CHAPTER I

INTRODUCTION

PART - I GENERAL BACKGROUND

1.1. PRELIMINARIES:

An optimization problem is that in which one chooses an optimal solution in one or the other sense from some set of possible solutions. Optimization problems can be of various types. The difference may be found in the goal i.e. maximization or minimization, in the constraints, i.e. inequalities or equalities and free or non-negative variables and in the mathematical properties of the functions involved in the objective or the constraints.

Mathematical programming is an integral and perhaps the most important group of the available quantitative techniques in Operations Research. A mathematical programming is an application of the optimization techniques to the planning in managerial, economic, industrial and other activities and is particularly efficient for solving "repetitive" problems at the operational level. Analytically, it consists of the optimization (maximization or minimization) of a numerical function of several variables which describe the levels of activities and which are subject to certain constraints.
Briefly, mathematical programming is effective in solving problems in which the decision maker must allocate scarce or limited resources in order to achieve the highest level of measurable goals or objectives. Depending upon the assumption and particular characteristics of a given problem, the number of solution may or may not be large and can be either finite or infinite. From the set of feasible solutions, one (sometimes more than one) is the best solution in the sense that it achieves the highest degree of the stated goal. Such a solution is referred to as optimal solution.

The first spark for the phenomenal growth of interest and the practical applications of programming problems came in 1947 when G.B. Dantzig formulated the general linear programming problem and developed the simplex method for its solution (1963). The early applications were primarily limited to problems involving military operations. Since that time, several useful extensions of the basic linear programming model have been developed.

Any situation in which a choice among available alternatives must be made defines a decision problem. Such decisions are made with the purpose of influencing future events, the decision maker has to make certain assumptions regarding the certainty or uncertainty of his own estimation of future states of nature. Problems involving certainty are called deterministic problems.

At present the deterministic approach seems to be
prevailing in the investigation of mathematical programming and
its application. However, this approach can often encounter
stumbling blocks and even an elaborate and sophisticated
deterministic programming model may have to be discarded
because the data it requires may not exist or may exist but be
of such a poor quality that the results obtained from it could
not be relied upon. Occasionally these difficulties arise
because the operation which we are attempting to stimulate
mathematically is performed by the presence of uncertainties
due to the occurrence of random events. The sources of
randomness may be many, depending on the nature and type of the
operation under study. For instance, in financial planning,
decisions have to be taken before the variable demands,
available capacities, prices and interest rates etc. are known
and as such these have to be treated as random variables. In
the design of mechanical system, the actual dimension of any
machined part has to be taken as a random variable since the
dimension may lie anywhere within a specified (permissible)
tolerance band. Another example is the designing of aircraft
and rockets in which the actual load acting on the vehicle is
uncertain and hence random, being dependent on the atmospheric
conditions at the time of the flight, which cannot be predicted
precisely in advance.

To ensure a certain class of reliability for the solutions
of optimization problems containing random data, it has become
an accepted approach to introduce probabilistic (or chance)
In fact deterministic development is based on assumption, constantly violated by random factors. If one wants to be more realistic, the assumption of the classical deterministic approach that the required data is completely known has to be relaxed and the effects of stochastic uncertainties have to be specifically taken care of.

The probabilistic (risk) problems on the other hand, occur when the decision maker does not assume certainty for the outcome of these courses of action. Uncertainty can arise in many ways. The outcome of a given action may depend on some chance event. Sometimes, the distribution of the chance events is known, sometimes it is unknown or partially known. In some cases uncertainty arises due to competitors or enemies.

Madansky [1960] pointed out that the area of programming under uncertainty cannot be usefully stated as a single problem, Dantzig [1955], Ferguson and Dantzig [1956]. The situations of decision maker facing random parameters in an optimization problem can be found in the literature. e.g. Sengupta [1972], and Vajda [1972], Kall [1976], Kolbin [1977], Kall and Prekopa [1980], Dempster [1980], Ermoliev and Wets [1988], Frauendorfer [1992] etc.

Stochastic uncertainty influences a programming model in two ways. The first is the direct effect stemming from the random phenomena whose probability distributions of anticipation are known with certainty. The second is the
indirect effect coming from the process of specifying the probability distributions of the random phenomenon. Tinter [1941] has distinguished the two as 'subjective risk' and 'subjective uncertainty' respectively. A technical distinction is, therefore, sometimes made between risk and uncertainty to indicate that the probability distributions of random variables involved are known and unknown respectively. The former field leads to the stochastic or probabilistic programming.

Throughout our discussion, unless otherwise stated, it shall be presumed that the joint probability distribution of the random variables involved, is known with certainty. This includes the case when they are independently distributed as also the case when some of them have fixed known values.

Stochastic programming problems are characterized by their difficulty of solutions; even the simplest linear problem can and usually does, become nonlinear (and hence more difficult to solve) when one or more of the parameters become random variables. One basic difficulty is that such a problem is capable of many formulations with only fragmentary results for each formulation, Madansky [1959].

There are also conceptual difficulties regarding the interpretation of probabilities and the meanings of feasibility and optimality. Besides, the technical difficulties regarding the mathematics of optimization are also not insignificant, since there is no universally applicable solution technique for stochastic problem analogous to the simplex algorithm for
there are solution methods that are applicable to a limited class of such problems. Algorithms for a number of important problems have been developed. For example, the expected value may be used instead of the random parameters; the problem can then be treated as deterministic and solved for the set of solution variables that gives the optimal value for the objective function, using standard techniques. This approach generally makes it possible to solve the stochastic problems with less effort. In general, however, the solution obtained by replacing random parameters by their expected values and solving the resulting deterministic problem results in a cost (profit) that is greater (or less) than the expected optimal value, Madansky [1962], Garvin [1960]. In fact, the stochastic programming models are considerably harder, tougher and complex as compared to their deterministic counterparts. It is because of these inherent difficulties that the progress in this field has been comparatively much slower. Had it not been for their theoretical and practical necessities in certain real decision making problems, there would have been only a limited interest in stochastic programming models.

This thesis is a modest contribution to the present state of knowledge about certain aspects of stochastic programming. Most of the work in this field has so far been concentrated on stochastic linear programming. However, in this thesis, a special class of stochastic non-linear programming problems
called the stochastic linear fractional functional programs are
concerned. These problems arise in a variety of situations. In
the cutting stock problem of paper industry with random
demands, the problem of minimizing the fraction of trim waste
in a typical example of stochastic linear fractional
programming, Wagner [1977]. These problems also arise in
maintenance and repair policies in the context of Markov
process formulation, Klein [1962]. Other examples can be found
in the works of Dantzig [1963], Fox [1966], Swarup ([1965],
[1966]) etc.

Besides, this thesis includes a special case of stochastic
transportation problem in which transshipment is permitted and
the cutting of rolls under uncertain demand.

The present chapter is a general introduction containing
the fundamentals of mathematical programming, stochastic
programming, chance constrained programming and two-stage
programming in Part-I, a review of the related work in Part-II
and a summary of the thesis in Part-III.

1.2. Mathematical Programming (MP):

Mathematical programming is a branch of optimization
time theory in which one has to determine the largest or smallest
value of a function of several variables subject possibly to
one or more constraints. Mathematical programming is effective
in solving problems in which the decision maker must allocate
scarce or limited resources in order to achieve the highest
level of measurable goals or objectives. A general mathematical programming problem can be stated as follows:

Problem F₁₁:

\[
\min_{x} \text{(or max.) } f(x) \\
\text{subject to } h_i(x) \leq 0, \quad i = 1, \ldots, m \\
\quad x \geq 0
\]  

where \( x \in \mathbb{R}^n \) is the vector of decision variables; (or activity levels) and \( f, h_i : \mathbb{R}^n \to \mathbb{R} \) \( \forall i = 1, \ldots, m \) i.e. \( f \) and \( h_i \) are one dimensional real functions defined on \( \mathbb{R}^n \).

The function \( f \) is the objective function and \( h_i \) \( (i = 1, \ldots, m) \) are the constraint functions. In deterministic models these functions are presumed to be completely known. Relations (1.2.1) and (1.2.2) are called constraints. In (1.2.1), only one of the signs \( \leq, =, \geq \) holds for any one constraint. Signs may, however, vary from constraint to constraint.

The set,

\[
X = \left\{ x \mid h_i(x) \leq 0, \quad i = 1, \ldots, m; \quad x \geq 0 \right\}
\]

is referred to as the set of all feasible solution or the constraint set. Any \( x_0 \in X \) which maximizes or minimizes the
objective function is known as the optimum solution and the value of the objective function at the point $x_0$, i.e. $f(x_0)$ as the optimum value. Since

$$\max_{x \in X} f(x) = -\min_{x \in X} [-f(x)]$$

a maximization problem can always be converted into a minimization problem and vice versa. For this reason we shall, hereinafter, use this conversion.

**The Linear Programming Problem (LPP):**

Linear programming is a mathematical programming technique most closely associated with operations research and management science. A linear programming arises when both the objective function and the constraints of a programming problem are linear. A linear programming problem is often referred to as an allocation problem because it deals with allocation of resources to alternative uses.

The problem in linear programming is that of determining the values of the decision variables which maximize (or minimize) the value of the objective function, subject to the linear side constraints. Hence, linear programming problem always contain the three major elements: objective function, non-negative decision variables, and side constraints. A general LPP may be expressed as:
Problem $P_{1.2}$:

$$\text{max. } (c^Tx)$$

subject to

$$Ax \preceq b$$

$$x \succeq 0$$

where $A \in \mathbb{R}^{m \times n}$; $c, x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

Linear programs have turned out to be appropriate models for solving practical problems in many fields. The first known method for solving LPPs is the simplex method developed by Dantzig in 1947. Dantzig simplex method solves a linear program by examining the extreme points of a convex feasible region.

Special methods have also been developed for solving LPPs with upper bound restrictions on decision variables, for example, Charnes et al. [1954], Dantzig [1955], Garvin [1960], Swarup [1970].

The ellipsoid method established by Khachiyan [1979], investigates the interior points of a feasible region until it reaches an optimal point on the boundary. The method proposed by Karmakar ([1984a], [1984b]) is the best algorithm for solving the linear programs, which also investigates the interior point algorithm. The Karmakar's algorithm is better and significantly faster than that of the ellipsoid and simplex.
method. Even since Karmakar proposed his algorithm, a number of variants of his algorithm for linear and special non-linear problems have appeared (e.g. Chandru and Kochar [1986], Anstreicher [1986], Gay [1987], Dennis [1987], and Karmakar et al. [1989]).

A significant amount of research is being carried out for implementation of the Karmakar-type algorithm, for example, Hockett and Stevenson [1987], Murty [1988] and Alder et al. [1989]. An important development is the discovery by Gill et al. [1986] that Karmakar’s algorithm belongs to a class of solution methods known as Projected Newton barrier methods.

The Non-Linear Programming Problem (NLPP):

Non-linear programming problem is described as the model in which either the objective function and/or one or more of the constraints are non-linear in decision variables. In non-linear programming the optimal solution can be any point along the boundaries of the feasible region, or it can exist within the feasible region. Work on NLPPs started almost simultaneously with LPPs when Kuhn and Tucker [1951] presented their important paper which laid down the foundations of most of the later investigations in NLP. There is no general efficient algorithm for the solution of non-linear problems. However, for problems with certain identifiable structures efficient algorithms have been developed.

The practical importance of some of the non-linear
Programming problems has inspired the rapid development of many algorithms for solving the NLPP. Of these the gradient projection method by Rosen [1960], cutting plane method by Mewly [1960], the method of feasible directions by Zountendijk [1960], penalty function method by Zangwill [1967], methods of centers by Huard [1967], and sequential unconstrained minimization techniques by Fiacco and McCormick [1968] are worth mentioning. For more sophisticated methods, see for example, Lasdon [1970], Mangasarain [1969] and Wilde et al. [1967] etc..

The Fractional Programming Problem (FPP):

Another important class of NLPPs known as fractional programming problems arises when the objective function appears as a quotient. A FPP may be represented as:

Problem \( P_{1.3} \):

\[
\max_{x \in X} \left[ \phi(x) = \frac{f(x)}{g(x)} \right]
\]

where \( f, g : \mathbb{R}^n \rightarrow \mathbb{R}, g \neq 0 \) and \( X \) is some set of feasible solutions. Problem \( P_{1.3} \) is called a fractional programming problem (FPP).

In particular, it is a linear fractional programming problem (LFPP) if \( f \) and \( g \) are linear, or affine, functions, the
constraints are linear, and the feasible region is polyhedral. A LFP deals with the problem of maximizing the ratio of two linear functions subject to a set of linear equalities and non-negativity constraints on the variables. A linear fractional programming problem may be stated as:

Problem $P_{1.4}$:

$$\max_{x \in X} \left[ \frac{(c^T x + \alpha)}{(d^T x + \beta)} \right]$$

where, $X = \{ x \in \mathbb{R}^n \mid A x \leq b, \ x \geq 0 \}$; $c, d, \in \mathbb{R}^n$; $\alpha, \beta \in \mathbb{R}$ and $(d^T x + \beta) > 0 \ \forall \ x \in X$.

(The case when the denominator may be zero is considered in Martos [1964]. A problem of type (LFPP) is known as hyperbolic optimization problem).

Solution methods have been devised which exploit the special form of the LFPPs. Isbell and Marlow [1956] solved the problem of maximizing the ratio of non-homogeneous linear forms subject to linear equality constraints by replacing it by a sequence of different linear programs provided the denominator does not vanish.

Charnes and Cooper [1962], after assuming the set $X$ to be
empty and bounded), by means of the transformation \( y = tx \), showed that the LFPP (\( P_{1,4} \)) can be solved by solving at most two LPPs which differ from each other only by change of sign in the objective function and in one constraint. Martos [1964] developed a simplex type computational technique for solving a LFPP and proved that the objective function attains a finite maximum on the constraint set \( X \) and this maximum exists on at least one vertex of \( X \).

Swarup [1965], attacked the problem directly without converting it into an equivalent LPP. Starting with a basic feasible solution, he developed a very efficient simplex like algorithm under the assumption that the set \( X \) is regular and that the denominator \( (d^T + \beta) > 0 \) on \( X \).

Wolf [1985] described an approach for determining the optimal solution of the linear fractional programming problem which is based mainly on a parametric analysis of a related linear substitution problem, where as existing algorithms for linear fractional programs concentrate solely on determining the optimal solution.

Mangasarain [1985] dealt with the characterization of optimal solution of a convex program in terms of the minimization of the exact penalty function connected with such a program. In another work, Xu [1988] discussed the saddle point type of optimality criteria for generalized fractional programming. Lal et al. [1990] extended the result of Mangasarain [1985] with reference to the exact penalty...
characterization of a solvable non-linear program to the case of fractional programs.

Snedovich [1989] proposed a solution strategy for Problem $P_{1.4}$, where the function satisfies certain convexity conditions. It is shown that subject to these conditions optimal solutions to this problem can be obtained from the solution of Problem $P_{1.3}$ is an exogenous parameter.

For other methods, references may be made, Dorn [1962], Gillmore and Gomory [1963], Schaible [1974] and Swarup [1970].

Linear fractional criteria are frequently encountered in finance as Corporate Planning and Bank Balance Sheet Management. Also, fractional objectives occur in other areas. For example, marine transportation by Bitran and Novaes [1973], maximizing profitability by Gupta and Swarup [1979].

The Non-Linear Fractional Programming Problem (NLFPP):

Non-linear fractional programming problem (NLFPPs) which fall under the category of Convex programs, (Bector's [1968]) can be solved by the usual techniques of Rosen's gradient projection methods ([1960], [1961]), Zoutendijk's method of feasible directions [1959] and Kelly's cutting plane method [1960] and Cheney et al. [1959].

However, since the NLFPPs are generally non-convex programs, special methods are required for solving them. Jagannathan [1966] and Dinkelbach [1967] related the fractional program $P_{1.3}$ to the following parametric program.
Problem \( P_{1,5} \):

\[
\max_{x \in X} \left[ f(X) - q \ g(x) \right]
\]

where \( q \in \mathbb{R} \) is a parameter.

If \( f \) and \( g \) are continuous and \( X \) is compact, then it is shown that \( x' \) solves \((P_{1,3})\) iff \( x' \) solves \((P_{1,5})\) for \( q = q_0 \), where \( q_0 \) is the unique solution of

\[
\max_{x \in X} \left[ f(X) - q \ g(x) \right] = 0
\]

and then

\[
q_0 = (x') = \max_{x' \in X} \left[ \frac{f(x) - q_0 g(x) - f(x')}{g(x') - q_0 g(x'')} \right]
\]

Other parametric approaches may be found in Geoffrion [1967] and Schaible [1976]. Kanti Swarup [1965] then extended the technique of variable transformations of Charnes and Cooper [1962] to the case of quadratic fractional program with linear constraints. After that Sharma [1967] extended it to a polynomial fractional program with linear constraints. Aggarwal [1968] further extended it to replace a standard error fractional program involving a non-differentiable term in the objective function by at most two convex programs. Mond and Graven [1973] considered Problem \( P_{1,3} \) with a still larger class of functions \( f \) and \( g \) with linear constraints. Schaible
(1973], [1974]), also exploited this technique to prove that a concave-convex fractional program is equivalent to a parameter free convex program. All these results were, however, subsumed by the later results of Mond and Graven [1975] and Schaible [1976].

For solving certain specific cases of NLFPPs, algorithms which are not based on any of these two techniques are also available in the literature, for example, Aggarwal and Swarup [1966] and Bector [1974].

1.3. Stochastic Programming (SP):

The observation that some data in real life optimization problems could be random, i.e. the origin of stochastic programming, dates back to the 1950s. Without any attempt at completeness, some early contributions to this field are mentioned, Avriel and Williams [1970], Beale ([1955], [1961]), Berneau [1967], Dantzig [1955], Dantzig and Madansky [1961], Tinter [1955] and Williams [1966] etc..

Stochastic linear programming deals with methods of characterizing and computing an optimal solution when the parameters of the problem are stochastic.

The mathematical formulation of an applied problem as a programming problem will incorporate certain parameters on which a model of the situation to be analyzed can be based. In the classical deterministic analysis these parameters are assumed to be (completely) known (fixed) constants. In many
cases of practical importance it turns out that some of the parameters appearing in the problem must be treated as random variables rather than as deterministic ones. Stochastic programming is concerned with programming problems in which some or all of the parameters are random variables with known (joint) probability distribution.

The objective of stochastic programming is to consider random effects explicitly in the solution of the model. The basic idea of all stochastic programming models is to convert the probabilistic nature of the problem into an equivalent deterministic solution. The counterpart of the stochastic programming is, of course, deterministic programming. Consider the following stochastic programming problem.

Problem $P_{1,6}$:

\[
\min_{x} \eta(x) = c^T x
\]

subject to

\[
Ax \leq b
\]
\[
x \geq 0
\]

where some or all of the components of $A$, $b$, $c$ are random variables with known (joint) probability distribution. Thus, given a probability space $(\Omega, \mathcal{F}, P_\omega)$ and a measurable transformation $(A_\omega, b_\omega, c_\omega): \Omega \rightarrow \mathbb{R}^{m \times n + m \times n}$ such that a random
realization \( \omega \in \Omega \) assigns to the random variables a value \((A_\omega, b_\omega, c_\omega)\) with a probability measure \(P_\omega\), then the point \((A_\omega, b_\omega, c_\omega)\) is said to be admissible iff at this particular point the constraint set,

\[
\{ x \mid A_\omega x \leq b_\omega, x \geq 0 \}
\]

is non-empty and bounded and if a finite optimum (i.e. min. \( x \)) is attained on it.

The determination of the distribution function of the optimum value was first formulated by Tinter [1955] as a problem of 'passive stochastic programming'.

The 'wait and see' problems are, however, not decision problems in the sense that a decision about the activity levels has to be made here and now. Wait and see problems have led to investigations of the distribution of the optimal value (and the optimal solution), e.g., Berneau [1967]. In fact, these problems require more of statistical distribution analysis than decision making. For this reason, these are, sometimes referred to as distribution problems, Kall [1976].

The mathematical models show the relationships that exist among the system variables. In practical applications, it is desirable that the estimates should be updated whenever a new available information is obtained. Since many models are assumed to have a structure expressed by a linear equation or by a system of simultaneous equations, the multivariate
Regression analysis is a useful statistical technique. It is desirable to update the optimization problem that contains an unknown solution by modifying the previous data. For such purposes, Morita and Ishii [1992] proposed a stochastic improvement method for the linear programming problem that contains unknown coefficients in the constraints, which are iteratively improved by using newly available data obtained one after another.

The non-parametric methods have great scope for application in many practical problems of decision making under conditions of risk and uncertainty. Rockafeller and Wets [1987] developed the method of scenario aggregation for obtaining robust solutions in management decision problems under conditions of incomplete knowledge and uncertainty. Kmietoweiz and Pearman [1981] suggested a realistic compromise between the two extremes of uncertainty and risk by postulating that the decision maker is able to rank future states of nature in terms of their probabilities, such that \( p_1 \geq p_2 \geq \ldots \geq p_t \).

Sengupta [1991] assumed a specific statistical distribution for the uncertain parameters and then generated a transformed model to obtain a cautious optimal solution. Various cases of this have been surveyed by Kolbin [1977], Sengupta [1980], and others. An alternative approach has been developed which emphasizes non-parametric methods in obtaining cautious optimal solutions. These methods are more heavily data-based and do not use any specific assumption about the
distribution of the stochastic elements. Several types of these methods are discussed by Charnes et al. [1986], Seiford and Thrall [1990], Sengupta [1990].

Many studies of statistical approach for stochastic programming are proposed, for example, a Bayesian analysis (Bracken and Soland [1966]); Jagannathan [1985]), a minimax model (Dupacova [1987]), prediction regions for the optimal decision (Cipra [1987]), and a confidence region method (Morita et al. [1987], [1988]).

Multistage stochastic programs are still under investigation, and were discussed early by Oslen ([1976], [1976]); [1976]) useful results on the deterministic equivalent of recourse problems and for the expectation functionals are due to Wets ([1974], [1989]). The idea of approximating stochastic programs with recourse (with a continuous type distribution) by discretizing the distribution is related to special convergence requirements for the (discretized) expected recourse function; Kall and Wallace [1994], Kall [1986].

Yahia et al. [1995] presented a technique which seeks the optimum (or near optimum) solution of a non-linear problem by searching through the sets of non-dominated solutions of the bicriterion linear programming problem. They presented an approximation formula for constructing two linear objective functions, based on the non-linear objective function of the equivalent deterministic form of the stochastic programming model.
II.A. Chance Constrained Programming (CCP):

In linear programming, when the presumed level of the reliability of the constraint(s) is less than one, then chance constrained programming can be employed as a means of describing that level of constraint violation. CCP has been introduced into the SP literature mainly through the exposition of Charnes and Cooper [1959], and since then it has been developed and applied by Kataoka [1963] and many others. Chance constrained technique is one which can be used to solve problems involving chance constraints, that is, constraints having finite probability of being violated. This chance constrained programming permits the constraints to be violated by a specific (small) probability. Thus the CCP formulation is as:

Problem $P_{1.7}$:

$$\min_{x} \phi = E(c_{\omega}^{T} x)$$

subject to

$$\text{Prob. } (A_{\omega} x \leq b_{\omega}) \geq p, \quad p \in [0,1]$$

$$x \geq 0$$

The interpretation of $P_{1.7}$ is that an activity $x \geq 0$ is to be chosen such that it satisfies $A_{\omega} x \leq b_{\omega}$ with at least probability $p$ and minimizes $E(c_{\omega}^{T} x)$. Symond [1967] formulated and founded condition for the
construction of deterministic equivalent to chance constrained stochastic problem. The working out of different problems for the qualitative analysis of chance constrained problems, were contributed by Miller and Wagner [1965], Sengupta ([1969], [1983]) and by many others.

Rakes et al. [1981] used a piece wise linear goal programming code for solving chance constrained models. Separable technique was applied to chance constrained first by Seppala and Orpana [1984] and non-linear constraints are linearized approximately. Oslon and Lee [1985] used a gradient algorithm for chance constrained non-linear goal programming. A linear approximation of CCP was given by Oslon and Scott [1987]. Weintraub and Vera [1991] solved the CCP using cutting plane algorithm.

Chance constrained programming models have found applications in diverse fields such as financial planning, stock investment, marketing, agriculture, industry production, network analysis, energy planning, Water system planning, (see Hogan et al. [1981]).

1.5. Two Stage Stochastic Programming (TSSP) Model:

Most of the problems in planning and management dealing with criterions of risk and uncertainty are considered and solved as two stage stochastic programming technique, such problems with compensation of divergencies in system with constraints have more applications in comparison to any other