T_2)\) is said to be \textbf{pairwise \( rT_1 \)} [Nou] if for any pair of distinct points of \( X \), there exists a \( T_1 \)-regularly open set \( U \) containing \( x \) but not \( y \) and a \( T_2 \)-regularly open set \( V \) containing \( y \) but not \( x \) or equivalently for any pair of
distinct points of X, there exists a $T_1$-$\delta$-open set $U$ containing $x$ but not $y$ and a $T_2$-$\delta$-open set $V$ containing $y$ but not $x$.

**DEFINITION 2.3:** A space $(X, T_1, T_2)$ is said to be **pairwise $rT_2$** [Nou] if for any pair of distinct points $x, y$ of $X$, there is a $T_1$-regularly open set $U$ and a $T_2$-regularly open set $V$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$ or equivalently for any pair of distinct points $x, y$ of $X$, there is a $T_1$-$\delta$-open set $U$ and a $T_2$-regularly open set $V$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

**DEFINITION 2.4:** A space $(X, T_1, T_2)$ is said to be **pairwise $R_0$** [MN] if for every $T_i$-open set $G$, $x \in G$ implies that $T_j$-$cl\{x\} \subseteq G; i \neq j, i, j = 1, 2$.

**DEFINITION 2.5:** A space $(X, T_1, T_2)$ is said to be **pairwise $R_1$** [MN] if for every pair of distinct points $x, y$ of $X$, with $T_i$-$cl\{x\} \neq T_j$-$cl\{y\}$ there exists a $T_i$-open set $U$ and a $T_j$-open set $V$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset; i \neq j, i, j = 1, 2$.

**LEMMA 2.6** [Shar]: If $X_1$ and $X_2$ be two topological spaces and $X = X_1 \times X_2$ be their topological product. If $S_1 \in VO(T_1), S_1 \in VO(T_1)$, then

$$S_1 \times S_2 \in VO(T_1 \times T_2).$$

By generalizing above lemma, we say that the product of finitely many $v$-open sets is again $v$-open.

**1. Pairwise $vT_0$-Spaces:**

**DEFINITION 2.1.1:** A space $(X, T_1, T_2)$ is said to be **pairwise $vT_0$** if for any pair of distinct points of $X$, there exists a $T_i$-$v$-open set containing one of the points but not the other; $i = 1$ or 2.

**EXAMPLE 2.1.2:** Let $X = \{a, b, c\}, T_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}, T_2 = \{\phi, \{a\}, X\}$. $VO(T_1) = \{\phi, \{a\}, \{b\}, \{a, c\}, \{b, c\}, X\}$, $VO(T_2) = \{\phi, X\}$. 
Then $(X, T_1, T_2)$ is a pairwise $\mathcal{v}T_0$-space.

**THEOREM 2.1.3:** Every pairwise $\mathcal{r}T_0$-space is pairwise $\mathcal{v}T_0$-space.

**PROOF:** Since every subset of $(X, T_1, T_2)$ which is $T_i$-regularly open is $T_i$-$v$-open, $i \in \{1, 2\}$. It follows that every pairwise $\mathcal{r}T_0$-space is pairwise $\mathcal{v}T_0$-space.

**EXAMPLE 2.1.4:** Let $X = \{a, b, c,\}$, $T_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$, $T_2 = \{\emptyset, \{b\}, X\}$.

$\mathcal{V}O (T_1) = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\mathcal{V}O (T_2) = \{\emptyset, X\}$.

Clearly $(X, T_1, T_2)$ is pairwise $T_0$-space but it is not pairwise $\mathcal{v}T_0$.

**THEOREM 2.1.5:** A space $(X, T_1, T_2)$ is pairwise $\mathcal{v}T_0$ if either $(X, T_1)$ or $(X, T_2)$ is $\mathcal{v}T_0$.

**PROOF:** Let $(X, T_1)$ be $\mathcal{v}T_0$ and let $x$ and $y$ be two distinct points of $X$, then there is a $T_1$-$v$-open set containing one of the points but not the other. Thus $(X, T_1, T_2)$ is pairwise $\mathcal{v}T_0$. Similarly $(X, T_1, T_2)$ is pairwise $\mathcal{v}T_0$ if $(X, T_2)$ is $\mathcal{v}T_0$.

Converse of the above theorem need not be true as can be seen from the following example.

**EXAMPLE 2.1.6:** Let $X = \{a, b, c,\}$, $T_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$, $T_2 = \{\emptyset, \{c\}, \{a, b\}, X\}$.

$\mathcal{V}O (T_1) = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\mathcal{V}O (T_2) = \{\emptyset, \{c\}, \{a, b\}, X\}$.

Clearly, $(X, T_1, T_2)$ is pairwise $\mathcal{v}T_0$ but neither $(X, T_1)$ nor $(X, T_2)$ is $\mathcal{v}T_0$.

**THEOREM 2.1.7:** A space $(X, T_1, T_2)$ is pairwise $\mathcal{v}T_0$ if and only if given two distinct points of $X$, either their $T_1$-$v$-closures are distinct or their $T_2$-$v$-closures are distinct.

**PROOF:** Let $(X, T_1, T_2)$ be a pairwise $\mathcal{v}T_0$-space and let $x$ and $y$ be two distinct points of $X$. Suppose $U$ is a $T_1$-$v$-open set containing $x$ but not $y$. Then $y \in T_1$-$vcl \{y\} \subseteq X - U$ and so $x \not\in T_1$-$vcl \{y\}$. Hence $T_1$-$vcl \{x\} \neq T_1$-$vcl \{y\}$. 

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Conversely, let \( x \) and \( y \) be any two distinct points of \( X \). Then either \( T_1-vcl\{x\} \neq T_1-vcl\{y\} \) or \( T_2-vcl\{x\} \neq T_2-vcl\{y\} \). In the first case let \( z \) be a point of \( X \) such that \( z \in T_1-vcl\{x\} \) and \( z \notin T_1-vcl\{y\} \). We claim that \( y \notin T_1-vcl\{x\} \). If \( y \in T_1-vcl\{x\} \) then \( T_1-vcl\{y\} \subseteq T_1-vcl\{x\} \). Therefore \( z \in T_1-vcl\{y\} \subseteq T_1-vcl\{x\} \). This contradicts the fact that \( z \notin T_1-vcl\{x\} \). Hence \( y \notin T_1-vcl\{x\} \). Thus \( U = X - T_1-vcl\{x\} \) which is \( T_1-v \)-open set containing \( y \) but not \( x \). The case \( T_2-vcl\{x\} \neq T_2-vcl\{y\} \) can be dealt with similarly.

**THEOREM 2.1.8:** Every pairwise \( vT_0 \)-space is pairwise semi-\( T_0 \)-space.

**PROOF:** Since every subset of \( (X, T_1, T_2) \) which is \( v \)-open is also semi-open [Shar]. It follows that every pairwise \( vT_0 \)-space is pairwise semi-\( T_0 \)-space. But converse need not true.

**THEOREM 2.1.9:** The product of a finite family of pairwise \( vT_0 \)-spaces is pairwise \( vT_0 \).

**PROOF:** Let \( (X, T_1, T_2) = \prod_{\alpha \in J} (X_\alpha, T_{1\alpha}, T_{2\alpha}) \), where \( T_1 \) and \( T_2 \) are product topologies on \( X \) generated by \( T_{1\alpha} \)'s and \( T_{2\alpha} \)'s respectively and \( X = \prod_{\alpha \in J} X_\alpha \). Let \( J = \{1, 2, 3, \ldots, n\} \). Let \( x = (x_\alpha)_{\alpha \in J} \) and \( y = (y_\alpha)_{\alpha \in J} \) be two distinct points of \( X \). Then \( y_\beta \neq x_\beta \), for each \( \beta \in J \). Since \( (X_\beta, T_{1\beta}, T_{2\beta}) \) is pairwise \( vT_0 \), there exists either a \( T_{1\beta} \)-open set \( U_\beta \) or a \( T_{2\beta} \)-open set \( V_\beta \) containing one of the points but not other. Let \( U = \prod_{\alpha \neq \beta} X_\alpha \times U_\beta \) or \( V = \prod_{\alpha \neq \beta} X_\alpha \times V_\beta \). Then \( U \) is a \( T_1 \)-open set or \( V \) is a \( T_2 \)-open set containing one of the points but not other. Hence \( (X, T_1, T_2) \) is pairwise \( vT_0 \).

2. **Pairwise \( vT_1 \) Spaces:**
DEFINITION 2.2.1: A space \((X, T_1, T_2)\) is said to be **pairwise vT** if for any pair of distinct points \(x, y\) of \(X\), there exists a \(T_i\)-v-open set containing \(x\) but not \(y\) and a \(T_j\)-v-open set containing \(y\) but not \(x\); \(i \neq j, i, j = 1, 2\).

EXAMPLE 2.2.2: Let \(X = \{a, b, c\}\), \(T_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\), \(T_2 = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{a, b\}, X\}\).

\(\text{VO (} T_1 \text{)} = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}, X\}\), \(\text{VO (} T_2 \text{)} = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{a, b\}, X\}\).

Then \((X, T_1, T_2)\) is a pairwise vT-space.

THEOREM 2.2.3: If \((X, T_1, T_2)\) is a pairwise vT-space, then for each \(x \in X\),
\[
\{x\} = T_1-v\text{cl} \{x\} \cap T_2-v\text{cl} \{x\}; i \neq j, i, j = 1, 2.
\]

PROOF: Let \(y \neq x \in X\). Since \((X, T_1, T_2)\) is a pairwise vT-space, there exists a \(T_1\)-v-open set \(U\) containing \(x\) but not \(y\) and a \(T_2\)-v-open set \(V\) containing \(y\) but not \(x\). Therefore \(y \notin T_1-v\text{cl} \{x\} \cap T_2-v\text{cl} \{x\}\). Thus \(\{x\} = T_1-v\text{cl} \{x\} \cap T_2-v\text{cl} \{x\}; i \neq j, i, j = 1, 2\).

THEOREM 2.2.4: If \((X, T_1, T_2)\) is a pairwise vT-space, then for each \(x \in X\), the intersection of all \(T_1\)-v-neighbourhoods of \(x\) and all \(T_2\)-v-neighbourhoods of \(x\) is \(\{x\}\).

PROOF: Let \(y \neq x \in X\). Since \((X, T_1, T_2)\) is a pairwise vT-space, there exists a \(T_1\)-v-open set \(U\) containing \(x\) but not \(y\) and a \(T_2\)-v-open set \(V\) containing \(y\) but not \(x\). Therefore \(y\) does not belong to the intersection of all \(T_1\)-v-neighbourhoods and \(T_2\)-v-neighbourhoods of \(x\). Hence the intersection of all \(T_1\)-v-neighbourhoods of \(x\) and all \(T_2\)-v-neighbourhoods of \(x\) is \(\{x\}\).

THEOREM 2.2.5: A space \((X, T_1, T_2)\) is pairwise vT if and only if both \((X, T_1)\) and \((X, T_2)\) are vT-spaces.

PROOF: Let \((X, T_1, T_2)\) be pairwise vT space and \(x \neq y \in X\). Then there exists a \(T_1\)-v-open set \(U\) containing \(x\) but not \(y\) and a \(T_2\)-v-open set \(V\) containing \(y\) but not \(x\).
Also for \( y \neq x \), there exists a \( T_1 \)-v-open set \( U_1 \) and a \( T_2 \)-v-open set \( V_1 \) such that \( y \in U_1, x \notin U_1 \) and \( x \in V_1, y \notin V_1 \). Thus we have \( T_1 \)-v-open set \( U \) and \( U_1 \) such that \( x \in U, y \notin U \) and \( y \in U_1, x \notin U_1 \), showing that \( (X, T_1) \) is \( vT_1 \)-space. Similarly \( V \) and \( V_1 \) are \( T_2 \)-v-open sets such that \( y \in V, x \notin V \) and \( x \in V_1, y \notin V_1 \), showing that \( (X, T_2) \) is \( vT_1 \)-space.

Conversely, Let \((X, T_1)\) and \((X, T_2)\) be \( vT_1 \)-spaces. Let \( x \neq y \in X \). Since \((X, T_1)\) is \( vT_1 \), there exist \( T_1 \)-v-open sets \( U \) and \( V \) such that \( x \in U, y \notin U \) and \( y \in V, x \notin V \). Similarly, since \((X, T_2)\) is \( vT_1 \), there exist \( T_2 \)-v-open sets \( U_1 \) and \( V_1 \) such that \( x \in U_1, y \notin U_1 \) and \( y \in V_1, x \notin V_1 \). Thus for \( x \neq y \), there are \( T_1 \)-v-open set \( U \) and a \( T_2 \)-v-open set \( V_1 \) such that \( x \in U, y \notin U \) and \( y \in V_1, x \notin V_1 \). Hence, \((X, T_1, T_2)\) is pairwise \( vT_1 \).

**Theorem 2.2.6:** If \((X, T_1, T_2)\) is pairwise \( vT_1 \)-space then for \( x, y \in X \), \( T_i \)-vcl \{x\} \( \neq \) \( T_j \)-vcl \{y\}; \( i \neq j \), \( i, j = 1, 2 \).

**Proof:** If \((X, T_1, T_2)\) is a pairwise \( vT_1 \)-space, then by theorem 2.2.3, for each \( x, y \in X \), \( \{x\} = T_1 \)-vcl \{x\} \( \cap \) \( T_2 \)-vcl \{x\} and \( \{y\} = T_1 \)-vcl \{y\} \( \cap \) \( T_2 \)-vcl \{y\}. Thus \( T_i \)-vcl \{x\} \( \neq \) \( T_j \)-vcl \{y\}; \( i \neq j \), \( i, j = 1, 2 \).

**Theorem 2.2.7:** Every pairwise \( rT_1 \)-space is pairwise \( vT_1 \)-space but converse need not be true.

**Proof:** Obvious as theorem 2.1.3.

**Example 2.2.8:** Let \( R \) be the set of real numbers and \( T \) be the co-countable topology. Then \((R, T, T)\) is pairwise \( T_1 \) but it is not pairwise \( vT_1 \) because the only \( v \)-open sets are \( \emptyset \) and \( R \).

**Theorem 2.2.9:** Every pairwise \( vT_1 \)-space is pairwise semi-\( T_1 \)-space.

**Proof:** Obvious as theorem 2.1.8.
THEOREM 2.2.10: Every pairwise vT₁-space is pairwise vT₀-space.

PROOF: Obvious.

THEOREM 2.2.11: The product of a finite family of pairwise vT₁-spaces is pairwise vT₁.

PROOF: An easy modification of the proof of the theorem 2.1.9.

3. Pairwise vT₂-Spaces:

DEFINITION 2.3.1: A space (X, T₁, T₂) is said to be pairwise vT₂ if for each pair of distinct points x, y of X, there is a T₁-v-open set U and a T₂-v-open V such that x ∈ U and y ∈ V and U ∩ V = ∅; i ≠ j, i, j =1, 2.

EXAMPLE 2.3.2: Let X= {a, b, c, d}, T₁={∅, {a}, {b}, {c}, {a, b}, {b, c}, {a, c}, {a, b, c}, X}, T₂={∅, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, d}, X}.

Clearly, (X, T₁, T₂) is a pairwise vT₂-space.

THEOREM 2.3.3: Every pairwise rT₂-space is pairwise vT₂-space but converse need not be true.

PROOF: Obvious as theorem 2.1.3.

THEOREM 2.3.4: Every pairwise vT₂-space is pairwise semi-T₂-space.

PROOF: Obvious as theorem 2.1.8.

THEOREM 2.3.5: Every pairwise vT₂-space is pairwise vT₁-space.

PROOF: Obvious.

THEOREM 2.3.6: The product of a finite family of pairwise vT₂-spaces is pairwise vT₂.

PROOF: Let (Xₐ, T₁ₐ, T₂ₐ): α ∈ {1, 2, 3,... n} = J be a finite family of pairwise vT₂-spaces and let (X, T₁, T₂) = ⋃_{α∈J} (Xₐ, T₁ₐ, T₂ₐ), where T₁ and T₂ are product
topologies on $X$ generated by $T_{1\alpha}$'s and $T_{2\alpha}$'s respectively and $X = \prod_{\alpha \in I} X_\alpha$. Let

$x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ be two distinct points of $X$. Then $x_\beta \neq y_\beta$, for some $\beta \in J$. Since $(X_\beta, T_{1\beta}, T_{2\beta})$ is pairwise $vT_2$. Therefore there exists a $T_{i\beta}$-v-open set $U_\beta$ and a $T_{j\beta}$-v-open set $V_\beta$ such that $x_\beta \in U_\beta, y_\beta \in V_\beta$ and $U_\beta \cap V_\beta = \phi$.

Let us define $U = \prod_{\alpha \neq \beta} X_\alpha \times U_\beta$ and $V = \prod_{\alpha \neq \beta} (X_\alpha \times V_\beta)$.

Then $U$ is $T_i$-v-open and $V$ is a $T_j$-v-open set such that $x \in U, y \in V$ and $U \cap V = \phi$.

Hence, $(X, T_1, T_2)$ is pairwise $vT_2$.

4. Pairwise $vR_0$ Spaces:

Here we generalize the concept of pairwise $R_0$ of of Murdeshwar and Naimpally [MN] using $v$-open sets and introduce pairwise $vR_0$.

DEFINITION 2.4.1: In a topological space $(X, T)$, the $v$-kernal [Shar] of a point $x$ of $X$ is the set

$v \ker \{x\} = \{y: x \in vcl\{y\}\};$ and the $v$-kernal of a subset $A$ of $X$ is the set

$v \ker A = \cap \{U: U$ is $v$-open and $A \subseteq U\}.$

LEMMA 2.4.2: Let $A$ be a subset of a space $(X, T)$. Then

$v \ker A = \{x \in X: vcl\{x\} \cap A \neq \phi\}.$

PROOF: $x \notin v \ker A$ implies $x \notin \cap \{U: U$ is $v$-open and $A \subseteq U\}$. Therefore, there is a $v$-open set $U$ such that $A \subseteq U$ and $x \notin U$. Hence $vcl\{x\} \cap A = \phi$. Now, $vcl\{x\} \cap A = \phi$ implies $X - vcl\{x\} = G$ (say) is a $v$-open set such that $A \subseteq G$. Also, $x$ does not belong to the intersection of all $v$-open neighbourhoods of $A$ and so $x \notin v \ker A$.

DEFINITION 2.4.3: A space $(X, T_1, T_2)$ is said to be pairwise $vR_0$ if for every $T_i$-v-open set $G, x \in G$ implies that $T_j$-$vcl\{x\} \subseteq G; i \neq j, i, j = 1, 2.$
EXAMPLE 2.4.4: Let \( X = \{a, b, c,\} \), \( T_1 = \{\emptyset, \{a\}, \{b, c\}, X\} \), \( T_2 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\} \). Clearly, \((X, T_1, T_2)\) is a pairwise vR\(_0\) space but it is not pairwise R\(_0\).

THEOREM 2.4.5: For a space \((X, T_1, T_2)\), the following statement are equivalent:

(a) \((X, T_1, T_2)\) is pairwise vR\(_0\) space.

(b) For each \( x \in X \), \( T_i\)-vcl \( \{x\} \subseteq T_j\)-vker \( \{x\} \); \( i \neq j \), \( i, j = 1, 2 \).

(c) For any \( x, y \in X \), \( y \in T_i\)-vker \( \{x\} \) if and only if \( x \in T_j\)-vcl \( \{y\} \);
   \( i \neq j \), \( i, j = 1, 2 \).

(d) For any \( x, y \in X \), \( y \in T_i\)-vcl \( \{x\} \) if and only if \( x \in T_j\)-vcl \( \{y\} \);
   \( i \neq j \), \( i, j = 1, 2 \).

(e) For any \( T_i\)-v-open set \( F \) and a point \( x \in X - F \), there exists a \( T_j\)-v-open set \( U \)
    such that \( x \not\in U \) and \( F \subseteq U \); \( i \neq j \), \( i, j = 1, 2 \).

(f) Each \( T_i\)-v-open set \( F \) can be expressed as \( F = \bigcap \{G: G \text{ is } T_j\text{-v-open and } F \subseteq G\} \);
   \( i \neq j \), \( i, j = 1, 2 \).

(g) Each \( T_i\)-v-open set \( G \) can be expressed as the union of \( T_j\)-v-open sets
    contained in \( G \); \( i \neq j \), \( i, j = 1, 2 \).

(h) For each \( T_i\)-v-open set \( F \), \( x \not\in F \) implies \( T_j\)-vcl \( \{x\} \cap F = \emptyset \).

PROOF (a) \( \rightarrow \) (b): By definition 2.4.1, for each \( x \in X \), we have \( T_j\)-vker \( \{x\} = \{y: y \in T_j\text{-vcl } \{x\}\} \), and by definition of pairwise vR\(_0\), each \( T_j\)-v-open set \( G \) containing \( x \) also contains \( T_i\)-vcl \( \{x\} \). Hence \( T_i\)-vcl \( \{x\} \subseteq T_j\)-vker \( \{x\} \) for each \( x \in X \); \( i \neq j \), \( i, j = 1, 2 \).

(b) \( \rightarrow \) (c): For any \( x, y \in X \), if \( y \in T_i\)-vker \( \{x\} \) then \( x \in T_i\)-vcl \( \{y\} \) and hence by (b)
   \( x \in T_j\)-vker \( \{y\} \).

(c) \( \rightarrow \) (d): For any \( x, y \in X \), if \( y \in T_i\)-vcl \( \{x\} \) then \( x \in T_i\)-vker \( \{y\} \) and hence by (c),
   \( y \in T_j\)-vker \( \{x\} \) implies \( x \in T_j\)-vcl \( \{y\} \).
(d) → (e): Let F be a $T_i$-v-open set and $x \notin F$. Then for any $y \in F$, $T_i$-$vcl\{y\} \subseteq F$ and so $x \notin T_i$-$vcl\{y\}$. Now by (d), $x \notin T_i$-$vcl\{y\}$ implies $y \notin T_j$-$vcl\{x\}$. This means that there exists a $T_j$-v-open set $G_y$ such that $y \in G_y$ and $x \notin G_y$. Let $G = \bigcup_{y \in F} \{G_y: G_y$ is $T_j$-v-open, $y \in G_y$ and $x \notin G_y\}$. Then $G$ is a $T_j$-v-open set such that $x \notin G$ and $F \subseteq G$; $i \neq j$, $i, j = 1, 2$.

(e) → (f): Let F be a $T_i$-v-open set and suppose that $H = \cap \{G: G$ is $T_j$-v-open and $F \subseteq G\}$. Then $F \subseteq H$ and it remains to show that $H \subseteq F$. Let $x \notin F$. Then by (e), there is a $T_j$-v-open set $G$ such that $x \notin G$ and $F \subseteq G$. Hence, $x \notin H$. Therefore, each $T_i$-v-open set $F$ can be written as $F = \cap \{G: G$ is $T_j$-v-open and $F \subseteq G\}; i \neq j$, $i, j = 1, 2$.

(f) → (g): Obvious.

(g) → (h): Let F be a $T_i$-v-open set and $x \notin F$. Then $X - F$ is a $T_i$-v-open set containing $x$. Then by (g), $X - F$ can be written as the union of $T_j$-v-open sets and so there is a $T_j$-v-open set $C$ such that $x \in C \subseteq X - F$. Hence $T_j$-$vcl\{x\} \subseteq X - F$. Thus $T_j$-$vcl\{x\} \cap F = \phi$.

(h) → (a): Let G be a $T_i$-v-open set containing a point $x$ of $X$. Then $x \notin X - G$ which is $T_i$-v-open set. By (h), $T_j$-$vcl\{x\} \cap X - G = \phi$ which implies that $T_j$-$vcl\{x\} \subseteq G$; $i \neq j$, $i, j = 1, 2$. Hence, $(X, T_1, T_2)$ is pairwise $vR_0$.

DEFINITION 2.4.6: In a bitopological space $(X, T_1, T_2)$ for any $x \in X$, we introduce the following notations:

(i) $bi$-$vcl\{x\} = T_1$-$vcl\{x\} \cap T_2$-$vcl\{x\}$

(ii) $bi$-$vker\{x\} = T_1$-$vker\{x\} \cap T_2$-$vker\{x\}$

THEOREM 2.4.7: In a pairwise $vR_0$ space $(X, T_1, T_2)$, for any $x, y$ in $X$, we have
either bi-vcl \{x\} = bi-vcl \{y\} or bi-vcl \{x\} \cap bi-vcl \{y\} = \phi.

PROOF: Suppose that bi-vcl \{x\} \cap bi-vcl \{y\} \neq \phi and let p \in [T_1-vcl \{x\} \cap T_2-vcl \{x\}] \cap [T_1-vcl \{y\} \cap T_2-vcl \{y\}]. Then T_1-vcl \{p\} \subseteq T_1-vcl \{x\} \cap T_1-vcl \{y\} and T_2-vcl \{p\} \subseteq T_2-vcl \{x\} \cap T_2-vcl \{y\}. Also, p \in T_1-vcl \{x\} implies T_2-vcl \{x\} \subseteq T_2-vcl \{y\} because by part (d) of theorem 2.4.5, p \in T_1-vcl \{x\} and so x \in T_2-vcl \{p\} implies T_2-vcl \{x\} \subseteq T_2-vcl \{p\} \subseteq T_2-vcl \{y\}. Similarly, p \in T_2-vcl \{x\} implies T_1-vcl \{x\} \subseteq T_1-vcl \{y\}.

Also, we have, p \in T_1-vcl \{y\} implies T_2-vcl \{y\} \subseteq T_2-vcl \{x\} and p \in T_2-vcl \{y\} implies T_1-vcl \{y\} \subseteq T_1-vcl \{x\}. Thus T_1-vcl \{x\} = T_1-vcl \{y\} and T_2-vcl \{x\} = T_2-vcl \{y\} and therefore, T_1-vcl \{x\} \cap T_2-vcl \{x\} = T_1-vcl \{y\} \cap T_2-vcl \{y\}. This proves the theorem.

THEOREM 2.4.8: In a pairwise vR_0 space \((X, T_1, T_2)\), for any x, y in X, we have either bi-vker \{x\} = bi-vker \{y\} or bi-vker \{x\} \cap bi-vker \{y\} = \phi.

PROOF: By virtue of the statement (c) of theorem 2.4.5, the proof is similar to that of theorem 2.4.7.

THEOREM 2.4.9: If a space \((X, T_1, T_2)\) is pairwise vT_1, then it is pairwise vR_0.

PROOF: If \((X, T_1, T_2)\) is pairwise vT_1, then it is b_1- vT_1. Hence T_1-vcl \{x\} = \{x\} = T_2-vcl \{x\} for each x \in X. Thus if x \in G and G is T_1-v-open then T_2-vcl \{x\} \subseteq G. Therefore \((X, T_1, T_2)\) is pairwise vR_0. Example 2.4.4 show that converse of this theorem is not true.

THEOREM 2.4.10: Every pairwise vT_0 and pairwise vR_0-space is pairwise vT_1.

PROOF: Let \((X, T_1, T_2)\) be pairwise vT_0 and pairwise vR_0. Let x, y be two distinct points of X, there exists a T_1-v-open G, containing one of the points but not the other.
Suppose $G$ contains $x$. Since $X$ is pairwise $vR_0$, $x \in G$ implies that $T_j$-vcl $\{x\} \subseteq G$.

Now $x \notin X - T_j$-vcl $\{x\} = M$ (say). Thus $G$ is a $T_i$-v-open set containing $x$ but not $y$ and $M$ is a $T_j$-v-open set containing $y$ but not $x$. Thus $(X, T_1, T_2)$ is pairwise $vT_1$.

THEOREM 2.4.11: A space $(X, T_1, T_2)$ is pairwise $vR_0$ if and only if for every pair of distinct points $x, y$ of $X$, either $T_1$-vcl $\{x\} \cap T_2$-vcl $\{x\} = \emptyset$ or $\{x, y\} \subseteq T_1$-vcl $\{x\} \cap T_2$-vcl $\{y\}$.

PROOF: Let $x, y$ be any two distinct points in $(X, T_1, T_2)$. Let $(X, T_1, T_2)$ be pairwise $vR_0$. Let $T_1$-vcl $\{x\} \cap T_2$-vcl $\{y\} \neq \emptyset$, and $p \in T_1$-vcl $\{x\} \cap T_2$-vcl $\{y\}$. Suppose $\{x, y\} \not\subseteq T_1$-vcl $\{x\} \cap T_2$-vcl $\{y\}$. Then $x \notin T_2$-vcl $\{y\}$ which implies that $x \in X - T_2$-vcl $\{y\}$ which is a $T_2$-v-open set. Since $p \in T_1$-vcl $\{x\} \cap T_2$-vcl $\{y\}$, therefore $T_1$-vcl $\{x\} \not\subseteq X - T_2$-vcl $\{y\}$. Hence the space $(X, T_1, T_2)$ is not pairwise $vR_0$.

Conversely, let $G$ be a $T_1$-v-open set containing a point $x$ of $X$. Suppose $T_2$-vcl $\{x\} \not\subseteq G$. Let $y$ be a point in $T_2$-vcl $\{x\}$ such that $y \notin G$. Then $T_1$-vcl $\{y\} \cap G = \emptyset$, since $X - G$ is $T_1$-v-open and $y \in X - G$. Therefore, $\{x, y\} \not\subseteq T_1$-vcl $\{x\} \cap T_2$-vcl $\{y\}$ and so $T_1$-vcl $\{x\} \cap T_2$-vcl $\{y\} = \emptyset$, which is a contradiction as $y \in T_1$-vcl $\{x\} \cap T_2$-vcl $\{y\}$. Hence $(X, T_1, T_2)$ is pairwise $vR_0$.

5. Pairwise $vR_1$ Spaces:

Here we generalize the concept of pairwise $R_1$ of Murdeshwar and Naimpally [MN] using $v$-open sets and introduce pairwise $vR_1$.

DEFINITION 2.5.1: A space $(X, T_1, T_2)$ is said to be **pairwise $vR_1$** if for every pair of distinct points $x, y$ of $X$, with $T_i$-vcl $\{x\} \neq T_j$-vcl $\{y\}$ there exits a $T_j$-v-open set $U$ and a $T_i$-v-open set $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$; $i \neq j$, $i, j = 1, 2$.

THEOREM 2.5.2: Every pairwise $vT_2$-space is pairwise $vR_1$-space.
PROOF: By definition, (X, T1, T2) is said to be pairwise vT2 if for each pair of
distinct points x, y of X, there is a T1-v-open set U and a Tj-v-open V such that x \in U
and y \in V and U \cap V = \emptyset. Therefore, y \notin T_l-vcl \{x\} and x \notin T_j-vcl \{y\}. Hence (X, T1, T2) is pairwise vR1.

THEOREM 2.5.3: Every pairwise vT1 and pairwise vR1-space is pairwise vT2.

PROOF: Let (X, T1, T2) be pairwise vT1 and pairwise vR1. Let x, y be two distinct
points of X. Since (X, T1, T2) is pairwise vT1 therefore by theorem 2.2.6, T_l-vcl \{x\} \neq T_j-vcl \{y\}. Since (X, T1, T2) be pairwise vR1, there exists a T_l-v-open set U and a Tj-v-open set V such that x \in U, y \in V and U \cap V = \emptyset; i \neq j, i, j = 1, 2. Hence
(X, T1, T2) is pairwise vT2.

THEOREM 2.5.4: Every pairwise vR1-space is pairwise vR0-space.

PROOF: Let (X, T1, T2) be pairwise vR1-space. Let G be any T1-v-open set and
x \in G. For each y \in X – G, T_l-vcl \{x\} \neq T_j-vcl \{y\}. Therefore, there exists a T_l-v-open set U_y and a Tj-v-open set V_y such that x \in U_y, y \in V_y and U_y \cap V_y = \emptyset.

If A = \{V_y: y \in X – G\}, then X – G \subseteq A and x \notin A. T_j-v-openness of A implies
T_l-vcl \{x\} \subseteq X – A \subseteq G. Hence, (X, T1, T2) is pairwise vR0.

THEOREM 2.5.5: A space (X, T1, T2) is pairwise vR1 if and only if for every pair of
distinct points x, y, of X such that T_l-vcl \{x\} \neq T_j-vcl \{y\}, there exists a T_l-v-open set
U and a Tj-v-open V such that T_l-vcl \{x\} \subseteq V, T_j-vcl \{y\} \subseteq U and U \cap V = \emptyset; i \neq j, i, j = 1, 2.

PROOF: Let (X, T1, T2) be pairwise vR1-space. Let x, y be two distinct points of X
such that T_l-vcl \{x\} \neq T_j-vcl \{y\}; i \neq j, i, j = 1, 2. Then there exists a T_l-v-open set U
and a Tj-v-open set V such that x \in V, y \in U and U \cap V = \emptyset. Since a pairwise vR1-
space is pairwise vRd, therefore x ∈ V implies T₁-vcl{x} ⊆ V and y ∈ U implies T₂-vcl{y} ⊆ U. Hence the result follows. The converse is obvious.

6. Pairwise Pᵥ-Spaces:

Here we generalize the concept of Pᵥ-spaces of Sharma [Shar] in bitopological spaces:

DEFINITION 2.6.1: A bitopological space (X, T₁, T₂) is said to be **pairwise Pᵥ-space** if x ∉ Tᵢ-vcl{y} ⇒ y ∉ Tⱼ-cl{x}; i ≠ j, i, j = 1, 2.

EXAMPLE 2.6.2: Let X = {a, b, c}, T₁ = {ϕ, {a}, {b, c}, X}, T₂ = {ϕ, {a}, {c}, {a, c}, {b, c}, X}. Then (X, T₁, T₂) is a pairwise Pᵥ-space.

THEOREM 2.6.3: A space X is a pairwise Pᵥ-space if and only if for each Tᵢ-v-open set S and each x ∈ S, Tⱼ-cl{x} ⊆ S; i ≠ j, i, j = 1, 2.

PROOF: Let S is a Tᵢ-v-open set containing x and let y ∉ S. Then x ∉ Tᵢ-vcl{y}. By pairwise Pᵥ-axiom, y ∉ Tⱼ-cl{x}. Hence Tⱼ-cl{x} ⊆ S.

Conversely, Let x ∉ Tᵢ-vcl{y}. So there is a Tᵢ-v-open set S (say) containing x which has empty intersection with {y}, i.e., y ∉ S. By hypothesis, Tⱼ-cl{x} ⊆ S and y ∉ Tⱼ-cl{x}. Hence X is a pairwise Pᵥ-space.

THEOREM 2.6.4: For a bitopological space X the following are equivalent:

(a) X is a pairwise Pᵥ-space.

(b) For each x ∈ X, Tⱼ-cl{x} ⊆ Tᵢ-v ker{x}; i ≠ j, i, j = 1, 2.

(c) If F is a Tᵢ-v-open set in X, then F is the intersection of all the Tⱼ-open set containing F; i ≠ j, i, j = 1, 2.

(d) If S is a Tᵢ-v-open set in X, then S the union of all Tⱼ-closed sets in X contained in S; i ≠ j, i, j = 1, 2.
(e) For \( A \neq \phi \), and a \( T_i \)-v-open set \( S \) in \( X \) such that \( S \cap A \neq \phi \), there exists a \( T_j \)-closed set \( F \subseteq S \) such that \( F \cap A \neq \phi \); \( i \neq j \), \( i, j = 1, 2 \).

(f) For any \( T_i \)-v-open set \( F \) in \( X \) and \( x \notin F \), \( T_j \)-cl \( \{ x \} \cap F = \phi \); \( i \neq j \), \( i, j = 1, 2 \).

**PROOF** (a) \( \rightarrow \) (b): Let \( y \in T_j \)-cl \( \{ x \} \) and \( S \) be a \( T_i \)-v-open set containing \( x \). By theorem 2.6.3, \( T_j \)-cl \( \{ x \} \subseteq S \) and thus \( y \in S \). Therefore \( x \in T_i \)-vcl \( \{ y \} \), i.e., \( y \in T_i \)-v ker \( \{ x \} \). Hence, \( T_j \)-cl \( \{ x \} \subseteq T_i \)-v ker \( \{ x \} \).

(b) \( \rightarrow \) (c): Let \( x \notin F \). Then \( X - F \) is a \( T_i \)-v-open set containing \( x \). If \( y \in T_j \)-cl \( \{ x \} \), then from (b), \( y \in T_i \)-v ker \( \{ x \} \) and therefore \( x \in T_i \)-vcl \( \{ y \} \). So \( y \in X - F \). Hence \( T_j \)-cl \( \{ x \} \subseteq X - F \), which implies, \( F \subseteq X - T_j \)-cl \( \{ x \} \). Therefore \( X - T_j \)-cl \( \{ x \} \) is a \( T_j \)-open set that does not contain \( x \). Thus \( x \) does not belong to the intersection of all the \( T_j \)-open sets, which contain \( F \). Hence (c) holds.

(c) \( \rightarrow \) (d): By taking complements of (c), we get (d).

(d) \( \rightarrow \) (e): Since \( S \cap A \neq \phi \), let \( x \in S \cap A \). Then \( x \in T_i \)-v-open set \( S \). Therefore, from (d), \( S \) is the union of all the \( T_j \)-closed sets contained in \( S \). Hence there exists a \( T_j \)-closed set \( F \) (say) such that \( x \in F \subseteq S \), which implies that \( F \cap S \neq \phi \). Thus (e) holds.

(e) \( \rightarrow \) (f): Let \( F \) be a \( T_i \)-v-open set in \( X \) and \( x \notin F \). Then \( X - F \) is a \( T_j \)-v-open set in \( X \) such that \( (X - F) \cap \{ x \} \neq \phi \). Therefore, from (e), there is a \( T_j \)-closed set \( K \) such that \( K \subseteq X - F \) and \( K \cap \{ x \} \neq \phi \). So, \( T_j \)-cl \( \{ x \} \subseteq X - F \). Hence, \( T_j \)-cl \( \{ x \} \cap F = \phi \). Thus, (f) is true.

(f) \( \rightarrow \) (a): Let \( S \) be a \( T_i \)-v-open set containing \( x \). Then, from (f), we have \( (X - S) \cap T_j \)-cl \( \{ x \} = \phi \) and hence cl \( \{ x \} \subseteq S \). Thus by theorem 2.6.3, \( X \) is a pairwise \( P_V \)-space.
THEOREM 2.6.5: A pairwise $P_V$-space $X$ is pairwise $T_1$ if it is pairwise $rT_0$.

PROOF: Let $(x, T_0, T_1)$ is pairwise $P_V$ and $rT_0$ spaces. Since $x$ is pairwise $rT_0$. Then there exists a $T_1$-regularly open set $G$ containing $x$ but not $y$. Since $X$ is a pairwise $P_V$-space by theorem 2.6.3, and the fact that every regularly open set is $v$-open, $T_j$-cl $\{x\} \subseteq G$. Also $y \not\in T_j$-cl $\{x\}$. Take $H = X - (T_j$-cl$\{x\})$, which is a $T_j$-open set containing $y$ but not $x$. Thus open sets $G$ and $H$ satisfy the requirement of pairwise $T_1$.

7. Pairwise $P_{1V}$-Spaces:

Here we generalize the concept of $P_{1V}$-spaces of Sharma [Shar] in bitopological spaces:

DEFINITION 2.7.1: A bitopological space $(X, T_1, T_2)$ is said to be **pairwise $P_{1V}$-space** if $x \not\in T_i$-$\delta$cl $\{y\} \Rightarrow y \not\in T_j$-vcl $\{x\}; i \neq j, i, j = 1, 2$.

EXAMPLE 2.7.2: Let $X = \{a, b, c\}, T_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}, T_2 = \{\emptyset, \{b\}, \{a, c\}, X\}$. Then $(X, T_1, T_2)$ is a pairwise $P_{1V}$-space.

THEOREM 2.7.3: A space $X$ is a pairwise $P_{1V}$-space if and only if for each $T_i$-regularly open set $S$ and each $x \in S$, $T_j$-vcl $\{x\} \subseteq S; i \neq j, i, j = 1, 2$.

PROOF: Let $S$ is a $T_i$-regularly open set containing $x$ and let $y \not\in S$. Then $x \not\in T_i$-$\delta$cl $\{y\}$. Since $X$ is a pairwise $P_{1V}$-space, therefore $y \not\in T_j$-vcl $\{x\}$. Hence $T_j$-vcl $\{x\} \subseteq S$.

Conversely, Let $x \not\in T_i$-$\delta$cl $\{y\}$. So there is a $T_i$-regularly open set $S$ (say) containing $x$ but not $y$. By hypothesis, $T_j$-cl $\{x\} \subseteq S$ and thus $y \not\in T_j$-vcl $\{x\}$. Hence $X$ is a pairwise $P_{1V}$-space.

THEOREM 2.7.4: A space $X$ is a pairwise $P_{1V}$-space if and only if for each $x \in S$, $T_j$-vcl $\{x\} \subseteq T_i$-$\delta$ ker $\{x\}; i \neq j, i, j = 1, 2$. 

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PROOF: Let \( y \in T_j - \text{vcl} \{x\} \) and \( S \) be a \( T_i \)-regularly open set containing \( x \). Since \( X \) is pairwise \( P_1V \), therefore, \( T_j - \text{vcl} \{x\} \subseteq S \). Hence \( y \in S \). So \( x \in T_i - \delta \text{cl} \{y\} \), i.e. \( y \in T_i - \delta \text{ker} \{x\} \). Thus, \( T_j - \text{vcl} \{x\} \subseteq T_i - \delta \text{ker} \{x\} \).

Conversely, Let \( x \not\in T_i - \delta \text{cl} \{y\} \). Then \( y \not\in T_i - \delta \text{ker} \{x\} \). Therefore, by hypothesis, \( y \not\in T_j - \text{vcl} \{x\} \). Hence, \( X \) is a pairwise \( P_1V \)-space.

(3) **Bitopological separation axioms using g-open sets**

This section, dealt with the applications of g-open sets to define new separation axioms. In this section, we introduce and study some new separation axioms using g-open sets. Using g-open sets Munshi [Mun] defined g-regular and g-normal spaces. By replacing the open sets by pre-open sets in g-regularity and g-normality due to Munshi [Mun], we introduce a new concept of gp-regularity and gp-normality to define some new separation axioms gp-regularity and gp-normality. Two generalized forms of \( P_1 \)-axioms, namely \( P_g \) and \( P_{1g} \)-spaces are introduced. From \( R_0 \)-axioms, \( gR_0 \)-spaces and from \( R_1 \)-axioms, \( gR_1 \)-spaces are derived.

In this section, we also generalized these axioms in bitopological spaces and introduce and study pairwise gp-regular, pairwise gp-normal, pairwise g\( R_0 \), pairwise g\( R_1 \), pairwise g\( P \) and pairwise g\( P_{1g} \)-spaces.

**DEFINITION 3.1 [Lev2]**: A subset \( A \) of \( X \) is said to be **g-closed** if \( \text{cl} (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, T)\). Clearly, every closed set is g-closed. Complement of a g-closed set is called a **g-open set**. The family of all g-open (resp. g-closed) sets in a space \((X, T)\) is denoted by \( GO (X, T) \) (resp. \( GC (X, T) \)).
DEFINITION 3.2 [Mun]: A set $U$ is said to be \textit{g-neighbourhood} of point $x \in X$ if $x \in U$ and $U$ is g-open.

DEFINITION 3.3 [Lev$_2$]: The \textit{g-closure} of a set $A$ in a space $X$ is denoted by $gcl\ A$ to be the intersection of all g-closed sets that contain $A$.

DEFINITION 3.4 [Lev$_2$]: A topological space $X$ is said to \textit{symmetric} if singletons are g-closed for each $x \in X$.

DEFINITION 3.5: A space $(X, T)$ is said to be \textit{g-regular} [Mun] (resp. \textit{gs-regular} [Nou], \textit{gg-regular} [Shar]) if for each g-closed set $F$ and a point $x \notin F$, there exists an open (resp. semi-open, g-open) set $U$ and an open (resp. semi-open, g-open) set $V$ such that $F \subseteq U$ and $x \in V$ and $U \cap V = \emptyset$.

DEFINITION 3.6: A space $(X, T)$ is said to be \textit{g-normal} [Mun] (\textit{gs-normal}, \textit{gg-normal} [Shar]) if for every pair of disjoint g-closed sets $A$ and $B$, there exist disjoint open (resp. semi-open, g-open) sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

DEFINITION 3.7 [Jan]: A mapping $f: (X, T) \rightarrow (Y, \sigma)$ is said to be \textit{pre-irresolute} if the inverse image of every pre-open set in $Y$ is pre-open in $X$.

LEMMA 3.8 [Lev$_2$]: If $f: X \rightarrow Y$ is a closed and continuous and if $A$ is g-closed set in $X$, then $f(A)$ is g-closed in $Y$.

LEMMA 3.9 [Lev$_2$]: If $f: X \rightarrow Y$ is a closed and continuous and if $A$ is g-closed (res. g-open) set in $Y$, then $f^{-1}(A)$ is g-closed (res. g-open) in $X$.

1. \textbf{gp-Regular Space}:

By replacing open set by pre-open set in g-regularity due to Munshi [Mun], we introduce a new concept of gp-regularity.
DEFINITION 3.1.1: A space $X$ is said to **gp-regular** if for every g-closed set $F$ and a point $x \notin F$, there exist disjoint pre-open sets $U$ and $V$ such that $F \subseteq U$, $x \in V$.

Clearly, every g-regular space is gp-regular but converse is not true.

EXAMPLE 3.1.2: Let $X = \{a, b, c\}$, $T = \{\emptyset, \{a, b\}, X\}$

$PO(X, T) = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$,

$GC(X, T) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$.

Then the space is gp-regular but not g-regular.

THEOREM 3.1.3: For a space $X$, the following are equivalent:

(a) $X$ is gp-regular.

(b) For each $x \in X$ and every g-open set $G$ containing $x$, there exists a pre-open set $O$ such that $x \in O \subseteq pcl O \subseteq G$.

(c) For every g-closed set $F$, the intersection of all pre-closed pre-neighborhoods of $F$ is exactly $F$.

(d) For every set $A$ and a g-open set $B$ such that $A \cap B \neq \emptyset$, there exists a pre-open set $G$ such that $A \cap G \neq \emptyset$, and $pcl G \subseteq B$.

(e) For every non empty set $A$ and any g-closed set $B$ satisfying $A \cap B = \emptyset$, there exist disjoint pre-open sets $G$ and $M$ such that $A \cap G \neq \emptyset$, and $B \subseteq M$.

PROOF (a) $\rightarrow$ (b): Let $x \in G$ and $G$ is g-open in $X$. Therefore $x \notin X - G$ and $X - G$ is g-closed in $X$. Since $X$ is gp-regular, there exist disjoint pre-open sets $O$ and $O_1$ such that $x \in O$ and $X - G \subseteq O_1$. Hence, $X - O_1 \subseteq G$ and $pcl O \subseteq X - O_1$. Hence $x \in O \subseteq pcl O \subseteq G$.

(b) $\rightarrow$ (c): Let $F$ be a g-closed subset of $X$ and $x \notin F$. Then $X - F$ is a g-open set containing $x$. Therefore by (b) there exists a pre-open set $O$ such that $x \in O \subseteq pcl O$
\( \subseteq X - F \). Hence, \( F \subseteq X - \text{pcl } O \subseteq X - O \) and \( x \not\in X - O \). Thus \( X - O \) is a pre-closed pre-neighborhood of \( F \) which does not contain \( x \). Hence, the intersection of all pre-closed pre-neighborhoods of \( F \) is exactly \( F \).

(c) \( \rightarrow \) (d): Let \( A \) be a non empty subset of \( X \) and \( B \) be a g-open set such that \( A \cap B \neq \emptyset \). Let \( x \in A \cap B \). Then \( X - B \) is a g-closed set such that \( x \not\in X - B \). Therefore, by (c), there exists a pre-closed pre-neighborhood of \( X - B \), say \( V \), such that \( x \not\in V \). Thus for the pre-closed set \( V \), there exists a pre-open set \( U \) such that \( X - B \subseteq U \subseteq V \). Take \( G = X - V \). Then \( G \) is a pre-open set containing \( x \). Also \( A \cap G \neq \emptyset \) and \( \text{pcl } G \subseteq X - U \subseteq B \).

(d) \( \rightarrow \) (e): Let \( A \cap B = \emptyset \), where \( A \) is non empty and \( B \) is a g-closed, then \( A \cap X - B \neq \emptyset \), where \( X - B \) is a g-open set. Therefore by (d), there exists a pre-open set \( G \) such that \( A \cap G \neq \emptyset \), and \( \text{pcl } G \subseteq X - B \). Now, put \( M = X - \text{pcl } G \). Then \( B \subseteq M \) and \( G \) and \( M \) are disjoint pre-open sets.

(e) \( \rightarrow \) (a): Let \( F \) be a g-closed subset of \( X \) and \( x \not\in F \). Then \( \{x\} \) and \( F \) are disjoint. Therefore, by (e), there exist disjoint pre-open sets \( G \) and \( M \) such that \( \{x\} \cap G \neq \emptyset \) and \( F \subseteq M \). Thus \( x \in G \) and \( F \subseteq M \). Hence \( X \) is gp-regular.

THEOREM 3.1.4: Every gp-regular, symmetric space is pre-T_2.

PROOF: Let \( x, y \) be any two distinct points of \( X \). Since \( X \) is symmetric implies \( \{x\} \) is g-closed. Also \( y \not\in \{x\} \). Since \( X \) is gp-regular, there exist pre-open sets \( U \) and \( V \) such that \( x \in V \), \( y \in U \) and \( U \cap V = \emptyset \). Hence \( X \) is pre-T_2.

THEOREM 3.1.5: Let \( f: X \rightarrow Y \) be a closed, continuous and pre-irresolute one-one mapping. Then \( X \) is gp-regular if \( Y \) is gp-regular.
PROOF: Let Y be gp-regular and let G be any g-closed set in X such that \( x \not\in G \). Then \( y \not\in f(G) \), where \( y = f(x) \) and \( f(G) \) is a g-closed set in Y since \( f \) is closed and continuous. By gp-regularity, there exist disjoint pre-open sets U and V in Y such that \( y \in U \) and \( f(G) \subseteq V \). Hence \( x \in f^{-1}(U) \) and \( G \subseteq f^{-1}(V) \) with \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \). Also \( f^{-1}(U) \) and \( f^{-1}(V) \) are pre-open sets since \( f \) is pre-irresolute. Hence X is gp-regular.

THEOREM 3.1.6: Let \( f: X \to Y \) be a closed, continuous and strongly pre-closed surjective mapping. Then Y is gp-regular if X is gp-regular.

PROOF: Let X be gp-regular. Let F be any g-closed subset in Y such that \( y \not\in F \). Then \( x \not\in f^{-1}(F) \), where \( y = f(x) \) and \( f^{-1}(F) \) is g-closed since \( f \) is closed, continuous. By gp-regularity, there exist disjoint pre-open sets U and V in Y such that \( x \in U \), \( f^{-1}(F) \subseteq V \). Let \( U_1 = Y - f(X - U) \), \( V_1 = Y - f(X - V) \). Since \( f \) is strongly pre-closed, therefore \( U_1 \) and \( V_1 \) are pre-open sets such that \( y \in U_1 \) and \( F \subseteq V_1 \) with \( U_1 \cap V_1 = \emptyset \). Hence Y is gp-regular.

2. **Pairwise gp-Regular Spaces:**

DEFINITION 3.2.1: A bitopological space \((X, T_1, T_2)\) is said to be pairwise gp-regular if for any point \( x \) and a \( T_i \)-g-closed set \( F \) not containing \( x \), there exists a \( T_i \)-pre-open sets \( U \) and a \( T_j \)-pre-open set \( V \) such that \( F \subseteq V \), \( x \in U \) and \( U \cap V = \emptyset ; i \neq j, i, j = 1, 2 \).

Clearly, every pairwise g-regular space is pairwise gp-regular but converse is not true.

EXAMPLE 3.2.2: Let \( X = \{a, b, c\}, T_1 = \{\phi, \{a\}, \{b, c\}, X\}, \)
\( T_2 = \{\phi, \{b\}, \{a, c\}, X\}. \)

PO \((X, T_1) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\),
PO \((X, T_2)\) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\},

GC \((X, T_1)\) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\},

GC \((X, T_2)\) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}.

Then the space \((X, T_1, T_2)\) is pairwise gp-regular but not pairwise g-regular.

THEOREM 3.2.3: For a bitopological space \((X, T_1, T_2)\), the following are equivalent:

(a) \((X, T_1, T_2)\) is pairwise gp-regular.

(b) For each \(x \in X\) and every \(T_i\)-g-open set \(U\) containing \(x\), there exist a \(T_i\)-pre-open set \(V\) such that \(x \in V \subseteq T_j\text{-pcl } V \subseteq U; \ i \neq j, i, j = 1, 2\).

(c) For every \(T_i\)-g-closed set \(F\), the intersection of all \(T_i\)-pre-closed, \(T_j\)-pre-neighbourhoods of \(F\), is exactly \(F; \ i \neq j, i, j = 1, 2\).

(d) For every set \(A\) and a \(T_i\)-g-open set \(B\) such that \(A \cap B \neq \phi\), there exists a \(T_i\)-pre-open set \(W\) such that \(A \cap W \neq \phi\), and \(T_j\text{-pcl } W \subseteq B; \ i \neq j, i, j = 1, 2\).

(e) For every non empty set \(A\) and any \(T_i\)-g-closed set \(B\) satisfying \(A \cap B = \phi\), there exist a \(T_i\)-pre-open set \(U\) and a \(T_j\)-pre-open set \(V\) such that \(A \cap U \neq \phi\), and \(B \subseteq V\) and \(U \cap V = \phi; \ i \neq j, i, j = 1, 2\).

PROOF (a) \(\rightarrow\) (b): Let \(x \in U\) and \(U\) is \(T_i\)-g-open in \(X\). Therefore \(x \not\in X - U\) and \(X - U\) is \(T_i\)-g-closed in \(X\). Since \(X\) is pairwise gp-regular, there exists a \(T_i\)-pre-open set \(V\) and a \(T_j\)-pre-open set \(W\) such that \(x \in V\) and \(X - U \subseteq W\) and \(V \cap W = \phi\).

Hence \(V \subseteq X - W\) and \(T_j\text{-pcl } V \subseteq X - W\). Hence \(x \in V \subseteq T_j\text{-pcl } V \subseteq U\).

(b) \(\rightarrow\) (c): Let \(F\) be a \(T_i\)-g-closed subset of \(X\) and \(x \not\in F\). Then \(X - F\) is a \(T_i\)-g-open set containing \(x\). Therefore, by (b), there exists a \(T_i\)-pre-open set \(V\) such that \(x \in V \subseteq T_j\text{-pcl } V \subseteq X - F\). Hence \(F \subseteq (X - T_j\text{-pcl } V) \subseteq X - V\) and \(x \not\in X - V\). Thus \(X - V\) is a
T$_1$-pre-closed, T$_j$-pre-neighbourhood of F which does not contain x. Hence the intersection of all T$_1$-pre-closed, T$_j$-pre-neighbourhoods of F is exactly F.

(c) $\rightarrow$ (d): Let A be a non empty subset of X and B be a T$_i$-g-open set such that A $\cap$ B $\neq$ $\phi$. Let x $\in$ A $\cap$ B. Then X – B is a T$_i$-g-closed such that x $\notin$ X – B. So, by (c), the intersection of all T$_i$-pre-closed, T$_j$-pre-neighbourhoods of X – B is exactly X – B, i.e., there exists a T$_i$-pre-closed, T$_j$-pre-neighbourhood V of X – B such that x $\notin$ V. Thus for the T$_i$-pre-closed set V, there exists a T$_j$-pre-open set U such that X – B $\subseteq$ U $\subseteq$ V. Let W = X – V. Then W is a T$_i$-pre-open set containing x, as x $\notin$ X – B and therefore x $\notin$ V. Hence x $\in$ A and x $\in$ W, which implies that A $\cap$ W $\neq$ $\phi$. Since X – V $\subseteq$ X – U $\subseteq$ B, therefore T$_j$-pcl (X – V) $\subseteq$ X – U $\subseteq$ B. Hence, T$_j$-pcl W $\subseteq$ B.

(d) $\rightarrow$ (e): Let A $\cap$ B = $\phi$, where A is non empty and B is a T$_i$-g-closed, then A $\cap$ (X – B) $\neq$ $\phi$, where X – B is a T$_i$-g-open set. Therefore by (d), there exists a T$_i$-pre-open set U such that A $\cap$ U $\neq$ $\phi$ and T$_j$-pcl U $\subseteq$ X – B. Now, put V = (X – T$_j$-pcl U).

Then B $\subseteq$ V and U $\cap$ V = $\phi$.

(e) $\rightarrow$ (a): Let F be a T$_i$-g-closed subset of X and x $\notin$ F. Then {x} and F are disjoint. So by (e), there exists a T$_i$-pre-open set U and a T$_j$-pre-open set V such that {x} $\cap$ U $\neq$ $\phi$, and F $\subseteq$ V and U $\cap$ V = $\phi$, i.e. x $\in$ U. Hence, X is pairwise gp-regular.

THEOREM 3.2.4: Every pairwise gp-regular bi-symmetric space is pairwise pre-T$_2$.

PROOF: Let X be a pairwise gp-regular bi-symmetric space. Let x, y be any two distinct points of X. Since X is bi-symmetric implies {x} is T$_i$-g-closed, i = 1, 2. Also y $\notin$ {x}. Since X is pairwise gp-regular, there exists a T$_j$-pre-open set U and a T$_i$-pre-open set V such that x $\in$ V, y $\in$ U and U $\cap$ V = $\phi$. Hence X is pairwise pre-T$_2$. 
THEOREM 3.2.5: Let \( f: (X, T_1, T_2) \to (Y, T_1^*, T_2^*) \) be a pairwise closed, pairwise continuous and pairwise pre-irresolute one-one mapping. Then \( X \) is pairwise gp-regular if \( Y \) is pairwise gp-regular.

PROOF: Let \( Y \) be pairwise gp-regular and let \( G \) be any \( T_1 \)-g-closed set in \( X \) such that \( x \notin G \). Then \( y \notin f(G) \), where \( y = f(x) \) and \( f(G) \) is a \( T_1^* \)-g-closed set in \( Y \) since \( f \) is pairwise closed and pairwise continuous. By pairwise gp-regularity of \( Y \), there exists a \( T_1^* \)-pre-open set \( U \) and a \( T_j^* \)-pre-open set \( V \) in \( Y \) such that \( y \in U \), \( f(G) \subseteq V \) and \( U \cap V = \emptyset \). Hence \( x \in f^{-1}(U) \) and \( G \subseteq f^{-1}(V) \) with \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \). Also \( f^{-1}(U) \) and \( f^{-1}(V) \) are respectively \( T_1 \)-pre-open and \( T_j \)-pre-open sets in \( X \), since \( f \) is pairwise pre-irresolute. Hence \( X \) be pairwise gp-regular.

THEOREM 3.2.6: Let \( f: (X, T_1, T_2) \to (Y, T_1^*, T_2^*) \) be a pairwise closed, pairwise continuous and pairwise strongly pre-closed surjection. Then \( Y \) is pairwise gp-regular if \( X \) is pairwise gp-regular.

PROOF: Let \( X \) be pairwise gp-regular. Let \( F \) be any \( T_1^* \)-g-closed subset in \( Y \) such that \( y \notin F \). Then \( x \notin f^{-1}(F) \), where \( y = f(x) \) and \( f^{-1}(F) \) is \( T_1 \)-g-closed since \( f \) is pairwise closed and pairwise continuous. By pairwise gp-regularity, there exists a \( T_1 \)-pre-open set \( U \) and a \( T_j \)-pre-open set \( V \) such that \( x \in U \), \( f^{-1}(F) \subseteq V \) and \( U \cap V = \emptyset \). Let \( U_1 = Y - f(X - U) \), \( V_1 = Y - f(X - V) \). Since \( f \) is strongly pre-closed, therefore \( U_1 \) and \( V_1 \) are respectively \( T_1^* \)-pre-open and \( T_j^* \)-pre-open sets such that \( y \in U_1 \) and \( F \subseteq V_1 \) with \( U_1 \cap V_1 = \emptyset \). Hence \( Y \) is pairwise gp-regular.

3. gp-Normal Spaces:

By replacing open set by pre-open set in g-normality due to Munshi [Mun], we introduce a new concept of gp-normality.
DEFINITION 3.3.1: A space $X$ is said to **gp-normal** if for any two disjoint g-closed sets $A$ and $B$, there exist disjoint pre-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

Clearly, every g-normal space is gp-normal but converse is not true.

EXAMPLE 3.3.2: Let $X = \{a, b, c\}$, $T = \{\emptyset, \{a\}, \{b, c\}, X\}$,

$GC (X, T) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$.

$PO (X, T) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$.

Then the space $X$ is gp-normal but not g-normal.

THEOREM 3.3.3: For a space $X$ the following are equivalent:

(a) $X$ is gp-normal.

(b) For each g-closed set $F$ and a g-open set $K$ containing $F$, there exists a pre-open set $U$ such that $F \subseteq U \subseteq pcl U \subseteq K$.

(c) For every g-closed set $A$ and a g-closed set $B$ disjoint from $A$, there exists a pre-open set $U$ containing $A$ such that $pcl U \cap B = \emptyset$.

PROOF (a) → (b): Let $X$ be gp-normal and let $K$ be a g-open set containing a g-closed set $F$. Then $F$ and $X - K$ are disjoint g-closed sets. So by (a), there exist a pre-open set $U$ and a pre-open set $V$ such that $F \subseteq U$, $X - K \subseteq V$ and $U \cap V = \emptyset$. Thus $U \subseteq X - V$, which implies that $pcl U \subseteq X - V$. Hence, $F \subseteq U \subseteq pcl U \subseteq K$.

(b) → (c): Let $A$ and $B$ be g-closed subsets of $X$ such that $A \cap B = \emptyset$, which implies $A \subseteq X - B$, a g-open set. So by (b), there exists a pre-open set $U$ such that $A \subseteq U \subseteq pcl U \subseteq X - B$. Hence $pcl U \cap B = \emptyset$.

(c) → (a): Let $A$ and $B$ be disjoint g-closed sets. Then, by (c), there is a pre-open set $U$ such that $A \subseteq U$ and $pcl U \cap B = \emptyset$. Now $pcl U$ is pre-closed. Hence, $B \subseteq X - pcl U$.
U, let \( V = X - \text{pcl} \ U \). Then \( V \) is a pre-open set such that \( B \subseteq V \) and \( U \cap V = \emptyset \).

Hence, \( X \) is gp-normal.

**THEOREM 3.3.4:** Let \( f : X \to Y \) be a closed, continuous, pre-irresolute one-one mapping. Then \( X \) is gp-normal if \( Y \) is gp-normal.

**PROOF:** Let \( Y \) be gp-normal. Let \( A \) and \( B \) be two disjoint g-closed subsets in \( X \). Then \( f \ (A) \) and \( f \ (B) \) are g-closed sets in \( Y \), since \( f \) is closed and continuous. By hypothesis, there exist disjoint pre-open sets \( U \) and \( V \) in \( Y \) such that \( f \ (A) \subseteq U, f \ (B) \subseteq V \). Hence \( A \subseteq f^{-1}(U), B \subseteq f^{-1}(V) \), and \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \) as \( U \cap V = \emptyset \).

Moreover \( f^{-1}(U) \) and \( f^{-1}(V) \) are pre-open sets as \( f \) is pre-irresolute. Hence \( X \) is gp-normal.

**THEOREM 3.3.5:** Let \( f : X \to Y \) be a closed, continuous and strongly pre-closed surjective mapping. Then \( Y \) is gp-normal if \( X \) is gp-normal.

**PROOF:** Let \( X \) be gp-normal. Let \( A \) and \( B \) be two disjoint g-closed subsets in \( Y \). Then \( f^{-1}(A) \) and \( f^{-1}(B) \) are g-closed sets in \( X \) since \( f \) is closed and continuous. By hypothesis, there exist disjoint pre-open sets \( U \) and \( V \) in \( X \) such that \( f^{-1}(A) \subseteq U, f^{-1}(B) \subseteq V \). Let \( U_1 = Y - f (X - U), V_1 = Y - f (X - V) \). Since \( f \) is strongly pre-closed, therefore \( U_1 \) and \( V_1 \) are pre-open sets such that \( A \subseteq U_1 \) and \( B \subseteq V_1 \) with \( U_1 \cap V_1 = \emptyset \).

Hence, \( Y \) is gp-normal.

**THEOREM 3.3.6:** Every gp-normal symmetric space \( X \) is gp-regular.

**PROOF:** Let \( F \) be a g-closed subset of \( X \) with \( x \not\in F \). Since \( X \) is symmetric, so \( \{x\} \) is g-closed. Therefore, \( \{x\} \) and \( F \) are disjoint g-closed sets in \( X \). Since \( X \) is gp-normal, there exist disjoint pre-open sets \( U \) and \( V \) such that \( \{x\} \subseteq U, F \subseteq V \). Hence \( X \) is gp-regular.
4. Pairwise gp-Normal Spaces:

DEFINITION 3.4.1: A bitopological space \((X, T_1, T_2)\) space is said to **pairwise gp-normal** if for every pair of disjoint \(T_i\)-g-closed set \(A\) and \(T_j\)-g-closed set \(B\), there exists a \(T_j\)-pre-open set \(U\) and a \(T_i\)-pre-open set \(V\) such that \(A \subseteq U\), \(B \subseteq V\) and \(U \cap V = \emptyset\); \(i \neq j\), \(i, j = 1, 2\). Evidently, every pairwise g-normal space is pairwise gp-normal. The converse is however not true.

EXAMPLE 3.4.2: Example 3.3.2 shows that the space \((X, T_1, T_2)\) is pairwise gp-normal but not pairwise g-normal.

THEOREM 3.4.3: For a bitopological space \((X, T_1, T_2)\), the following are equivalent:

(a) \((X, T_1, T_2)\) is pairwise gp-normal.

(b) For each \(T_i\)-g-closed set \(F\) and a \(T_j\)-g-open set \(K\) containing \(F\), there exists a \(T_j\)-pre-open set \(U\) such that \(F \subseteq U \subseteq T_i\)-pcl \(U \subseteq K\); \(i \neq j\), \(i, j = 1, 2\).

(c) For every \(T_i\)-g-closed set \(A\) and a \(T_j\)-g-closed set \(B\) disjoint from \(A\), there exists a \(T_j\)-pre-open set \(U\) containing \(A\) such that \((T_i\)-pcl \(U) \cap B = \emptyset\); \(i \neq j\), \(i, j = 1, 2\).

PROOF (a) \(\rightarrow\) (b): Let \(X\) be pairwise gp-normal and let \(K\) be a \(T_j\)-g-open set containing a \(T_i\)-g-closed set \(F\). Then \(F\) and \(X - K\) are disjoint \(T_i\)-g-closed and \(T_j\)-g-closed sets respectively. So by (a), there exists a \(T_i\)-pre-open set \(U\) and a \(T_i\)-pre-open set \(V\) such that \(F \subseteq U\), \(X - K \subseteq V\) and \(U \cap V = \emptyset\). Thus \(U \subseteq X - V\), which implies that \(T_i\)-pcl \(U \subseteq X - V\). Hence \(F \subseteq U \subseteq T_i\)-pcl \(U \subseteq K\).

(b) \(\rightarrow\) (c): Let \(A\) and \(B\) are respectively \(T_i\)-g-closed and \(T_j\)-g-closed subsets of \(X\) such that \(A \cap B = \emptyset\), which implies \(A \subseteq X - B\), a \(T_j\)-g-open set. So by (b), there exists a \(T_j\)-pre-open set \(U\) such that \(A \subseteq U \subseteq T_i\)-pcl \(U \subseteq X - B\). Hence \(T_i\)-pcl \(U \cap B = \emptyset\).
(c) $\rightarrow$ (a): Let A be $T_i$-$g$-closed and B be a $T_j$-$g$-closed set disjoint from A. Then, by (c), there is a $T_j$-pre-open set U such that $A \subseteq U$ and $(T_i$-pcl $U) \cap B = \emptyset$.

Now $T_i$-pcl U is $T_i$-pre-closed. Hence $B \subseteq X - T_i$-pcl U, let $V = X - T_i$-pcl U. Then $V$ is a $T_i$-pre-open set such that $B \subseteq V$ and $U \cap V = \emptyset$. Hence X is pairwise gp-normal.

THEOREM 3.4.4: Let $f: (X, T_1, T_2) \rightarrow (Y, T_1^*, T_2^*)$ be a pairwise closed, pairwise continuous, pairwise pre-irresolute one-one mapping. Then X is pairwise gp-normal if Y is pairwise gp-normal.

PROOF: Let Y be pairwise gp-normal. Let A and B be two disjoint $T_i$-$g$-closed and $T_j$-$g$-closed sets in X respectively. Then $f(A)$ is $T_i$-$g^*$-closed and $f(B)$ $T_j$-$g^*$-closed in Y since f is pairwise closed and pairwise continuous. Since Y is pairwise gp-normal, there exists a $T_j$-$g^*$-pre-open set U and a $T_i$-$g^*$-pre-open set V in Y such that $f^{-1}(A) \subseteq U$, $f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. Hence $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$, and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ as $U \cap V = \emptyset$. Moreover $f^{-1}(U)$ and $f^{-1}(V)$ are $T_i$-pre-open and $T_j$-pre-open sets respectively as f is pairwise pre-irresolute. Hence X is pairwise gp-normal.

THEOREM 3.4.5: Let $f: (X, T_1, T_2) \rightarrow (Y, T_1^*, T_2^*)$ be a pairwise closed, pairwise continuous and pairwise strongly pre-closed surjection. Then Y is pairwise gp-normal if X is pairwise gp-normal.

PROOF: Let X is pairwise gp-normal. Let A and B be two disjoint $T_i$-$g$-closed and $T_j$-$g$-closed sets in Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are $T_i$-$g$-closed and $T_j$-$g$-closed sets in X, since f is pairwise closed and pairwise continuous. Since X is pairwise gp-normal, there exists a $T_j$-pre-open set U and a $T_i$-pre-open sets V in X such that $f^{-1}(A) \subseteq U$, $f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. Hence Y is pairwise gp-normal.
f^{-1}(B) \subseteq V and U \cap V = \phi. Let U_1 = Y - f(X - U), V_1 = Y - f(X - V). Since f is pairwise strongly pre-closed, therefore U_1 and V_1 are respectively T_j^*-pre-open and T_i^*-pre-open sets such that A \subseteq U_1 and B \subseteq V_1 with U_1 \cap V_1 = \phi. Hence Y is pairwise gp-normal.

THEOREM 3.4.6: Every pairwise gp-normal bi-symmetric space X is pairwise gp-regular.

PROOF: Let F be a T_j-g-closed subset of X with x \notin F. Since X is bi-symmetric so \{x\} is T_i-g-closed, i = 1, 2. So \{x\} and F are disjoint T_i-g-closed and T_j-g-closed sets respectively in X. Since X is pairwise gp-normal, there exist disjoint T_j-pre-open set U and T_i-pre-open set V such that \{x\} \subseteq U, F \subseteq V. Hence X is pairwise gp-regular.

5. gR_0-spaces:

By replacing ‘cl’ by ‘gcl’ in the definition of R_0-spaces [Def. 1.1], we introduce the concept of gR_0-spaces as follows:

DEFINITION 3.5.1: A space X is said to be a gR_0-space if x \notin gcl \{y\} implies that y \notin gcl \{x\}. Clearly, every R_0 space is a gR_0-space but not conversely.

EXAMPLE 3.5.2: Let X = \{a, b, c\}, T = \{\phi, \{a\}, X\}.

GC (X, T) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}.

Then the space X is a gR_0-space but not a R_0-space.

THEOREM 3.5.3: A space X is a gR_0-space if and only if for each g-open set S and each x \in S, gcl \{x\} \subseteq S.

PROOF: Let S be a g-open set containing x and let y \notin S. Then x \notin gcl \{y\}. By gR_0 axiom, y \notin gcl \{x\}. Thus, gcl \{x\} \subseteq S.
Conversely, let \( x \notin \text{gcl} \{y\} \). So there is a g-open set \( S \) (say) containing \( x \), which has empty intersection with \( \{y\} \), i.e., \( y \notin S \). By hypothesis, \( \text{gcl} \{x\} \subseteq S \) and thus, \( y \notin \text{gcl} \{x\} \). Hence \( X \) is a gR\(_0\)-space.

We introduce the following definition of g-kernal:

**DEFINITION 3.5.4:** In a topological space \((X, T)\), the **g-kernal** of a point \( x \) of \( X \) is the set

\[
g-\text{ker} \{x\} = \{y: x \in \text{gcl}\{y\}\};
\]

and the g-kernal of a subset \( A \) of \( X \) is the set

\[
g-\text{ker} A = \cap \{U: U \text{ is g-open and } A \subseteq U\}.
\]

**THEOREM 3.5.5:** For a space \( X \), the following are equivalent:

(a) \( X \) is a gR\(_0\)-space.

(b) For each \( x \in X \), \( \text{gcl} \{x\} \subseteq g-\text{ker} \{x\} \).

(c) If \( F \) is a g-closed set in \( X \), then \( F \) is the intersection of all the g-open sets containing \( F \).

(d) If \( S \) is a g-open set in \( X \), then \( S \) is the union of all g-closed sets in \( X \) contained in \( S \).

(e) For \( A \neq \emptyset \), and a g-open set \( S \) in \( X \) such that \( S \cap A \neq \emptyset \), there exists a g-closed set \( F \subseteq S \) such that \( F \cap A \neq \emptyset \).

(f) For any g-open set \( F \) in \( X \) and \( x \notin F \), \( \text{gcl} \{x\} \cap F = \emptyset \).

**PROOF** (a) \( \rightarrow \) (b): Let \( y \in \text{gcl} \{x\} \) and \( S \) be a g-open set containing \( x \). Since \( X \) is a gR\(_0\)-space therefore by theorem 3.5.3, \( \text{gcl} \{x\} \subseteq S \) and thus \( y \in S \). Therefore, \( x \in \text{gcl} \{y\} \), i.e., \( y \in \text{g-ker} \{x\} \). Hence, \( \text{gcl} \{x\} \subseteq \text{g-ker} \{x\} \).

(b) \( \rightarrow \) (c): Let \( x \notin F \). Then \( X - F \) is a g-open set containing \( x \). If \( y \in \text{gcl} \{x\} \), then from (b), \( y \in \text{g-ker} \{x\} \) and therefore \( x \in \text{gcl} \{y\} \). So \( y \in X - F \). Hence, \( \text{gcl} \{x\} \subseteq \text{gcl} \{y\} \).
X – F, which implies, F ⊆ X – gcl {x} is a g-open set that does not contain x. Thus, x does not belong to the intersection of all the g-open sets, which contain F. Hence, (c) holds.

(c) → (d): By taking complements of (c), we get (d).

(d) → (e): Since S ∩ A ≠ ∅, therefore let x ∈ S ∩ A. Then x ∈ g-open set S.
Therefore, from (d), S is the union of all the g-closed sets of X contained in S. Hence there exists a g-closed set F (say) such that x ∈ F ⊆ S, which implies that F ∩ A ≠ ∅. Thus, (e) holds.

(e) → (f): Let F be a g-open set in X and x ∉ F. Then X – F is a g-closed set in X such that (X – F) ∩ {x} ≠ ∅. Therefore, from (e), there is a g-closed set K such that K ⊆ X – F and K ∩ {x} ≠ ∅. So gcl {x} ⊆ X – F. Hence gcl {x} ∩ F = ∅. Thus, (f) is true.

(f) → (a): Let S be a g-open set containing x. Then, from (f), we have (X – S) ∩ gcl {x} = ∅ and hence gcl {x} ⊆ S. Thus by theorem 3.5.3, X is a gR₀-space.

THEOREM 3.5.6: A gR₀-space X is g₁ if it is T₀.

PROOF: Let x ≠ y ∈ T₀-space X. Then, there exists an open set G containing x but not y. Since X is gR₀-space therefore, by theorem 3.5.3, gcl {x} ⊆ G. Also y ∉ gcl {x}. Take H = X – gcl {x}, which is a g-open set containing y but not x. In addition, every open set is g-open. Thus g-open sets G and H satisfy the requirement of g₁-axiom for the space X.

THEOREM 3.5.7: If f is a closed and continuous mapping from a gR₀-space X to a Space Y, then Y is also a gR₀-space.
PROOF: Let \( y_1 \) and \( y_2 \in Y \) and \( y_1 \not\in \text{gcl} \{y_2\} \). Then there exists a g-open set \( V_1 \) in \( Y \) such that \( y_1 \in V_1 \) and \( y_2 \not\in V_1 \). Put \( f^{-1}(V_1) = G \). Since \( f \) is closed and continuous therefore \( G \) is g-open in \( X \) [Lev2]. Also \( f^{-1}(y_1) \in G \), \( f^{-1}(y_2) \cap G = \phi \). Let \( x_1 \in f^{-1}(y_1) \) and \( x_2 \in f^{-1}(y_2) \). Then \( x_1 \in G \) and \( x_2 \not\in G \). Therefore, \( x_1 \not\in \text{gcl} \{x_2\} \). By gR0-axiom on \( X \), \( x_2 \not\in \text{gcl} \{x_1\} \). Thus there is a g-open set \( X - \text{gcl} \{x_1\} = V_{x_2} \) (say) containing \( x_2 \) but not \( x_1 \). Let \( V = \bigcup \{ V_{x_2} : x_2 \in f^{-1}(y_2) \} \). Then \( V \) is a g-open set in \( X \) containing \( f^{-1}(y_2) \) but not \( x_1 \). So \( X - V \) is a g-closed set in \( X \). Since \( f \) is closed and continuous, so \( f(X - V) \) is g-closed in \( Y \) [Lev2] not containing \( y_1 \) but not \( y_2 \). Hence \( y_2 \not\in \text{gcl} \{y_1\} \). Thus, \( Y \) is a gR0-space.

6. Pairwise gR0-Spaces:

Here we generalize the concept of pairwise R0 of of Murdeshwar and Naimpally [MN] using g-open sets and introduce pairwise gR0.

**LEMMA 3.6.1:** Let \( A \) be a subset of a space \((X, T)\). Then

\[
g\ker A = \{ x \in X : \text{gcl} \{x\} \cap A \neq \phi \}.
\]

**PROOF:** By def 3.5.4 \( x \not\in g\ker A \) implies \( x \not\in \bigcap \{ U : U \text{ is g-open and } A \subseteq U \} \).

Therefore, there is a g-open set \( U \) such that \( A \subseteq U \) and \( x \not\in U \). Hence \( \text{gcl} \{x\} \cap A = \phi \). Now, \( \text{gcl} \{x\} \cap A = \phi \) implies \( X - \text{gcl} \{x\} = G \) (say) is a g-open set such that \( A \subseteq G \). Also, \( x \) does not belong to the intersection of all g-open neighbourhoods of \( A \) and so \( x \not\in g\ker A \).

**DEFINITION 3.6.2:** A space \((X, T_1, T_2)\) is said to be pairwise gR0 if for every \( T_i \)-g-open set \( G \), \( x \in G \) implies that \( T_j\text{-gcl} \{x\} \subseteq G \); \( i \neq j, i, j = 1, 2 \).
Every pairwise R₀-space is pairwise gR₀-space but converse is not true as shown in the following example.

EXAMPLE 3.6.3: Let \( X = \{a, b, c\} \), \( T_1 = \{\emptyset, \{a\}, X\} \), \( T_2 = \{\emptyset, \{b\}, X\} \).

\[
\text{GC}(X, T_1) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\},
\]

\[
\text{GC}(X, T_2) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}.
\]

Clearly, \((X, T_1, T_2)\) is a pairwise gR₀-space. But it is not pairwise R₀.

THEOREM 3.6.4: For a bitopological space \( X \) the following are equivalent:

(a) \( X \) is a pairwise gR₀-space.

(b) For each \( x \in X \), \( T_i\)-gcl \( \{x\} \subseteq T_j\)-g-ker \( \{x\} \); \( i \neq j, i, j = 1, 2 \).

(c) For any \( x, y \in X \), \( y \in T_i\)-g-ker \( \{x\} \) if and only if \( x \in T_j\)-gcl \( \{y\} \);

\[
i \neq j, i, j = 1, 2.
\]

(d) For any \( x, y \in X \), \( y \in T_i\)-gcl \( \{x\} \) if and only if \( x \in T_j\)-gcl \( \{y\} \);

\[
i \neq j, i, j = 1, 2.
\]

(e) For any \( T_i\)-g-closed set \( F \) and a point \( x \in X - F \), there exists a \( T_j\)-g-open set \( U \) such that \( x \notin U \) and \( F \subseteq U \); \( i \neq j, i, j = 1, 2 \).

(f) Each \( T_i\)-g-closed set \( F \) can be expressed as \( F = \cap \{G: G \text{ is } T_j\text{-g-open and } F \subseteq G\} \); \( i \neq j, i, j = 1, 2 \).

(g) Each \( T_i\)-g-open set \( G \) can be expressed as the union of \( T_j\)-g-closed sets contained in \( G \); \( i \neq j, i, j = 1, 2 \).

(h) For each \( T_i\)-g-closed set \( F, x \notin F \) implies \( T_i\)-gcl \( \{x\} \cap F = \emptyset \); \( i \neq j, i, j = 1, 2 \).

PROOF (a) \( \rightarrow \) (b): By definition 3.5.4, for each \( x \in X \), we have \( T_j\)-gker \( \{x\} = \{y: y \in T_j\text{-gcl } \{x\}\} \), and by definition of pairwise gR₀, each \( T_j\)-g-open set \( G \) containing \( x \) also contains \( T_i\)-gcl \( \{x\} \). Hence \( T_i\)-gcl \( \{x\} \subseteq T_j\text{-gker } \{x\} \) for each \( x \in X \); \( i \neq j, i, j = 1, 2 \).
(b) → (c): For any \( x, y \in X \), if \( y \in T_i\text{-}\ker \{x\} \) then \( x \in T_i\text{-}\ker \{y\} \) and hence by (b), \( x \in T_j\text{-}\ker \{y\} \).

(c) → (d): For any \( x, y \in X \), if \( y \in T_i\text{-}\cl \{x\} \) then \( x \in T_i\text{-}\ker \{y\} \) and hence by (c), \( y \in T_j\text{-}\ker \{x\} \) implies \( x \in T_j\text{-}\cl \{y\} \).

(d) → (e): Let \( F \) be a \( T_i\text{-}g \)-closed set and \( x \not\in F \). Then for any \( y \in F \), \( T_i\text{-}\cl \{y\} \subseteq F \) and so \( x \not\in T_i\text{-}\cl \{y\} \). Now by (d), \( x \not\in T_i\text{-}\cl \{y\} \) implies \( y \not\in T_j\text{-}\cl \{x\} \). This means that there exists a \( T_j\text{-}g \)-open set \( G_y \) such that \( y \in G_y \) and \( x \not\in G_y \). Let \( G = \bigcup_{y \in F} \{G_y: G_y \text{ is } T_j\text{-}g \text{-open, } y \in G_y \text{ and } x \not\in G_y \} \). Then \( G \) is a \( T_j\text{-}g \)-open set such that \( x \not\in G \) and \( F \subseteq G; i \neq j, i, j = 1, 2 \).

(e) → (f): Let \( F \) be a \( T_i\text{-}g \)-closed set and suppose that \( H = \bigcap \{G: G \text{ is } T_j\text{-}g \text{-open and } F \subseteq G\} \). Then \( F \subseteq H \) and it remains to show that \( H \subseteq F \). Let \( x \not\in F \). Then by (e), there is a \( T_j\text{-}g \)-open set \( G \) such that \( x \not\in G \) and \( F \subseteq G \). Hence \( x \not\in H \). Therefore, each \( T_i\text{-}g \)-closed set \( F \) can be written as \( F = \bigcap \{G: G \text{ is } T_j\text{-}g \text{-open and } F \subseteq G\}; i \neq j, i, j = 1, 2 \).

(f) → (g): Obvious.

(g) → (h): Let \( F \) be a \( T_i\text{-}g \)-closed set and \( x \not\in F \). Then \( X - F \) is a \( T_i\text{-}g \)-open set containing \( x \). Then by (g), \( X - F \) can be written as the union of \( T_j\text{-}g \)-closed sets and so there is a \( T_j\text{-}g \)-closed set \( C \) such that \( x \in C \subseteq X - F \). Hence, \( T_j\text{-}\cl \{x\} \subseteq X - F \). Thus \( T_j\text{-}\cl \{x\} \cap F = \phi \).

(h) → (a): Let \( G \) be a \( T_i\text{-}g \)-open set containing a point \( x \) of \( X \). Then \( x \not\in X - G \) which is \( T_i\text{-}g \)-closed set. By (h), \( T_j\text{-}\cl \{x\} \cap X - G = \phi \) which implies that \( T_j\text{-}\cl \{x\} \subseteq G; i \neq j, i, j = 1, 2 \). Hence \( (X, T_1, T_2) \) is pairwise gR0.

THEOREM 3.6.5: A pairwise gR0-space \( X \) is pairwise g_1 if it is pairwise T_0.
PROOF: Let X be a pairwise gR₀ and pairwise T₀-space. Let \( x \neq y \in X \). Since X is pairwise T₀-space, then there exists a T₁-open set G containing x but not y. Since X is a pairwise gR₀-space, then \( T_j\text{-gcl} \{x\} \subseteq G \) and the fact that every open set is g-open. Also \( y \notin T_j\text{-gcl} \{x\} \). Take \( H = X - T_j\text{-gcl} \{x\} \), which is a g-open set containing y but not x. Thus open sets G and H satisfy the requirement of pairwise g₁.

DEFINITION 3.6.6: In a bitopological space \((X, T_1, T_2)\) for any \( x \in X \), we introduce the following notations:

(i) \( \text{bi-gcl} \{x\} = T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{x\} \)

(ii) \( \text{bi-gker} \{x\} = T_1\text{-gker} \{x\} \cap T_2\text{-gker} \{x\} \)

THEOREM 3.6.7: In a pairwise gR₀-space \((X, T_1, T_2)\), for any \( x, y \in X \), we have either \( \text{bi-gcl} \{x\} = \text{bi-gcl} \{y\} \) or \( \text{bi-gcl} \{x\} \cap \text{bi-gcl} \{y\} = \emptyset \).

PROOF: Suppose that \( \text{bi-gcl} \{x\} \cap \text{bi-gcl} \{y\} \neq \emptyset \) and let \( p \in [T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{x\}] \cap [T_1\text{-gcl} \{y\} \cap T_2\text{-gcl} \{y\}] \). Then \( T_1\text{-gcl} \{p\} \subseteq T_1\text{-gcl} \{x\} \cap T_1\text{-gcl} \{y\} \) and \( T_2\text{-gcl} \{p\} \subseteq T_2\text{-gcl} \{x\} \cap T_2\text{-gcl} \{y\} \). Also, \( p \in T_1\text{-gcl} \{x\} \) implies \( T_2\text{-gcl} \{x\} \subseteq T_2\text{-gcl} \{p\} \subseteq T_2\text{-gcl} \{y\} \). Similarly, \( p \in T_2\text{-gcl} \{x\} \) implies \( T_1\text{-gcl} \{x\} \subseteq T_1\text{-gcl} \{y\} \).

Also, we have, \( p \in T_1\text{-gcl} \{y\} \) implies \( T_2\text{-gcl} \{y\} \subseteq T_2\text{-gcl} \{x\} \) and \( p \in T_2\text{-gcl} \{y\} \) implies \( T_1\text{-gcl} \{y\} \subseteq T_1\text{-gcl} \{x\} \). Thus \( T_1\text{-gcl} \{x\} = T_1\text{-gcl} \{y\} \) and \( T_2\text{-gcl} \{x\} = T_2\text{-gcl} \{y\} \) and therefore, \( T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{x\} = T_1\text{-gcl} \{y\} \cap T_2\text{-gcl} \{y\} \). This proves the theorem.

THEOREM 3.6.8: In a pairwise gR₀-space \((X, T_1, T_2)\) for any \( x, y \in X \), we have either \( \text{bi-g ker} \{x\} = \text{bi-g ker} \{y\} \) or \( \text{bi-g ker} \{x\} \cap \text{bi-g ker} \{y\} = \emptyset \).
PROOF: By virtue of the statement (c) of theorem 3.6.4, the proof is similar to that of theorem 3.6.7.

THEOREM 3.6.9: If a space \((X, T_1, T_2)\) is pairwise \(g_1\), then it is pairwise \(gR_0\).

PROOF: If \((X, T_1, T_2)\) is pairwise \(g_1\), implies it is bi-\(g_1\). Hence \(T_1\text{-gcl} \{x\} = \{x\} = T_2\text{-gcl} \{x\}\) for each \(x \in X\). Thus if \(x \in G\) and \(G\) is \(T_1\text{-g-open}\) then \(T_1\text{-gcl} \{x\} \subseteq G\). Therefore, \((X, T_1, T_2)\) is pairwise \(gR_0\).

THEOREM 3.6.10: A space \((X, T_1, T_2)\) is pairwise \(gR_0\) if and only if for every pair of distinct points \(x, y\) of \(X\), either \(T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{x\} = \emptyset\) or \(\{x, y\} \subseteq T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{y\}\).

PROOF: Let \(x, y\) be any two distinct points in \((X, T_1, T_2)\). Let \((X, T_1, T_2)\) be pairwise \(gR_0\). Let \(T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{y\} \neq \emptyset\), and \(p \in T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{y\}\). Suppose \(\{x, y\} \not\subseteq T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{y\}\). Then \(x \notin T_2\text{-gcl} \{y\}\) which implies that \(x \in X - T_2\text{-gcl} \{y\}\) which is a \(T_2\text{-g-open}\) set. Since \(p \in T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{y\}\), therefore \(T_1\text{-gcl} \{x\} \not\subseteq X - T_2\text{-gcl} \{y\}\). Hence, the space \((X, T_1, T_2)\) is not pairwise \(gR_0\).

Conversely, let \(G\) be a \(T_1\text{-g-open}\) set containing a point \(x\) of \(X\). Suppose \(T_2\text{-gcl}\{x\} \not\subseteq G\). Let \(y\) be a point in \(T_2\text{-gcl} \{x\}\) such that \(y \notin G\). Then \(T_1\text{-gcl} \{y\} \cap G = \emptyset\), since \(X - G\) is \(T_1\text{-g-closed}\) and \(y \in X - G\). Therefore, \(\{x, y\} \not\subseteq T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{y\}\) and so \(T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{y\} = \emptyset\), which is a contradiction as \(y \in T_1\text{-gcl} \{x\} \cap T_2\text{-gcl} \{y\}\). Hence \((X, T_1, T_2)\) is pairwise \(vR_0\).

6. \(P_g\)-Spaces:

Here we generalize \(R_0\)-spaces of Shanin [Shan] by using \(gcl\).

DEFINITION 3.7.1: A space \(X\) is said to be a \(P_g\text{-space}\) if \(x \notin \text{cl} \{y\}\) implies that \(y \notin \text{gcl} \{x\}\). Clearly, every \(R_0\) space is a \(P_g\text{-space}\) but not conversely.
EXAMPLE 3.7.2: Example 3.5.2 shows that the space X is a $P_g$-space but not a $R_0$-space.

THEOREM 3.7.3: A space X is a $P_g$-space if and only if for each open set S and each $x \in S$, $\text{gcl} \{x\} \subseteq S$.

PROOF: Let S be an open set containing x and let $y \notin S$. Then $x \notin \text{cl} \{y\}$. By $P_g$-axiom, $y \notin \text{gcl} \{x\}$. Thus, $\text{gcl} \{x\} \subseteq S$.

Conversely, let $x \notin \text{cl} \{y\}$. Then there is an open set S (say) containing x, which has empty intersection with {y}, i.e., $y \notin S$. By hypothesis, $\text{gcl} \{x\} \subseteq S$ and thus, $y \notin \text{gcl} \{x\}$. Hence X is a $P_g$-space.

THEOREM 3.7.4: For a space X, the following are equivalent:

(a) X is a $P_g$-space.

(b) For each $x \in X$, $\text{gcl} \{x\} \subseteq \text{ker} \{x\}$.

(c) If F is a closed set in X, then F is the intersection of all the g-open sets containing F.

(d) If S is an open set in X, then S is the union of all the g-closed sets in X contained in S.

(e) For a non-empty set A, and an open set S in X such that $S \cap A \neq \emptyset$, there is a g-closed set $K \subseteq S$ such that $K \cap A \neq \emptyset$.

(f) For any closed set F in X and $x \notin F$, $\text{gcl} \{x\} \cap F = \emptyset$.

PROOF (a) $\rightarrow$ (b): Let $y \in \text{gcl} \{x\}$ and S be an open set containing x. Since X is a $P_g$-space therefore by theorem 3.7.3, $\text{gcl} \{x\} \subseteq S$ and thus $y \in S$. Therefore $x \in \text{gcl} \{y\}$, i.e., $y \in \text{ker} \{x\}$. Hence, $\text{gcl} \{x\} \subseteq \text{ker} \{x\}$.
(b) → (c): Let \( F \) be a closed set. Let \( x \not\in F \). Then \( X - F \) is an open set containing \( x \). If \( y \in \text{gcl} \{x\} \), then from (b), \( y \in \ker \{x\} \) and therefore \( x \in \text{cl} \{y\} \). So \( y \in X - F \). Hence, \( \text{gcl} \{x\} \subseteq X - F \), which implies, \( F \subseteq X - \text{gcl} \{x\} \) is a g-open set that does not contain \( x \). Thus, \( x \) does not belong to the intersection of all the g-open sets, which contain \( F \). Hence, (c) holds.

(c) → (d): By taking complements of (c), we get (d).

(d) → (e): Since \( S \cap A \neq \emptyset \), therefore let \( x \in S \cap A \). Then \( x \in \text{open set } S \). Therefore, from (d), \( S \) is the union of all the g-closed sets of \( X \) contained in \( S \). Hence there exists a g-closed set \( K \) (say) such that \( x \in K \subseteq S \), which implies that \( K \cap A \neq \emptyset \). Thus, (e) holds.

(e) → (f): Let \( F \) be a closed set in \( X \) and \( x \not\in F \). Then \( X - F \) is an open set in \( X \) such that \( (X - F) \cap \{x\} \neq \emptyset \). Therefore, from (e), there is a g-closed set \( K \) such that \( K \subseteq X - F \) and \( K \cap \{x\} \neq \emptyset \). So \( \text{gcl} \{x\} \subseteq X - F \). Hence \( \text{gcl} \{x\} \cap F = \emptyset \). Thus, (f) is true.

(f) → (a): Let \( S \) be an open set containing \( x \). Then, from (f), we have \( (X - S) \cap \text{gcl} \{x\} = \emptyset \) and hence \( \text{gcl} \{x\} \subseteq S \). Thus by theorem 3.7.3, \( X \) is a \( P_g \)-space.

**THEOREM 3.7.5:** A \( P_g \)-space \( X \) is \( g_1 \) if it is \( T_0 \).

**PROOF:** Let \( x \neq y \in T_0 \)-space \( X \). Then, there exists an open set \( G \) containing \( x \) but not \( y \). Since \( X \) is \( P_g \)-space therefore, by theorem 3.7.3, \( \text{gcl} \{x\} \subseteq G \). Therefore, \( y \notin \text{gcl} \{x\} \). Take \( H = X - \text{gcl} \{x\} \) that is a g-open set containing \( y \) but not \( x \). In addition, every open set is g-open. Thus g-open sets \( G \) and \( H \) satisfy the requirement of \( g_1 \)-axiom for the space \( X \).
THEOREM 3.7.6: If $f$ is a closed and continuous mapping from a $P_g$-space $X$ to a $Space Y$, then $Y$ is also a $P_g$-space.

PROOF: Let $y_1$ and $y_2 \in Y$ and $y_1 \not\in \text{cl} \{y_2\}$. Then there exists an open set $V_1$ such that $y_1 \in V_1$ and $y_2 \not\in V_1$. Put $f^{-1}(V_1) = G$. Since $f$ is continuous, therefore $G$ is an open set in $X$. Also $f^{-1}(y_1) \in G$, $f^{-1}(y_2) \cap G = \emptyset$. Let $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$.

Therefore, $x_1 \not\in \text{cl} \{x_2\}$. By $P_g$-axiom on $X$, $x_2 \not\in g\text{cl} \{x_1\}$. Thus, there is a $g$-open set $X - g\text{cl} \{x_1\} = V x_2$ (say) containing $x_2$ but not $x_1$. Let $V = \cup \{V x_2 : x_2 \in f^{-1}(y_2)\}$.

Then $V$ is a $g$-open set in $X$ containing $f^{-1}(y_2)$ but not $x_1$. So $X - V$ is a $g$-closed set in $X$. Since $f$ is closed and continuous, therefore $f(X - V)$ is $g$-closed in $Y$ [Lev2] containing $y_1$ but not $y_2$. Hence $y_2 \not\in \text{gcl} \{y_1\}$. Thus $Y$ is a $P_g$-space.

7. **Pairwise $P_g$-Spaces:**

Here we introduce the bitopological analogue of $P_g$-spaces.

DEFINITION 3.8.1: A bitopological space $(X, T_1, T_2)$ is said to be **pairwise $P_g$-space** if $x \not\in T_i\text{-cl} \{y\} \Rightarrow y \not\in T_j\text{-gcl} \{x\}; i \neq j, i, j = 1, 2$. Clearly, every pairwise $R_0$-space is pairwise $P_g$-space but not conversely.

EXAMPLE 3.8.2: Let $X = \{a, b, c\}$, $T_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$,

$T_2 = \{\emptyset, \{a, b\}, X\}$.

$GC(X, T_1) = \{\emptyset, \{c\}, \{b, c\}, \{a, c\}, X\}$, $GC(X, T_2) = \{\emptyset, \{c\}, \{b, c\}, \{a, c\}, X\}$.

Then $(X, T_1, T_2)$ is a pairwise $P_g$-space but not pairwise $R_0$-space.

THEOREM 3.8.3: A space $X$ is a pairwise $P_g$-space if and only if for each $T_i$-open set $S$ and each $x \in S$, $T_j\text{-gcl} \{x\} \subseteq S; i \neq j, i, j = 1, 2$. 
PROOF: Let $S$ be a $T_i$-open set containing $x$ and let $y \not\in S$. Then $x \not\in T_i\text{-cl} \{y\}$. Since $X$ is a pairwise $P_g$-space, therefore $y \not\in T_j\text{-gcl} \{x\}$. Hence $T_j\text{-gcl} \{x\} \subseteq S$.

Conversely, let $x \not\in T_i\text{-cl} \{y\}$. So there is a $T_i$-open set $S$ (say) containing $x$ but not $y$. By hypothesis, $T_j\text{-gcl} \{x\} \subseteq S$ and thus $y \not\in T_j\text{-gcl} \{x\}$. Hence $X$ is a pairwise $P_g$-space.

THEOREM 3.8.4: For a bitopological space $X$ the following are equivalent.

(a) $X$ is a pairwise $P_g$-space.

(b) For each $x \in X$, $T_j\text{-gcl} \{x\} \subseteq T_i\text{-ker} \{x\}$.

(c) If $F$ is a $T_i$-closed set in $X$, then $F$ is the intersection of all the $T_j$-g-open sets containing $F$.

(d) If $S$ is a $T_i$-open set in $X$, then $S$ the union of all $T_j$-g-closed sets in $X$ contained in $S$.

(e) For $A \neq \emptyset$, and a $T_i$-open set $S$ in $X$ such that $S \cap A \neq \emptyset$, there exists a $T_j$-g-closed set $F \subseteq S$ such that $F \cap A \neq \emptyset$.

(f) For any $T_i$-closed set $F$ in $X$ and $x \not\in F$, $T_j\text{-gcl} \{x\} \cap F = \emptyset$.

PROOF (a) $\rightarrow$ (b): Let $y \in T_j\text{-gcl} \{x\}$ and $S$ be a $T_i$-open set containing $x$. Since $X$ is a pairwise $P_g$-space. Therefore by theorem 3.8.3, $T_j\text{-scl} \{x\} \subseteq S$ and thus $y \in S$. Therefore, $x \in T_i\text{-cl} \{y\}$, i.e. $y \in T_i\text{-ker} \{x\}$. Hence $T_j\text{-gcl} \{x\} \subseteq T_i\text{-ker} \{x\}$.

(b) $\rightarrow$ (c): Let $F$ be a $T_i$-closed set. Let $x \not\in F$. Then $X - F$ is a $T_i$-open set containing $x$. If $y \in T_j\text{-gcl} \{x\}$, then from (b), $y \in T_i\text{-ker} \{x\}$. Therefore $x \in T_i\text{-cl} \{y\}$. So $y \in X - F$. Hence $T_j\text{-gcl} \{x\} \subseteq X - F$, which implies, $F \subseteq X - T_j\text{-gcl} \{x\}$. Therefore
X – $T_{j}$-gcl \{x\} is a $T_{j}$-g-open set that does not contain x. Thus x does not belong to the intersection of all the $T_{j}$-g-open sets, which contain F. Hence (c) holds.

(c) → (d): By taking complements of (c), we get (d).

(d) → (e): Since S ∩ A ≠ ∅, let x ∈ S ∩ A. Then x ∈ $T_{j}$-open set S. Therefore, from (d), S is the union of all the $T_{j}$-g-closed sets contained in S. Hence there exists a $T_{j}$-g-closed set F (say) such that x ∈ F ⊆ S, which implies that F ∩ A ≠ ∅. Thus (e) holds.

(e) → (f): Let F be a $T_{i}$-closed set in X and x ∈ F. Then $X – F$ is a $T_{i}$-open set in X such that $(X – F) \cap \{x\} \neq ∅$. Therefore, from (e), there is a $T_{j}$-g-closed set K such that K ⊆ $X – F$ and K ∩ \{x\} ≠ ∅. So $T_{j}$-gcl\{x\} ⊆ $X – F$. Hence $T_{j}$-gcl\{x\} ∩ F = ∅. Thus (f) is true.

(f) → (a): Let S be a $T_{i}$-open set containing x. Then, from (f), we have $(X – S) \cap T_{j}$-gcl\{x\} = ∅ and hence $T_{j}$-gcl\{x\} ⊆ S. Thus by theorem 3.4.3, X is a pairwise $P_{g}$-space.

THEOREM 3.8.5: A pairwise $P_{g}$-space X is pairwise $g_{1}$ if it is pairwise $T_{0}$.

PROOF: Let x ≠ y ∈ pairwise $T_{0}$-space. Then there exists a $T_{i}$-open set G containing x but not y. Since X is a pairwise $P_{g}$-space by theorem 3.8.3 and the fact that every open set is g-open, $T_{j}$-gcl \{x\} ⊆ G. Also y ∈ $T_{j}$-gcl \{x\}. Take H = $X – T_{j}$-gcl \{x\}, which is a $T_{j}$-g-open set containing y but not x. Thus open sets G and H satisfy the requirement of pairwise $g_{1}$.

THEOREM 3.8.6: If f: (X, $T_{1}$, $T_{2}$) → (Y, $T_{1}^{*}$, $T_{2}^{*}$) is a pairwise closed and pairwise continuous mapping from a $P_{g}$-space X to a space Y, then Y is also a $P_{g}$-space.

PROOF: Let $y_{1}$ and $y_{2}$ ∈ Y and $y_{1} \notin T_{1}^{*}$-cl \{y_{2}\}. Then there exists a $T_{1}^{*}$-open set $V_{1}$ such that $y_{1} \in V_{1}$ and $y_{2} \notin V_{1}$. Put $f^{-1}(V_{1}) = G$. Since f is pairwise continuous therefore G is a $T_{j}$-open set in X. Also $f^{-1}(y_{1}) \in G$, $f^{-1}(y_{2}) \cap G = ∅$. Let $x_{1} \in f^{-1}(y_{1})$.
and \(x_2 \in f^{-1}(y_2)\). Therefore \(x_1 \not\in T_i\text{-cl}\{x_2\}\). By pairwise \(P_g\)-axiom on \(X\), \(x_2 \not\in T_j\text{-gcl}\{x_1\}\). Thus there is a \(T_j\text{-g}\)-open set \(V_x\) in \(X\) containing \(x_2\) but not \(x_1\). \(X - T_j\text{-gcl}\{x_1\} = V_{x_2}\) (say) containing \(x_2\) but not \(x_1\). Let \(V = \cup\{V_{x_2} : x_2 \in f^{-1}(y_2)\}\). Then \(V\) is a \(T_j\text{-g}\)-open set in \(X\) containing \(f^{-1}(y_2)\) but not \(x_1\). So \(X - V\) is a \(T_j\text{-g}\)-closed set in \(X\). Since \(f\) is pairwise closed and pairwise continuous, \(f^{-1}(X - V)\) is \(T_j^*\text{-g}\)-closed in \(Y\) not containing \(y_2\). Hence \(Y - f^{-1}(X - V)\) is a \(T_j^*\text{-g}\)-open set in \(Y\) containing \(y_2\) but not \(y_1\). Hence \(y_2 \not\in T_j^*\text{-gcl}\{y_1\}\). Thus \(Y\) is a pairwise \(P_g\)-space.

9. \(\text{gR}_1\)-Spaces:

By replacing ‘cl’ by ‘gcl’ in the definition of \(R_1\)-spaces [Def 1.3], we introduce the concept of \(\text{gR}_1\)-spaces as follows:

DEFINITION 3.9.1: A space \(X\) is said to be \(\text{gR}_1\) if for every pair of distinct points \(x, y\) of \(X\), with \(\text{gcl}\{x\} \neq \text{gcl}\{y\}\) there exists a \(g\)-open set \(U\) and a \(g\)-open set \(V\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\). Clearly, every \(R_1\)-space is \(\text{gR}_1\) but not conversely.

EXAMPLE 3.9.2: Let \(X = \{a, b, c, d\}, T = \{\emptyset, \{a, b\}, X\}\), \(\text{GO}(X, T) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}\).

Clearly, \((X, T)\) is a \(\text{gR}_1\) space. However, it is not \(R_1\).

THEOREM 3.9.3: Every \(g_2\)-space is \(\text{gR}_1\)-space.

PROOF: By definition, \(X\) is said to be \(g_2\) if for each pair of distinct points \(x, y\) of \(X\), there is a \(g\)-open set \(U\) and a \(g\)-open \(V\) such that \(x \in U\) and \(y \in V\) and \(U \cap V = \emptyset\).

Therefore, \(y \not\in \text{gcl}\{x\}\) and \(x \not\in \text{gcl}\{y\}\). Hence, \(X\) is \(\text{gR}_1\).

THEOREM 3.9.4: Every \(g_1\) and \(\text{gR}_1\)-space is \(g_2\).
PROOF: Let X be g\(_1\) and gR\(_1\)-space. Let x, y be two distinct points of X. Since X is g\(_1\) therefore gcl \{x\} \neq gcl \{y\}. Since X be gR\(_1\), there exists a g-open set U and a g-open set V such that x \in U, y \in V and U \cap V = \emptyset. Hence, X is g\(_2\).

THEOREM 3.9.5: Every gR\(_1\)-space is gR\(_0\)-space.

PROOF: Let X be gR\(_1\)-space. Let G be any g-open set and x \in G. For each y \in X \setminus G, gcl \{x\} \neq gcl \{y\}. Therefore there exists a g-open set U\(_y\) and a g-open set V\(_y\) such that x \in U\(_y\), y \in V\(_y\) and U\(_y\) \cap V\(_y\) = \emptyset. If A = \bigcup \{V\(_y\) : y \in X \setminus G\}, then X \setminus G \subseteq A and x \notin A. g-openness of A implies gcl \{x\} \subseteq X \setminus A \subseteq G. Hence, X is gR\(_0\).

THEOREM 3.9.6: A space X is gR\(_1\) if and only if for every pair of distinct points x, y of X such that gcl \{x\} \neq gcl \{y\}, there exists a g-open set U and a g-open V such that gcl \{x\} \subseteq V, gcl \{x\} \subseteq U and U \cap V = \emptyset.

PROOF: Let X be gR\(_1\)-space. Let x, y be two distinct points of X such that gcl \{x\} \neq gcl \{y\}, then there exists a g-open set U and a g-open set V such that x \in V, y \in U and U \cap V = \emptyset. Since a gR\(_1\)-space is gR\(_0\), therefore x \in V implies gcl\{x\} \subseteq V and y \in U implies gcl\{y\} \subseteq U. Hence, the result follows. The converse is obvious.

10. **Pairwise gR\(_1\)-Spaces:**

Here we generalize the concept of pairwise R\(_1\) of of Murdeshwar and Naimpally [MN] using g-open sets and introduce pairwise gR\(_1\).

DEFINITION 3.10.1: A space \((X, T_1, T_2)\) is said to be **pairwise gR\(_1\)** if for every pair of distinct points x, y of X, with T\(_1\)-gcl \{x\} \neq T\(_j\)-gcl \{y\}, there exists a T\(_j\)-g-open set U and a T\(_i\)-g-open set V such that x \in U, y \in V and U \cap V = \emptyset; i \neq j, i, j = 1, 2.

Every pairwise R\(_1\) space is pairwise gR\(_1\) but converse is not true.

EXAMPLE 3.10.2: Let \(X = \{a, b, c\}, T_1 = \{\emptyset, \{a\}, \{b, c\}, X\}\),
\[ T_2 = \{ \phi, \{ b \}, \{ a, c \}, X \}. \]

\[
\text{GC} (X, T_1) = \{ \phi, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ b, c \}, \{ a, c \}, X \},
\]

\[
\text{GC} (X, T_2) = \{ \phi, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ b, c \}, \{ a, c \}, X \},
\]

Clearly \((X, T_1, T_2)\) is a pairwise gR\(_1\) space. But it is not pairwise R\(_1\).

**THEOREM 3.10.3:** Every pairwise g\(_2\)-space is pairwise gR\(_1\)-space.

**PROOF:** By definition, \((X, T_1, T_2)\) is said to be pairwise g\(_2\) if for each pair of distinct points \(x, y\) of \(X\), there is a \(T_i\)-g-open set \(U\) and a \(T_j\)-g-open \(V\) such that \(x \in U\) and \(y \in V\) and \(U \cap V = \phi\). Therefore \(y \notin T_i\text{-gcl }\{x\}\) and \(x \notin T_j\text{-gcl }\{y\}\). Hence \((X, T_1, T_2)\) is pairwise gR\(_1\).

**THEOREM 3.10.4:** Every pairwise g\(_1\) and pairwise gR\(_1\)-space is pairwise g\(_2\).

**PROOF:** Let \((X, T_1, T_2)\) be pairwise g\(_1\) and pairwise gR\(_1\). Let \(x, y\) be two distinct points of \(X\). Since the pairwise g\(_1\) space is bi-g\(_1\), therefore \(\{x\}\) is \(T_2\)-g-closed and \(\{y\}\) is \(T_1\)-g-closed. Hence \(T_2\text{-gcl }\{x\} \neq T_1\text{-gcl }\{y\}\). Since \((X, T_1, T_2)\) be pairwise gR\(_1\), there exist a \(T_1\)-g-open set \(U\) and a \(T_2\)-g-open set \(V\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \phi\). Hence \((X, T_1, T_2)\) is pairwise g\(_2\).

**THEOREM 3.10.5:** Every pairwise gR\(_1\)-space is pairwise gR\(_0\)-space.

**PROOF:** Let \((X, T_1, T_2)\) be pairwise gR\(_1\)-space. Let \(G\) be any \(T_i\)-g-open set and \(x \in G\). For each \(y \in X - G\), \(T_j\text{-gcl }\{x\} \neq T_i\text{-gcl }\{y\}\). Therefore, there exists a \(T_i\)-g-open set \(U_y\) and a \(T_j\)-g-open set \(V_y\) such that \(x \in U_y\), \(y \in V_y\) and \(U_y \cap V_y = \phi\). If \(A = \{ V_y: y \in X - G \}\), then \(X - G \subseteq A\) and \(x \notin A\). \(T_j\text{-gopenness of } A \text{ implies } T_j\text{-gcl }\{x\} \subseteq X - A \subseteq G\). Hence \((X, T_1, T_2)\) is pairwise gR\(_0\).

**THEOREM 3.10.6:** A space \((X, T_1, T_2)\) is pairwise gR\(_1\) if and only if for every pair of distinct points \(x, y\) of \(X\) such that \(T_i\text{-gcl }\{x\} \neq T_j\text{-gcl }\{y\}\), there exists a \(T_i\)-g-open set.
U and a $T_j$-g-open $V$ such that $T_i$-gcl $\{x\} \subseteq V$, $T_j$-gcl $\{y\} \subseteq U$ and $U \cap V = \phi$; $i \neq j$, $i$, $j = 1$, 2.

PROOF: Let $(X, T_1, T_2)$ be pairwise $gR_1$-space. Let $x$, $y$ be two distinct points of $X$ such that $T_i$-gcl $\{x\} \neq T_j$-gcl $\{y\}$, then there exists a $T_i$-g-open set $U$ and a $T_j$-g-open set $V$ such that $x \in V$, $y \in U$ and $U \cap V = \phi$. Since a pairwise $gR_1$-space is pairwise $gR_0$, therefore $x \in V$ implies $T_i$-gcl $\{x\} \subseteq V$ and $y \in U$ implies $T_j$-gcl $\{y\} \subseteq U$. Hence the result follows. The converse is obvious.
**REFERENCES:**

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