CHAPTER 1

PRELIMINARIES

1.1. Introduction

A special function is a function - usually of a single variable - that arises in sufficiently many applications as to warrant its own name, an investigation of its basic properties and the development of special algorithms for computing its values. The first “special functions” one meets up with are the exponential function, the natural logarithm and the trigonometric functions. However, since these common functions, along with polynomials, rational functions, algebraic functions (combinations of roots) and the hyperbolic functions appear much earlier in one’s mathematical education, they are usually referred to as 

**elementary functions.** True **special functions**, such as the Gamma function, Bessel functions, Legendre functions, Airy functions, hypergeometric function and many more, await a more advanced mathematical training. These special functions play a starring role in more advanced applications in physics, engineering and mathematics. They initially appear when one tries to solve linear partial differential equations in higher dimensions in non-rectangular coordinate systems. Application of the method of separation of variables reduces the partial differential equation to an ordinary differential equation of a non-elementary type. The solutions to these special ordinary differential equations are the aforementioned special functions.

The theory of special functions with its numerous beautiful formulae is very well suited to an algorithmic approach to mathematics. Although, special functions can be defined in different ways such as (Rodrigue’s formulae, generating functions, summation formulae, integral representations **et cetera**), but it is usually shown to be expressible as a series, because this is frequently the most practical way to obtain numerical values for the functions.

The subject of special functions has been continuously developed with contribution of Chebyshev, Euler, Gauss, Hardy, Hermite, Kummer, Laplace, Legendre, Ramanujan, Riemann, Watson and other classical authors. In the past several years the discoveries of new special functions and applications of this kind of functions to new areas of
mathematics have initiated a great interest in this field. Moreover, in recent years, particular cases of long familiar special functions have been clearly defined and applied as orthogonal polynomials. A number of books consisting of the theory and applications of special functions are available, see for example Abramowitz and Stegun [1], Andrews [3,4], Andrews et al. [5], Askey [8], Erdélyi et al. [58-60], Iwasaki et al. [74], Lebedev [95], Rainville [109], Sneddon [120] et cetera.

Generalized and multi-variables forms of the special functions of mathematical physics have witnessed a significant evolution during the recent years. Various generalized and multi-variable forms of special functions are introduced, see for example [14,25,31,32,44,50,51,80,90]. This further advancement in the theory of special functions serves as an analytical foundation for the majority of problems in mathematical physics that have been solved exactly and finds broad practical applications. For some physical problems the use of new classes of special functions provided solutions hardly achievable with conventional analytical and numerical means. For example, the use of generalized Bessel functions is now a well established tool to treat synchrotron radiation [41] and crystallographic [103] problems. Further, the importance of generalized Hermite polynomials has been recognized [28,36] and has been exploited to deal with quantum mechanical and optical beam transport problems. The usefulness of the generalized Laguerre polynomials to treat radiation physics problems such as wave propagation and quantum beam life time in storage rings due to the quantum fluctuations is a well established fact, see [127].

Operational methods provide the tools to treat various special polynomials from a unified point of view and offer the keynote to construct/introduce new families of special functions. In the case of multi-variable generalized special functions, the use of operational techniques, combined with the principle of monomiality [22,124] provides new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems.

Operational techniques also provide a general framework to derive generating relations and summation formulae involving multi-variable special functions. The interest in applications of operational methods in the study of special functions relies upon the
fact that, most of the operational identities appear quite naturally in quantum mechanical problems or in applications concerning wave propagation in classical optics. The special functions of generalized nature are the suited tool to solve equations related to classical electromagnetism, like radiation emission by accelerated charge. A large body of the formalism, associated with the monomiality principle, is ideally suited for the study of differential equations of different nature, involving parabolic equations related to evolution problems and thus the combined use of special functions and operational techniques becomes a unique tool to solve Schrödinger and Liouville type equations by means of the evolution operator method.

Dattoli and his co-workers have shown that by using operational techniques, many properties of ordinary and multi-variable special functions are simply derived and framed in a more general context, see for example [22,23,32-34,37,39,40,47-50]. Recently, Subuhi khan and her co-authors introduced and studied new families of mixed type special polynomials by employing certain operational methods, see for example [85,86,90-92].

In this chapter, the basic definitions and concepts related to hypergeometric and other special functions, generating functions, operational methods and Hermite and Laguerre polynomials of two variables are given. These definitions and concepts will be used in subsequent chapters. In Section 1.2, the definitions of Gamma, Beta and hypergeometric functions are given. In Section 1.3, the concepts of generating functions are reviewed and definitions of certain classical special functions are given. In Section 1.4, operational identities, inverse derivative operator and monomiality principle are discussed. Finally, in Section 1.5, the Hermite and Laguerre polynomials of two variables are framed within the context of monomiality principle.

1.2. Gamma, Beta and Hypergeometric Functions

The theory of hypergeometric functions is fundamental for mathematical physics, since almost all elementary functions can be expressed as either hypergeometric or ratios of hypergeometric functions and many non elementary functions in this field can be expressed as hypergeometric functions. The general theory of hypergeometric
functions is very powerful and it is worthwhile to check if a given series is hypergeometric, because we may gain a lot of insight into a function by first recognizing that it is hypergeometric, then identifying its parameters and finally by using known results about such functions.

A fairly wide range of special functions can be represented in terms of the hypergeometric and confluent hypergeometric functions. We first give the definitions and important properties of some elementary functions such as the Gamma, Beta and related functions [123].

The Gamma function

The Gamma function is a generalization of the factorial function from the domain of positive integers to the domain of all real and complex numbers (except for \( z > 0 \) and \( \text{Re}(z) > 0 \), respectively). For a complex number \( z \) with positive real part the Gamma function is defined by

\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \quad (\text{Re}(z) > 0).
\] (1.2.1)

The Gamma function is distinguished by being analytic (except at the non-positive integers) and by being the most useful solution in practice and can be characterized in several ways. However, the integral representation of \( \Gamma(z) \) given in equation (1.2.1) is the most common way in which the Gamma function is defined.

It appears occasionally by itself in physical applications, but much of its importance stems from its usefulness in developing other functions such as hypergeometric functions and Bessel functions, which have more direct physical application. Further, the Gamma function is a component in various probability-distribution functions and as such it is applicable in the fields of probability and statistics, as well as combinatorics.

Upon integration by parts, definition (1.2.1) yields the recurrence relation for \( \Gamma(z) \):

\[
\Gamma(z + 1) = z \Gamma(z),
\] (1.2.2)
which enables us to use definition (1.2.1) to define \( \Gamma(z) \) on the entire \( z \)-plane except when \( z \) is zero or a negative integer as follows:

\[
\Gamma(z) = \begin{cases} 
\int_0^\infty t^{z-1} e^{-t} \, dt & (\text{Re}(z) > 0), \\
\frac{\Gamma(z+1)}{z} & (\text{Re}(z) < 0; \ z \neq -1, -2, -3, \ldots). 
\end{cases} 
\]

(1.2.3)

The recurrence relation (1.2.2) yields the useful result

\[
\Gamma(n + 1) = n! \quad (n = 0, 1, 2, \ldots). 
\]

(1.2.4)

**Pochhammer’s symbol and the factorial function**

The Pochhammer symbol \((\lambda)_n\) is defined by

\[
(\lambda)_n = \begin{cases} 
1 & (n = 0), \\
\lambda(\lambda + 1) \ldots (\lambda + n - 1) & (n = 1, 2, \ldots). 
\end{cases} 
\]

(1.2.5)

Since \((1)_n = n!\), the symbol \((\lambda)_n\) is also referred to as the factorial function.

In terms of Gamma functions, we have

\[
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \neq 0, -1, -2, \ldots), 
\]

(1.2.6)

which can easily be verified. Furthermore, the binomial coefficient may be expressed as

\[
\binom{\lambda}{n} = \frac{\lambda(\lambda - 1) \ldots (\lambda - n + 1)}{n!} = \frac{(-1)^n (-\lambda)_n}{n!}, 
\]

(1.2.7)

or, equivalently as

\[
\binom{\lambda}{n} = \frac{\Gamma(\lambda + 1)}{n! \, \Gamma(\lambda - n + 1)}. 
\]

(1.2.8)

It follows from (1.2.7) and (1.2.8) that

\[
\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - n + 1)} = (-1)^n (-\lambda)_n, 
\]

(1.2.9)
which, for $\lambda = \alpha - 1$, yields
\[
\frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1 - \alpha)_n} \quad (\alpha \neq 0, \pm 1, \pm 2, \ldots). 
\] (1.2.10)

Equations (1.2.6) and (1.2.10) suggest the definition
\[
(\lambda)_{-n} = \frac{(-1)^n}{(1 - \lambda)_n} \quad (n = 1, 2, \ldots; \lambda \neq 0, \pm 1, \pm 2, \ldots). 
\] (1.2.11)

Equation (1.2.6) also yields
\[
(\lambda)_{m+n} = (\lambda)_m (\lambda + m)_n ,
\] (1.2.12)

which, in conjunction with (1.2.11), gives
\[
(\lambda)_{n-k} = \frac{(-1)^k(\lambda)_n}{(1 - \lambda - n)_k} \quad (0 \leq k \leq n). 
\] (1.2.13)

For $\lambda = 1$, we have
\[
(n - k)! = \frac{(-1)^k n!}{(-n)_k} \quad (0 \leq k \leq n),
\] (1.2.14)

which may alternatively be written in the form:
\[
(-n)_k = \begin{cases} 
\frac{(-1)^k n!}{(n - k)!} & (0 \leq k \leq n), \\
0 & (k > n).
\end{cases} 
\] (1.2.15)

In view of above equations, we note the following general formulae:
\[
(-n)_{mk} = \frac{(-1)^{mk} n!}{(n - mk)!} \quad (0 \leq k \leq \left[ \frac{n}{m} \right])
\] (1.2.16)

and
\[
(-n)_M = \frac{(-1)^M n!}{(n - M)!} \quad (0 \leq M \leq n),
\] (1.2.17)

where the symbol $\left[ \frac{n}{m} \right]$ denotes the greatest integer less than or equal $\frac{n}{m}$ and $M := m_1k_1 + m_2k_2 + \cdots + m_jk_j$, $m_1, m_2, \ldots, m_j \in \mathbb{N}$; $k_1, k_2, \ldots, k_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. 

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Also, for every positive integer \( m \), we have Gaussian multiplication theorem

\[
(\lambda)_{mn} = m^{mn} \prod_{j=1}^{m} \left( \frac{\lambda + j - 1}{m} \right) \quad (n = 0, 1, 2, \ldots), \quad (1.2.18)
\]

which for \( m = 2 \), reduces to the useful relation

\[
(\lambda)_{2n} = 2^{2n} \left( \frac{1}{2} \lambda \right) \left( \frac{1}{2} (\lambda + 1) \right) \quad (n = 0, 1, 2, \ldots). \quad (1.2.19)
\]

The Beta function

The Beta function \( B(\alpha, \beta) \) is a function of two complex variables \( \alpha \) and \( \beta \), defined by the Eulerian integral of the first kind

\[
B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1 - t)^{\beta-1} \, dt \quad (\text{Re}(\alpha), \text{Re}(\beta) > 0). \quad (1.2.20)
\]

The Beta function is closely related to the gamma function; in fact, we have

\[
B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (\alpha, \beta \neq 0, -1, -2, \ldots). \quad (1.2.21)
\]

In view of equation (1.2.21), we obtain the symmetry property

\[
B(\alpha, \beta) = B(\beta, \alpha). \quad (1.2.22)
\]

Gaussian hypergeometric function

A hypergeometric series, in the most general sense, is a power series in which the ratio of successive coefficients indexed by \( n \) is a rational function of \( n \). The series, if convergent, will define a hypergeometric function, which may then turn out to be defined over a wider domain of the argument by analytic continuation. The term hypergeometric series also refers to a specific type of the series, also known as Gauss’s series (Carl Friedrich Gauss), which were the object of a great deal of interest in the nineteenth century.

The hypergeometric function \( _2F_1[\alpha, \beta; \gamma; z] \) is defined by

\[
_2F_1[\alpha, \beta; \gamma; z] = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} \frac{z^n}{n!} \quad (|z| < 1; \ \gamma \neq 0, -1, -2, \ldots), \quad (1.2.23)
\]
where $\alpha, \beta$ and $\gamma$ are real or complex parameters.

The first person to use the term “hypergeometric” was John Wallis in his work *Arithmetica Infinitorum* (1655). He used it to denote a series which was beyond the ordinary geometric series

$$1 + z + z^2 + z^3 + \ldots$$

Gauss’s series were studied by Euler in 1769, he established an integral representation, a series expansion, a differential equation and several other properties including reduction and transformation formulae for hypergeometric function. But the first full systematic treatment is found in Gauss’s seminal paper of 1812, who introduced the hypergeometric series into analysis and gave $F$-notation for it. Gauss’s work was of great historical importance because it initiated far reaching development in many branches of analysis not only in infinite series, but also in the general theories of linear differential equations and functions of a complex variable. The hypergeometric function has retained its significance in modern mathematics because of its powerful unifying influence since many of the principal special functions of higher analysis are also related to it.

By d’Alembert’s ratio test, it is easily seen that the hypergeometric series in (1.2.23) converges absolutely within the unit circle, that is, when $|z| < 1$, provided that the denominator parameter $\gamma$ is neither zero nor a negative integer. However, if either or both of the numerator parameters $\alpha$ and $\beta$ in (1.2.23) is zero or a negative integer, the hypergeometric series terminates.

When $|z| = 1$ (that is, on the unit circle), the hypergeometric series is:

(1) absolutely convergent if $\text{Re} (\gamma - \alpha - \beta) > 0$;

(2) conditionally convergent if $-1 < \text{Re} (\gamma - \alpha - \beta) \leq 0$, $z \neq 1$;

(3) divergent if $\text{Re} (\gamma - \alpha - \beta) \leq -1$.

$2F_1[\alpha, \beta; \gamma; z]$ is a solution of the differential equation

$$z(1-z)\frac{d^2u}{dz^2} + (\gamma - (\alpha + \beta + 1)z)\frac{du}{dz} - \alpha\beta u = 0,$$  

(1.2.24)
in which \( \alpha, \beta \) and \( \gamma \) are independent of \( z \). This is a homogeneous linear differential equation of the second order and is called the hypergeometric equation. It has at most three singularities 0, \( \infty \), and 1 which are all regular \[106\]. This function has the following integral representation:

\[
2F_1[\alpha, \beta; \gamma; z] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta} \, dt \quad (1.2.25)
\]

(\( \text{Re}(\gamma) > \text{Re}(\alpha) > 0; \ |\arg(1-z)| < \pi \)).

There are several varieties of functions of the hypergeometric type, but the most common are the standard hypergeometric function and the confluent hypergeometric function. Also, a natural generalization of these functions is the generalized hypergeometric function, which is accomplished by the introduction of an arbitrary number of numerator and denominator parameters.

**Confluent hypergeometric function**

If in hypergeometric equation (1.2.24), we replace \( z \) by \( z/\beta \), the resulting equation will have three singularities at \( z = 0, \beta, \infty \).

By letting \( |\beta| \to \infty \), this transformed equation reduces to

\[
\frac{d^2 u}{dz^2} + \left( \gamma - z \right) \frac{du}{dz} - \alpha u = 0. \quad (1.2.26)
\]

Equation (1.2.26) has a regular singularity at \( z = 0 \) and an irregular singularity at \( z = \infty \), which is formed by the confluence of two regular singularities at \( \beta \) and \( \infty \) of equation (1.2.24) with \( z \) replaced by \( \frac{z}{\beta} \).

Consequently, equation (1.2.26) is called the confluent hypergeometric equation or Kummer’s differential equation after E.E. Kummer, who presented a detailed study of its solutions in 1836, see \[93\].

The simplest solution of equation (1.2.26) is confluent hypergeometric function or Kummer’s function \( _1F_1[\alpha; \gamma; z] \) which is given as

\[
_1F_1[\alpha; \gamma; z] = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n n!} z^n \quad (\gamma \neq 0, -1, -2, \cdots; \ |z| < \infty), \quad (1.2.27)
\]
which can also be deduced as a special case of hypergeometric function \( _2F_1[\alpha, \beta; \gamma; z] \).

In fact, we have

\[
\lim_{\beta \to \infty} _2F_1\left[\alpha, \beta; \gamma; \frac{z}{\beta}\right] = _1F_1[\alpha; \gamma; z].
\] (1.2.28)

A great number of common mathematical functions are expressible in terms of hypergeometric functions. For example

\[
_1F_1[\alpha; \alpha; z] = _0F_0[-; -; z] = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z,
\] (1.2.29)

\[
_2F_1[\alpha; \beta; \beta; z] = _1F_0[\alpha; -; z] = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \frac{z^n}{n!} = (1 - z)^{-\alpha},
\] (1.2.30)

where \( e^z \) is the well-known exponential function and \( (1 - z)^{-\alpha} \) is the familiar binomial expansion.

The generalized hypergeometric function

A natural generalization of the Gaussian hypergeometric series \( _2F_1[\alpha, \beta; \gamma; z] \), is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

\[
_pF_q\left[\begin{array}{c}
\alpha_1, \alpha_2, \ldots, \alpha_p; \\
\beta_1, \beta_2, \ldots, \beta_q;
\end{array} \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}
\] (1.2.31)

\[
= _pF_q[\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z],
\]

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here \( p \) and \( q \) are positive integers or zero and we assume that the variable \( z \), the numerator parameters \( \alpha_1, \alpha_2, \ldots, \alpha_p \) and the denominator parameters \( \beta_1, \beta_2, \ldots, \beta_q \) take on complex values, provided that

\[
\beta_j \neq 0, -1, -2, \ldots ; \quad j = 1, 2, \ldots, q.
\]

Supposing that none of numerator parameters is zero or a negative integer and for \( \beta_j \neq 0, -1, -2, \ldots ; \ j = 1, 2, \ldots, q \), we note that the \( _pF_q \) series defined by equation (1.2.31):
(i) converges for $|z| < \infty$, if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q + 1$ and

(iii) diverges for all $z$, $z \neq 0$, if $p > q + 1$.

Furthermore, if we set

$$w = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

it is known that the $pF_q$ series, with $p = q + 1$, is

(I) absolutely convergent for $|z| = 1$, if $\text{Re}(w) > 0$,

(II) conditionally convergent for $|z| = 1$, $|z| \neq 1$, if $-1 < \text{Re}(w) \leq 0$ and

(III) divergent for all $|z| = 1$, if $\text{Re}(w) \leq -1$.

The hypergeometric function (1.2.31) is a solution of the differential equation

$$z \frac{d}{dz} \prod_{j=1}^{q} \left( z \frac{dy}{dz} + (\beta_j - 1)y \right) - z \prod_{j=1}^{p} \left( z \frac{dy}{dz} + \alpha_jy \right) = 0, \quad (1.2.32)$$

which for $p = 2$ and $q = 1$ reduces to equation (1.2.24).

**Hypergeometric functions of two and more variables**

Continuation to the great success of the theory of hypergeometric functions in a single variable has stimulated the development of a corresponding theory in two and more variables. A multiple Gaussian hypergeometric series is a hypergeometric series in two and more variables, which reduces to the familiar Gaussian hypergeometric series (1.2.23), whenever only one variable is non-zero. Fourteen distinct double Gaussian series exist: Appell [6] introduced $F_1, F_2, F_3, F_4$, but the set was not completed until after Horn [72] gave the remaining ten series $G_1, G_2, G_3, H_1, H_2, \ldots, H_7$. Lauricella [94] introduced 14 triple Gaussian series $F_1, F_2, \ldots, F_{14}$. More precisely, he defined four $n$-dimensional series $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ and ten further triple series $F_3, F_4, F_6, F_7, F_8, F_{10}, \ldots, F_{14}$. Saran [115] initiated a systematic study of these ten
triple Gaussian series of Lauricella’s set and gave his notations \( F_E, F_F, \ldots, F_T \). Exton [61] introduced 21 quadruple Gaussian series \( K_1, K_2, \ldots, K_{21} \). Further, Sharma and Parihar [117] introduced 83 complete quadruple Gaussian series \( \text{F}^{(4)}_1, \text{F}^{(4)}_2, \ldots, \text{F}^{(4)}_{83} \). It is remarkable that out of these 83 series, 19 series are introduced by Exton [61]. We give the definitions of the functions which are used in our work.

**Appell functions**

A formal extension of the hypergeometric function \( _2F_1 \) to two variables, resulting in four kinds of functions called Appell functions and denoted by \( F_1, F_2, F_3 \) and \( F_4 \). Appell defined the functions in 1880 [6] and they were subsequently studied by Picard in 1881. Appell functions are defined as:

\[
F_1(a, b, b'; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m! n!} \quad (\text{max}\{|x|, |y|\} < 1), \quad (1.2.33)
\]

\[
F_2(a, b, b'; c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)m(b')_n}{(c)m(c')_n} \frac{x^m y^n}{m! n!} \quad (|x| + |y| < 1), \quad (1.2.34)
\]

\[
F_3(a, a', b, b'; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m! n!} \quad (\text{max}\{|x|, |y|\} < 1), \quad (1.2.35)
\]

\[
F_4(a, b; c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)m(b')_n}{(c)m(c')_n} \frac{x^m y^n}{m! n!} \quad ((\sqrt{|x|} + \sqrt{|y|}) < 1), \quad (1.2.36)
\]

where, as usual, the denominator parameters \( c \) and \( c' \) are neither zero nor a negative integer.

The standard work on the theory of Appell functions is the monograph by Appell and Kampé de Fériet [7]. For a review of the subsequent work on the subject, see Erdélyi et al. [58-60], Bailey [9], Slater [118], Exton [62] and Srivastava and Karlsson [122].

**Kampé de Fériet function**

The four Appell functions were unified and generalized by Kampé de Fériet [79], who defined a general hypergeometric function of two variables, see [123, p. 63(16)].
The Kampé de Fériet function is defined by

\[
F_{\ell;m,n}^{p;q;k}(a_p : (b_q); (c_k) ; x, y) = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{r+s} \prod_{j=1}^{q} (b_j)_r \prod_{j=1}^{k} (c_j)_s}{\prod_{j=1}^{l} (\alpha_j)_{r+s} \prod_{j=1}^{m} (\beta_j)_r \prod_{j=1}^{n} (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!}, \tag{1.2.37}
\]

where, for convergence,

(i) \( p + q < l + m + 1, \ p + k < l + n + 1, \ |x| < \infty, \ |y| < \infty, \) or

(ii) \( p + q = l + m + 1, \ p + k = l + n + 1, \) and

\[
\begin{cases}
|x|^{\frac{1}{p-l}} + |y|^{\frac{1}{p-l}} < 1, & \text{if } p > l, \\
\max\{|x|, |y|\} < 1, & \text{if } p \leq l.
\end{cases}
\]

The notation \( F_{\ell;m,n}^{p;q;k} \) for Kampé de Fériet general double hypergeometric series of superior order applies successfully to the Appell double hypergeometric series \( F_1, F_2, F_3, F_4. \) Thus, for example, we have

\[
F_1 = F_{1:0;0}^{1:1;1}, \quad F_2 = F_{0:1;1}^{1:1;1}, \quad F_3 = F_{1:0;0}^{0:2;2}, \quad F_4 = F_{0:1;1}^{2:0;0}.
\tag{1.2.38}
\]

**Lauricella function of \( n \) variables**

Lauricella [94] further generalized the four Appell functions \( F_1, F_2, F_3 \) and \( F_4 \) to functions of \( n \) variables. Let \( n \) be the number of variables, then the Lauricella functions are defined as follows:

\[
F_A^{(n)}[a, b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_n; x_1, x_2, \ldots, x_n] = \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\ldots+m_n} (b_1)_{m_1} (b_2)_{m_2} \ldots (b_n)_{m_n} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}}{(c)_{m_1+m_2+\ldots+m_n} m_1! m_2! \ldots m_n!} (|x_1| + |x_2| + \ldots + |x_n| < 1), \tag{1.2.39}
\]

\[
F_B^{(n)}[a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n; c; x_1, x_2, \ldots, x_n] = \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a_1)_{m_1} (a_2)_{m_2} \ldots (a_n)_{m_n} (b_1)_{m_1} (b_2)_{m_2} \ldots (b_n)_{m_n} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}}{(c)_{m_1+m_2+\ldots+m_n} m_1! m_2! \ldots m_n!} (\max\{|x_1|, |x_2|, \ldots, |x_n|\} < 1), \tag{1.2.40}
\]

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where Φ in the resultant equation, we get

\[
F_{C}^{(n)}[a, b; c_{1}, c_{2}, \ldots, c_{n}; x_{1}, x_{2}, \ldots, x_{n}]
\]

\[
= \sum_{m_{1}, m_{2}, \ldots, m_{n}=0}^{\infty} \frac{(a)_{m_{1}+m_{2}+\ldots+m_{n}}(b)_{m_{1}+m_{2}+\ldots+m_{n}}}{(c_{1})_{m_{1}}(c_{2})_{m_{2}}\ldots(c_{n})_{m_{n}}} \frac{x_{1}^{m_{1}}}{m_{1}!} \frac{x_{2}^{m_{2}}}{m_{2}!} \cdots \frac{x_{n}^{m_{n}}}{m_{n}!}
\]

(1.2.41)

\[
|a \lim_{n \to \infty} n \left( \frac{b}{c_{1}} \right)^{n} = \Phi^{n} (n \in \mathbb{N}_{0} := \mathbb{N} \cup \{0\}),
\]

(1.2.43)

in the resultant equation, we get

\[
\lim_{|a| \to \infty} F_{D}^{(n)}(a, b_{1}, b_{2}, \ldots, b_{n}; c; x_{1}/a, x_{2}/a, \ldots, x_{n}/a) = \Phi_{2}^{(n)}(b_{1}, b_{2}, \ldots, b_{n}; c; x_{1}, x_{2}, \ldots, x_{n}),
\]

(1.2.44)

where \(\Phi_{2}^{(n)}\) denotes the confluent hypergeometric function of \(n\) variables defined by

\[
\Phi_{2}^{(n)}(b_{1}, b_{2}, \ldots, b_{n}; c; x_{1}, x_{2}, \ldots, x_{n})
\]

\[
= \sum_{m_{1}, m_{2}, \ldots, m_{n}=0}^{\infty} \frac{(b_{1})_{m_{1}}(b_{2})_{m_{2}} \cdots (b_{n})_{m_{n}}}{(c)_{m_{1}+m_{2}+\ldots+m_{n}}} \frac{x_{1}^{m_{1}}}{m_{1}!} \frac{x_{2}^{m_{2}}}{m_{2}!} \cdots \frac{x_{n}^{m_{n}}}{m_{n}!}
\]

(1.2.45)

\(\max\{|x_{1}|, |x_{2}|, \ldots, |x_{n}|\} < \infty\).

**Generalized Lauricella function of several variables**

Srivastava and Daoust [121] further generalized the Kampé de Fériet function (1.2.37) to function in several variables, which is referred to in the literature as the generalized Lauricella function of several variables and defined by [30, p.64(18)]
and the coefficients
\[
\theta_j^{(k)}, j = 1, 2, \ldots, A; \quad \phi_j^{(k)}, j = 1, 2, \ldots, B^{(k)}; \quad \psi_j^{(k)}, j = 1, 2, \ldots, C; \quad \delta_j^{(k)}, j = 1, 2, \ldots, D^{(k)};
\]
for all \( k \in \{1, 2, \ldots, n\} \) are real and positive, \((a)\) abbreviates the array of \( A \) parameters \( a_1, a_2, \ldots, a_A \), \((b^{(k)})\) abbreviates the array of \( B^{(k)} \) parameters \( b_j^{(k)}, j = 1, 2, \ldots, B^{(k)} \) for all \( k \in \{1, 2, \ldots, n\} \) with similar interpretations for \((c)\) and \((d^{(k)})\), \( k = 1, 2, \ldots, n; \) \textit{et cetera}.

Note that, when the coefficients in equation (1.2.46) equal to 1, the generalized Lauricella function (1.2.46) reduces to the following multivariable extension of the Kampé de Fériet function (1.2.37):

\[
\begin{align*}
F^{A;B^p,B^q;…,B^{(n)}}_{C;D^p,D^q;…,D^{(n)}} & \left[ \begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_n \\ \end{array} \right] \\
\equiv & \sum_{m_1,m_2,\ldots,m_n=0}^{\infty} \Omega(m_1, m_2, \ldots, m_n) \frac{z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}}{m_1! m_2! \cdots m_n!} \tag{1.2.46}
\end{align*}
\]

where
\[
\Omega(m_1, m_2, \ldots, m_n) = \frac{\prod_{j=1}^{A} (a_j)_{m_1 \theta_j^{(1)} + m_2 \theta_j^{(2)} + \cdots + m_n \theta_j^{(n)}} \prod_{j=1}^{B^p} (b_j^p)_{m_1 \phi_j^{(1)} + m_2 \phi_j^{(2)} + \cdots + m_n \phi_j^{(n)}} \prod_{j=1}^{B^q} (b_j^q)_{m_1 \phi_j^{(1)} + m_2 \phi_j^{(2)} + \cdots + m_n \phi_j^{(n)}} \prod_{j=1}^{B^{(n)}} (b_j^{(n)})_{m_1 \delta_j^{(1)} + m_2 \delta_j^{(2)} + \cdots + m_n \delta_j^{(n)}}} {\prod_{j=1}^{C} (c_j)_{m_1 \psi_j^{(1)} + m_2 \psi_j^{(2)} + \cdots + m_n \psi_j^{(n)}} \prod_{j=1}^{D^p} (d_j^p)_{m_1 \delta_j^{(1)} + m_2 \delta_j^{(2)} + \cdots + m_n \delta_j^{(n)}} \prod_{j=1}^{D^q} (d_j^q)_{m_1 \delta_j^{(1)} + m_2 \delta_j^{(2)} + \cdots + m_n \delta_j^{(n)}} \prod_{j=1}^{D^{(n)}} (d_j^{(n)})_{m_1 \delta_j^{(1)} + m_2 \delta_j^{(2)} + \cdots + m_n \delta_j^{(n)}}}
\]

and the coefficients
\[
\theta_j^{(k)}, j = 1, 2, \ldots, A; \quad \phi_j^{(k)}, j = 1, 2, \ldots, B^{(k)}; \quad \psi_j^{(k)}, j = 1, 2, \ldots, C; \quad \delta_j^{(k)}, j = 1, 2, \ldots, D^{(k)};
\]
for all \( k \in \{1, 2, \ldots, n\} \) are real and positive, \((a)\) abbreviates the array of \( A \) parameters \( a_1, a_2, \ldots, a_A \), \((b^{(k)})\) abbreviates the array of \( B^{(k)} \) parameters \( b_j^{(k)}, j = 1, 2, \ldots, B^{(k)} \) for all \( k \in \{1, 2, \ldots, n\} \) with similar interpretations for \((c)\) and \((d^{(k)})\), \( k = 1, 2, \ldots, n; \) \textit{et cetera}.

Note that, when the coefficients in equation (1.2.46) equal to 1, the generalized Lauricella function (1.2.46) reduces to the following multivariable extension of the Kampé de Fériet function (1.2.37):

\[
\begin{align*}
F^{p;p_1;p_2;\ldots;p_n}_{r;\delta_1;\delta_2;\ldots;\delta_n} & \left[ \begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_n \\ \end{array} \right] \\
\equiv & \sum_{m_1,m_2,\ldots,m_n=0}^{\infty} \Omega(m_1, m_2, \ldots, m_n) \frac{z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}}{m_1! m_2! \cdots m_n!} \tag{1.2.47}
\end{align*}
\]

In the next section, we review the concept of generating functions and give the definitions of certain classical special functions. We also mention the link of these special functions with the hypergeometric functions.
1.3. Generating Functions and Certain Classical Special Functions

Laplace formulated the calculus of generating functions in 1779 and he introduced the concept of “generating function” in 1812. Since then the theory of generating functions has been developed in various directions. A generating function may be used to define a set of functions, to determine differential or pure recurrence relations, to evaluate certain integrals et cetera. Generating relations of special functions arise in a diverse range of applications in quantum physics, molecular chemistry, harmonic analysis, multivariate statistics, number theory et cetera. Generating functions play an important role in the investigation of various useful properties of the sequences which they generate. There are various methods of obtaining generating functions for a fairly wide variety of sequences of special functions (and polynomials), see for example [98] and [123].

In our work, we use the generating functions of the following types:

I. Linear generating functions

Consider a two-variable function $F(x, t)$ which possesses a formal (not necessarily convergent for $t \neq 0$) power series expansion in $t$ such that

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x) t^n,$$

(1.3.1)

where each member of the coefficient set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of $t$. Then, the expansion (1.3.1) of $F(x, t)$ is said to have generated the set $\{f_n(x)\}$ and $F(x, t)$ is called a linear generating function (or, simply, a generating function) for the set $\{f_n(x)\}$.

This definition may be extended slightly to include a generating function of the type:

$$G(x, t) = \sum_{n=0}^{\infty} c_n g_n(x) t^n,$$

(1.3.2)

where the sequence $\{c_n\}_{n=0}^{\infty}$ may contain the parameters of the set $g_n(x)$, but is independent of $x$ and $t$. 

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If \( c_n \) and \( g_n(x) \) in expansion (1.3.2) are prescribed and if we can formally determine the sum function \( G(x,t) \) in terms of known special functions, then we say that the generating functions \( G(x,t) \) has been found.

Further, if the set \( \{ f_n(x) \} \) is defined for \( n = 0, \pm 1, \pm 2, \cdots \), then the definition (1.3.2) may be extended in terms of the Laurent series expansion:

\[
F^*(x,t) = \sum_{n=-\infty}^{\infty} \gamma_n f_n(x) t^n, \tag{1.3.3}
\]

where the sequence \( \{ \gamma_n \}_{n=-\infty}^{\infty} \) is independent of \( x \) and \( t \).

**II. Bilinear generating functions**

If a three-variable function \( F(x,y,t) \) possesses a formal power series expansion in \( t \) such that

\[
F(x,y,t) = \sum_{n=0}^{\infty} \gamma_n f_n(x) f_n(y) t^n, \tag{1.3.4}
\]

where the sequence \( \{ \gamma_n \} \) is independent of \( x, y \) and \( t \), then \( F(x,y,t) \) is called a bilinear generating function for the set \( \{ f_n(x) \} \).

More generally, if \( \mathcal{F}(x,y,t) \) can be expanded in powers of \( t \) in the form

\[
\mathcal{F}(x,y,t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) f_{\beta(n)}(y) t^n, \tag{1.3.5}
\]

where \( \alpha(n) \) and \( \beta(n) \) are functions of \( n \) which are not necessarily equal, then also \( \mathcal{F}(x,y,t) \) is called a bilinear generating function for the set \( \{ f_n(x) \} \).

**III. Bilateral generating functions**

If a three-variable function \( H(x,y,t) \) has a formal power series expansion in \( t \) such that

\[
H(x,y,t) = \sum_{n=0}^{\infty} h_n f_n(x) g_n(y) t^n, \tag{1.3.6}
\]

where the sequence \( \{ h_n \} \) is independent of \( x, y \) and \( t \) and the sets of functions \( \{ f_n(x) \}_{n=0}^{\infty} \) and \( \{ g_n(x) \}_{n=0}^{\infty} \) are different. Then \( H(x,y,t) \) is called a bilateral generating function for the set of \( \{ f_n(x) \} \) or \( \{ g_n(x) \} \).
The above definition of a bilateral generating function, may be extended to include bilateral generating function of the type:

\[ H(x, y, t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x) g_{\beta(n)}(y) t^n \]  

where the sequence \( \{\gamma_n\} \) is independent of \( x, y \) and \( t \), the sets of functions \( \{f_n(x)\}_{n=0}^{\infty} \) and \( \{g_n(x)\}_{n=0}^{\infty} \) are different and \( \alpha(n) \) and \( \beta(n) \) are functions of \( n \) which are not necessarily equal.

IV. Multi-variable generating functions

If \( G(x_1, x_2, \cdots, x_r; t) \) is a function of \( r + 1 \) variables, which has a formal expansion in powers of \( t \) such that

\[ G(x_1, x_2, \cdots, x_r; t) = \sum_{n=0}^{\infty} c_n g_n(x_1, x_2, \cdots, x_r) t^n, \]  

where the sequence \( \{c_n\} \) is independent of the variables \( x_1, x_2, \cdots, x_r \) and \( t \). Then \( G(x_1, x_2, \cdots, x_r; t) \) is called a multi-variable generating function for the set \( \{g_n(x_1, x_2, \cdots, x_r)\}_{n=0}^{\infty} \) corresponding to the nonzero coefficients \( c_n \).

Similarly, we extend the definition of bilinear and bilateral generating functions to include such multi-variable generating functions as:

\[ F(x_1, x_2, \cdots, x_r; y_1, y_2, \cdots, y_s; t) = \sum_{n=0}^{\infty} \gamma_n f_{\alpha(n)}(x_1, x_2, \cdots, x_r) f_{\beta(n)}(y_1, y_2, \cdots, y_s) t^n \]  

and

\[ H(x_1, x_2, \cdots, x_r; y_1, y_2, \cdots, y_s; t) = \sum_{n=0}^{\infty} h_n f_{\alpha(n)}(x_1, x_2, \cdots, x_r) g_{\beta(n)}(y_1, y_2, \cdots, y_s) t^n, \]  

respectively.

The Gauss hypergeometric function \( _2F_1 \) and the confluent hypergeometric function \( _1F_1 \) form the core of the special functions and include as special cases most of the commonly used functions. The \( _2F_1 \) includes as its special cases, many elementary functions,
Legendre functions of the first and second kinds, the incomplete beta function, complete elliptic integrals of the first and second kinds, Jacobi polynomials, Gegenbauer (or ultra spherical) polynomials, Legendre (or spherical) polynomials, Tchebycheff polynomials of the first and second kinds et cetera (see [123, p. 34-36]). On the other hand, \( \text{1F1} \) includes as its special cases, Bessel functions, Whittaker functions, incomplete gamma functions, error functions, parabolic cylinder (or Weber) functions, Bateman’s \( k \)-function, Hermite polynomials, Laguerre polynomials and functions, Poisson-Charlier polynomials et cetera (see [123, p. 39-41]).

We give the definitions of certain special functions and mention their relationship with the hypergeometric functions. (we consider only those special functions which will be used in our work).

1. Bessel functions

Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. For example, they arise in the study of free vibrations of a circular membrane and in finding the temperature distribution in a circular cylinder. They also occur in electromagnetic theory and numerous other areas of physics and engineering. Because of their close association with cylindrical-shaped domains, all solutions of Bessel’s equation are collectively called cylinder functions.

The Bessel functions \( J_n(x) \) are defined by means of the generating function

\[
\exp \left( \frac{x}{2} \left( t - \frac{1}{t} \right) \right) = \sum_{n=-\infty}^{\infty} J_n(x)t^n \quad (t \neq 0; \ |x| < \infty). \tag{1.3.11}
\]

The Bessel functions \( J_n(x) \) are also defined by the series

\[
J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{x}{2} \right)^{n+2k}}{k! \Gamma(1+n+k)} \quad (|x| < \infty), \tag{1.3.12}
\]

where \( n \) is a positive integer or zero and

\[
J_n(x) = (-1)^n J_{-n}(x), \tag{1.3.13}
\]

where \( n \) is a negative integer.
The Bessel functions $J_n(x)$ are solutions of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0. \quad (1.3.14)$$

We note that

$$J_n(x) = \frac{(x/2)^n}{\Gamma(1 + n)} {}_0F_1 \left[ -; n + 1; -\frac{x^2}{4} \right]. \quad (1.3.15)$$

Also, we recall that the generalized Bessel functions or the Bessel-Wright functions are defined by

$$J_n^{(m)}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!(n + mk)!}, \quad (1.3.16)$$

which for $x \to -x$ becomes

$$W_n(x; m) = \sum_{k=0}^{\infty} \frac{x^k}{k!(n + mk)!} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; m \in \mathbb{N}). \quad (1.3.17)$$

2. Tricomi functions

Tricomi functions are Bessel like functions. The Tricomi functions $C_n(x)$ are defined by means of the generating function

$$\exp \left( t - \frac{x}{t} \right) = \sum_{n=-\infty}^{\infty} C_n(x) t^n \quad (t \neq 0; \ |x| < \infty), \quad (1.3.18)$$

The Tricomi functions $C_n(x)$ are also defined by the series

$$C_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k! (n + k)!} \quad (n = 0, 1, 2, \ldots) \quad (1.3.19)$$

and are characterized by the following link with the ordinary Bessel functions $J_n(x)$:

$$C_n(x) = x^{-\frac{1}{2}} J_n(2 \sqrt{x}). \quad (1.3.20)$$

The Tricomi functions $C_n(x)$ are solutions of the differential equation

$$x \frac{d^2 y}{dx^2} + (n + 1) \frac{dy}{dx} + y = 0. \quad (1.3.21)$$
We note that
\[ C_n(x) = \frac{1}{\Gamma(1+n)} \, _0F_1[-; n+1; -x]. \] (1.3.22)

3. Legendre polynomials

The Legendre polynomials are closely associated with physical phenomena for which spherical geometry is important. In particular, these polynomials first arose in the problem of expressing the Newtonian potential of a conservative force field in an infinite series involving the distance variables of two points and their included central angle. Other similar problems dealing with either gravitational potentials or electrostatic potentials also lead to Legendre polynomials, as do certain steady-state heat-conduction problems in spherical-shaped solids and so forth.

The Legendre polynomials \( P_n(x) \) are defined by means of the generating function
\[
\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x) t^n \quad (|t| < 1; |x| \leq 1). \] (1.3.23)

The Legendre polynomials \( P_n(x) \) are also defined by the series
\[
P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)! (2x)^{n-2k}}{2^n k! (n-k)! (n-2k)!}. \] (1.3.24)

The Legendre polynomials \( P_n(x) \) are solutions of the differential equation
\[
(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0. \] (1.3.25)

We note that
\[
P_n(x) = \, _2F_1 \left[ -n, n+1; 1; \frac{1-x}{2} \right]. \] (1.3.26)

4. Hermite polynomials

The Hermite polynomials play an important role in problems involving Laplace’s equation in parabolic coordinates, in various problems in quantum mechanics and in probability theory.
The Hermite polynomials $H_n(x)$ are defined by means of the generating function
\[
\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \quad (|t| < \infty; \ |x| < \infty).
\] (1.3.27)

The Hermite polynomials $H_n(x)$ are also defined by the series
\[
H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (2x)^{n-2k}}{k! (n-2k)!}.
\] (1.3.28)

The Hermite polynomials $H_n(x)$ are solutions of the differential equation
\[
\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0.
\] (1.3.29)

We note that
\[
H_n(x) = (2x)^n \ 2F_0 \left[ -\frac{n}{2}, -\frac{n+1}{2}; -; -\frac{1}{x^2} \right].
\] (1.3.30)

Also, a second form of the Hermite polynomials denoted by $He_n(x)$ is defined by means of the generating function
\[
\exp \left( xt - \frac{t^2}{2} \right) = \sum_{n=0}^{\infty} He_n(x) \frac{t^n}{n!} \quad (|t| < \infty; \ |x| < \infty).
\] (1.3.31)

The Hermite polynomials $He_n(x)$ are also defined by the series
\[
He_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! x^{n-2k}}{2^k k! (n-2k)!}.
\] (1.3.32)

The polynomials $H_n(x)$ and $He_n(x)$ are both known in the literature as the Hermite polynomials. They are related to each other by
\[
He_n(x) = 2^{-n/2} \ H_n \left( \frac{x}{\sqrt{2}} \right), \quad H_n(x) = 2^{n/2} \ He_n(x\sqrt{2}).
\] (1.3.33)

In view of the above relations, we conclude that all the properties of the $He_n(x)$ can be deduced from the corresponding ones for $H_n(x)$.
5. Laguerre polynomials

The Laguerre polynomials play a key role in applied mathematics and physics, they are involved in the solutions to the wave equation of the hydrogen atom.

The Laguerre polynomials $L_n(x)$ are defined by means of the generating function
\[
\frac{1}{(1-t)} \exp \left( -\frac{xt}{1-t} \right) = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!} \quad (|t| < 1; \ 0 \leq x < \infty). \tag{1.3.34}
\]

The Laguerre polynomials $L_n(x)$ are also defined by the series
\[
L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(k!)^2 (n-k)!} x^k. \tag{1.3.35}
\]

The Laguerre polynomials $L_n(x)$ are solutions of the differential equation
\[
x \frac{d^2 y}{dx^2} + (1-x) \frac{dy}{dx} + ny = 0. \tag{1.3.36}
\]

We note that
\[
L_n(x) = _1F_1[-n; 1; x]. \tag{1.3.37}
\]

In many applications, particularly in quantum-mechanical problems, a generalization of the Laguerre polynomials $L_n(x)$ called the associated Laguerre polynomials is needed.

The associated Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by means of the generating function
\[
\frac{1}{(1-t)^{\alpha+1}} \exp \left( -\frac{xt}{1-t} \right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) \frac{t^n}{n!} \quad (|t| < 1; \ 0 \leq x < \infty). \tag{1.3.38}
\]

The associated Laguerre polynomials $L_n^{(\alpha)}(x)$ are also defined by the series
\[
L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (n+\alpha)!}{(n-k)! (\alpha+k)!} x^k. \tag{1.3.39}
\]

The associated Laguerre polynomials $L_n^{(\alpha)}(x)$ are solutions of the differential equation
\[
x \frac{d^2 y}{dx^2} + (1+\alpha-x) \frac{dy}{dx} + ny = 0. \tag{1.3.40}
\]
We note that
\[
L_n^{(\alpha)}(x) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1) \Gamma(\alpha + 1)} \, _1F_1[-n; \alpha + 1; x].
\] (1.3.41)

Also, we note that for \( \alpha = 0 \), we have
\[
L_n^{(0)}(x) = L_n(x).
\] (1.3.42)

6. Gegenbauer and Chebyshev polynomials

The Gegenbauer polynomials are closely connected with axially symmetric potential in \( n \) dimensions and are the generalization of the Legendre polynomials.

The Gegenbauer polynomials \( C_n^{(\nu)}(x) \) are defined by means of the generating function
\[
\frac{1}{(1 - 2xt + t^2)^\nu} = \sum_{n=0}^{\infty} C_n^{(\nu)}(x) t^n \quad (|t| < 1; \ |x| \leq 1).
\] (1.3.43)

The Gegenbauer polynomials \( C_n^{(\nu)}(x) \) are also defined by the series
\[
C_n^{(\nu)}(x) = \frac{1}{\Gamma(\nu)} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{\Gamma(n - k + \nu)(2x)^{n-2k}}{k!(n-2k)!}.
\] (1.3.44)

From equations (1.3.23) and (1.3.43), we have
\[
C_n^{(\frac{1}{2})}(x) = P_n(x).
\] (1.3.45)

The Gegenbauer polynomials \( C_n^{(\nu)}(x) \) are solutions of the differential equation
\[
(1 - x^2) \frac{d^2 y}{dx^2} - (2\nu + 1)x \frac{dy}{dx} + n(n + 2\nu)y = 0.
\] (1.3.46)

We note that
\[
C_n^{(\nu)}(x) = \left( \frac{n + 2\nu - 1}{n} \right) \, _2F_1 \left[ -n, n + 2\nu; \nu + \frac{1}{2}; \frac{1 - x}{2} \right].
\] (1.3.47)

The Gegenbauer polynomials contain Chebyshev polynomials, of which there are two kinds. The Chebyshev polynomials of the first kind \( T_n(x) \) are defined by means of the generating function
\[
\frac{(1 - xt)}{(1 - 2xt + t^2)} = \sum_{n=0}^{\infty} T_n(x) \, t^n \quad (|t| < 1; \ |x| \leq 1).
\] (1.3.48)
The Chebyshev polynomials of the first kind $T_n(x)$ are also defined by the series

$$T_n(x) = \frac{n \left(\frac{n}{2}\right)}{2} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \frac{(n-k-1)!}{k!} \frac{(2x)^{n-2k}}{(n-2k)!}. \quad (1.3.49)$$

Also, we have

$$T_0(x) = 1, \quad T_n(x) = \frac{n}{2} \lim_{\nu \to 0} \frac{C_n^{(\nu)}(x)}{\nu} \quad (n = 0, 1, 2, ...). \quad (1.3.50)$$

The Chebyshev polynomials of the second kind $U_n(x)$ are defined by means of the generating function

$$\frac{1}{(1 - 2xt + t^2)} = \sum_{n=0}^{\infty} U_n(x) t^n \quad (|t| < 1; \ |x| \leq 1). \quad (1.3.51)$$

The Chebyshev polynomials of the second kind $U_n(x)$ are also defined by the series

$$U_n(x) = \sum_{k=0}^{|n|} \frac{(-1)^k (n-k)!}{k!(n-2k)!} \frac{(2x)^{n-2k}}{n!}. \quad (1.3.52)$$

Also, we have

$$C_n^{(1)}(x) = U_n(x) \quad (n = 0, 1, 2, ...). \quad (1.3.53)$$

The polynomials $T_n(x)$ and $U_n(x)$ satisfy the differential equations

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0 \quad (1.3.54)$$

and

$$(1 - x^2) \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + n(n+2)y = 0, \quad (1.3.55)$$

respectively.

We note that

$$T_n(x) = _2F_1 \left[ -n, \frac{1}{2}; \frac{1-x}{2} \right] \quad (1.3.56)$$

and

$$U_n(x) = (n+1) _2F_1 \left[ -n, \frac{1}{2}; \frac{3-x}{2} \right]. \quad (1.3.57)$$

In the next section, we give some operational identities and the concepts related to inverse derivative operators and monomiality principle.
1.4. Operational Methods and Special Functions

Operational methods provide a systematic and analytic approach to study special functions. The operational techniques are based upon single, double and multiple integral transforms and upon certain operators involving derivatives. Methods connected with the use of integral transforms have been successfully applied to the solutions of differential and integral equations, the study of generalized special functions and the evaluation of integrals. Operational techniques are important because they are closer to implementations and language definitions than more abstract mathematical techniques.

Operational methods provide powerful techniques to solve problems both in classical and quantum mechanics. The distinctive feature of these tools is their versatility and the possibility of exploiting them in absolutely different contexts, from the time-dependent Schrödinger problems to the charged beam transport problems in accelerators. Differential equations have been the primary motivation for the introduction of these techniques. Operational methods have become “popular” in applied science for their wide flexibility and have stimulated the development of new computer languages, useful for symbolic manipulation.

Dattoli and his co-authors proposed the use of operational methods in connection with the study of classical and new sets of the special functions, including the multi-dimensional and multi-index case. It has been shown that the operational methods can be used to simplify the derivation of the properties of ordinary and generalized special functions and also provide a unique tool to treat various polynomials from a general and unified point of view, see for example [22,23,32-34,37,39,40]. A useful introductory exposition of operational techniques and special polynomials is given by Ricci and Tavkhelidze [110].

We deal with the formalism of the exponential operators, which are the most commonly used operators to treat evolution problems. We require the rules for the disentanglement of exponential operators. We recall some useful operational results contained in [40].
Shift operator and disentanglement identities

Consider an analytic function \( f(x) \) so that the corresponding Taylor’s series expansion

\[
f(x + \lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} f^{(k)}(x),
\]

converges to the corresponding values of \( f \) in a suitable neighborhood of \( x \). The most commonly used exponential operator is the shift or translation operator \( \exp(\lambda \frac{d}{dx}) \), where \( \lambda \) is a parameter. The action of the shift operator on a function of \( x \) produces a shift of the variable by \( \lambda \) and thus it is defined as follows:

\[
\exp \left( \lambda \frac{d}{dx} \right) \{ f(x) \} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \frac{d^n}{dx^n} f(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f^{(n)}(x) = f(x + \lambda).
\]

To derive equation (1.4.2), we made the requirement, assumed valid for all the exponential operators we will deal with, that \( \exp(\lambda \frac{d}{dx}) \) can be expanded in Taylor’s series and \( f(x) \) is continuous and infinitely differentiable. As a consequence of identity (1.4.2), we have the following identities:

\[
\exp \left( \lambda x \left( \frac{d}{dx} \right) \right) \{ f(x) \} = f \left( e^\lambda x \right), \quad (1.4.3a)
\]

\[
\exp \left( \lambda x^2 \left( \frac{d}{dx} \right) \right) \{ f(x) \} = f \left( \frac{x}{1 - \lambda x} \right) \left( |x| < \frac{1}{\lambda} \right), \quad (1.4.3b)
\]

\[
\exp \left( \lambda x^n \left( \frac{d}{dx} \right) \right) \{ f(x) \} = f \left( \frac{x}{\sqrt[n]{1 - \lambda (n-1)x^{n-1}}} \right) \left( |x| < \frac{1}{\lambda(n-1)} \right). \quad (1.4.3c)
\]

In the case, when we consider the exponential of operators \( \hat{A} \) and \( \hat{B} \), we have, unlike with respect to the scalar case

\[
e^{\hat{A}+\hat{B}} \neq e^{\hat{A}} e^{\hat{B}}. \quad (1.4.4)
\]

Therefore, disentanglement techniques correcting the difference between the two sides of relation (1.4.4) are introduced in [40]. The disentanglement depends on the given operators \( \hat{A} \) and \( \hat{B} \) and their relevant commutator \([\hat{A}, \hat{B}]\), where

\[
[\hat{A}, \hat{B}]f = \hat{A}(\hat{B}f) - \hat{B}(\hat{A}f). \quad (1.4.5)
\]
We recall some of the commonly used disentanglement identities ([40,110]).

**Weyl identity**

Let \( \hat{A} \) and \( \hat{B} \) be two operators satisfying the commutation relations

\[
[\hat{A}, \hat{B}] = k, \quad [k, \hat{A}] = [k, \hat{B}] = 0.
\]  

(1.4.6)

Then the Weyl identity holds:

\[
e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-k/2} \quad (k \in \mathbb{C}).
\]  

(1.4.7)

**Sack identity**

Let \( \hat{A} \) and \( \hat{B} \) be two operators satisfying the commutation relation

\[
[\hat{A}, \hat{B}] = -\lambda \hat{A}.
\]  

(1.4.8)

Then the Sack identity holds:

\[
e^{\hat{A} + \hat{B}} = e^{\frac{e^{-1}}{\lambda} \hat{A} \hat{B}}.
\]  

(1.4.9)

**Berry identity**

Let \( \hat{A} \) and \( \hat{B} \) be two operators satisfying the commutation relation

\[
[\hat{A}, \hat{B}] = m\hat{A}^{1/2}.
\]  

(1.4.10)

Then the Berry identity holds:

\[
e^{\hat{A} + \hat{B}} = e^{m^2/12} e^{-(m/2)\hat{A}^{1/2} + \hat{A}} e^{\hat{B}}.
\]  

(1.4.11)

**Hausdorff identity**

Let \( \hat{A} \) and \( \hat{B} \) be two operators independent of the parameter \( \lambda \). Then the Hausdorff identity holds:

\[
e^{\lambda \hat{A} \hat{B} e^{-\lambda \hat{A}}} = \hat{B} + \lambda [\hat{A}, \hat{B}] + \frac{\lambda^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{\lambda^3}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \ldots.
\]  

(1.4.12)
Crofton identity

The Crofton identity is given in the form

\[
\exp\left(\lambda \frac{d^m}{dx^m}\right) \left\{ f(x) \right\} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} f^{(m)}(x) = f \left( x + m \lambda \frac{d^{m-1}}{dx^{m-1}} \right) \{1\}. \tag{1.4.13}
\]

Note that, for \( m = 2 \), Crofton identity (1.4.13) becomes

\[
\exp\left(\lambda \frac{d^2}{dx^2}\right) \left\{ f(x) \right\} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} f^{(2)}(x) = f \left( x + 2 \lambda \frac{d}{dx} \right) \{1\} \tag{1.4.14}
\]

and for \( m = 1 \), Crofton identity (1.4.13) reduces to the shift identity (1.4.2).

Glaisher identity

The Glaisher identity is given in the form [21] (see [33])

\[
\exp\left(\lambda \frac{d^2}{dx^2}\right) \left\{ \exp(-x^2) \right\} = \frac{1}{\sqrt{1+4\lambda}} \exp\left(-\frac{x^2}{1+4\lambda}\right). \tag{1.4.15}
\]

The generalization of the Glaisher identity (1.4.15) is given as [33]:

\[
\exp\left(\lambda \frac{d^2}{dx^2}\right) \left\{ \exp(-ax^2 + bx) \right\} = \frac{1}{\sqrt{1+4a\lambda}} \exp\left(-\frac{ax^2 - bx - b^2\lambda}{1+4a\lambda}\right). \tag{1.4.16}
\]

The other useful forms of the Glaisher identity are given as [33]:

\[
\exp\left(\lambda \frac{\partial^2}{\partial x \partial y}\right) \left\{ \exp(-x^2 - y^2) \right\} = \frac{1}{\sqrt{1-4\lambda^2}} \exp\left(-\frac{x^2 + y^2 - 4\lambda xy}{1-4\lambda^2}\right) \quad (|\lambda| < \frac{1}{2}), \tag{1.4.17}
\]

\[
\exp\left(-\lambda \left( \frac{d}{dx} x \frac{d}{dx} \right) \right) \left\{ \exp(-xt) \right\} = \frac{1}{1-\lambda t} \exp\left(-\frac{xt}{1-\lambda t}\right) \quad (|\lambda t| < 1). \tag{1.4.18}
\]

Also, we note the following disentanglement identity:

\[
e^{\hat{A} + \hat{B}} = (1 + m\hat{A})^{1/m} e^{\hat{B}}, \quad [\hat{A}, \hat{B}] = m\hat{A}^2. \tag{1.4.19}
\]

The operational methods can be used to develop the methodology of inverse differential operators and to derive a number of operational identities involving various inverse differential operators.
Inverse derivative operator

There are both practical and theoretical reasons for examining the process of inverting differential operators. Indeed, the inverse or integral form of a differential equation displays explicitly the input-output relationship of the system. From theoretical point of view, it may be advantageous to apply computational procedures to differential systems, based on the inverse or integral description of the system.

In mathematics, an inverse function is a function that undoes another function: If an input $x$ into the function $f$ produces an output $y$, then putting $y$ into the inverse function $g$ produces the output $x$ and vice-versa, i.e., $f(x) = y$ and $g(y) = x$ or $g(f(x)) = x$. If a function $f$ has an inverse $f^{-1}$, it is called invertible and the inverse function is then uniquely determined by $f$. A relation can be determined to have an inverse, if it is a one-to-one function. We can develop similar approach with regard to differential operators. The notation commonly used for the study of differential equations is designed rather for treating boundary conditions than for understanding of differential operators. Consequently, the concept of the inverse of a differential operator is not common. We discuss the inverse derivative operator, in order to make use of operational techniques for solving a variety of differential equations and producing useful relations involving differential operators, special functions and series of extended forms of polynomials.

Denoting the inverse of the derivative operator $\hat{D}_x := \frac{\partial}{\partial x}$ by $\hat{D}_x^{-1}$, we recall that the symbol $a\hat{D}_x^{-1}$ denotes an operator whose action on a function $f(x)$ is such that

$$a\hat{D}_x^{-1}\{f(x)\} = \int_a^x f(\xi)d\xi.$$  \hfill (1.4.20)

The subscript on the l.h.s. of equation (1.4.20) will be omitted, if the lower integration limit $a$ is zero. The action of the inverse derivative of the $n^{th}$ order

$$\hat{D}_x^{-n}\{f(x)\} = \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1}f(\xi)d\xi,$$  \hfill (1.4.21)

can be complemented with the definition for its $0^{th}$ order action as follows:

$$\hat{D}_x^0\{f(x)\} = f(x).$$  \hfill (1.4.22)
Hence, we can write

\[
\hat{D}_x^{-n}\{1\} = \frac{x^n}{n!} \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).
\] 

(1.4.23)

**Monomiality principle**

The combination of operational techniques and monomiality principle in the case of multi-variable special functions provides new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems. The suggestion of the concept of poweroid by Steffensen [124] is behind the idea of monomiality. This idea is reformulated and systematically used by Dattoli [22].

It turns out that all polynomial families and in particular all special polynomials, are essentially the same, since it is possible to obtain each of them transforming a basic monomial set by means of suitable operators (called the multiplicative and derivative operators respectively of the considered family). This was shown by theoretical proofs in [11,12] and can be viewed as the basis of the umbral calculus [112] - a term invented by Sylvester - since the exponent, for example in \(x^n\), is transformed into its “shadow” in \(p_n(x)\).

But, the multiplicative and derivative operators for a general set of polynomials are given by formal series of the ordinary derivative operator. It is therefore not possible to obtain sufficiently simple formulas to work with. However, for particular polynomials, related to suitable classes of generating functions, the above mentioned formal series reduce to finite sums, so that the relevant properties can be easily derived.

Recalling the leading work of Dattoli [22], we state the following definition:

**Definition 1.4.1** The polynomial set \(\{p_n(x)\}_{n\in\mathbb{N}}\) is quasi-monomial, if there exist two operators \(\hat{M}\) and \(\hat{P}\), called respectively the multiplicative operator and derivative operator, satisfying for all \(n \in \mathbb{N}\), the identities

\[
\hat{M}\{p_n(x)\} = p_{n+1}(x),
\]

(1.4.24a)

\[
\hat{P}\{p_n(x)\} = n \, p_{n-1}(x).
\]

(1.4.24b)
The operators $\hat{M}$ and $\hat{P}$ satisfy the commutation relation

$$[\hat{P}, \hat{M}] = \hat{P} \hat{M} - \hat{M} \hat{P} = \hat{1}$$  \hspace{1cm} (1.4.25)

and thus display a Weyl group structure.

If the considered polynomial set $\{p_n(x)\}$ is quasi-monomial, its properties can be easily derived from those of the $\hat{M}$ and $\hat{P}$ operators. In fact the following holds:

(i) If $\hat{M}$ and $\hat{P}$ have a differential realization, then the polynomials $p_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}\{p_n(x)\} = n \ p_n(x).$$  \hspace{1cm} (1.4.26)

(ii) Assuming here and in the following $p_0(x) = 1$, then the polynomials $p_n(x)$ can be explicitly constructed as:

$$p_n(x) = \hat{M}^n\{p_0(x)\} = \hat{M}^n\{1\}.$$  \hspace{1cm} (1.4.27)

(iii) The last identity implies that the exponential generating function of the polynomials $p_n(x)$ can be cast in the form

$$\exp(t\hat{M})\{1\} = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!} \quad (|t| < \infty).$$  \hspace{1cm} (1.4.28)

The concepts and the formalism associated with the monomiality treatment can be exploited in different ways. On one side, they can be used to study the properties of ordinary or generalized special polynomials by means of a formalism closer to that of natural monomials. On the other side, they can be useful to establish rules of operational nature, framing the special polynomials within the context of particular solutions of generalized forms of partial differential equations.

Most of the properties of families of polynomials, recognized as quasi-monomial, can be deduced by using operational rules associated with the relevant multiplicative and derivative operators. The notion of quasi-monomiality has been exploited within different contexts to deal with isospectral problems [119] and to study the properties
of new families of special functions, see for example [22,46,90,91]. Thus, we can define families of isospectral problems by exploiting the correspondence

\[ \hat{M} \leftrightarrow x, \quad \hat{P} \leftrightarrow \frac{\partial}{\partial x}, \quad p_n(x) \leftrightarrow x^n. \]  

(1.4.29)

There is a continuous demand of operational techniques in research fields like classical and quantum optics and in these fields the use of operational techniques has provided powerful and efficient means of investigation. Most of the interest is relevant to operational identities associated with ordinary and multi-variable forms of Hermite and Laguerre polynomials, see for example [22,32,49,50,80].

The two basic examples of the special polynomials, which can be framed within the context of monomiality principle formalism are the 2-variable Hermite-Kampé de Fériet polynomials (2VHKdFP) \( H_n(x, y) \) [7] and the 2-variable Laguerre polynomials (2VLP) \( L_n(x, y) \) [23,47].

In the next section, we show how the properties of special polynomials can be derived by using the monomiality principle. We consider the 2VHKdFP \( H_n(x, y) \) and the 2VLP \( L_n(x, y) \).

1.5. Hermite and Laguerre Polynomials of Two Variables

Generalized forms of Laguerre and Hermite polynomials provide a possibility to write solutions in most general cases in the form of series or sums and allow for compact form of the solutions, facilitating their analysis. Moreover, they frequently give links to special functions, which is sometimes even more advantageous.

2-variable Hermite-Kampé de Fériet polynomials

Hermite polynomials are frequently exploited in many branches of pure and applied mathematics and physics. The importance of multi-variable Hermite polynomials has been recognized in [26,36] and these polynomials have been exploited to deal with quantum mechanical and optical beam transport problems.
The 2VHKdFP $H_n(x,y)$ are defined by the series [7, p. 341]:

$$H_n(x,y) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{y^k x^{n-2k}}{k!(n-2k)}.$$  \hspace{1cm} (1.5.1)

There exists the following relationship [7, p. 341]:

$$H_n(x,y) = (-i)^n y^{n/2} H_n\left(\frac{ix}{2\sqrt{y}}\right) = i^n (2y)^{n/2} He_n\left(\frac{x}{i\sqrt{2y}}\right),$$  \hspace{1cm} (1.5.2)

of the 2VHKdFP $H_n(x,y)$ with the classical Hermite polynomials $H_n(x)$ and $He_n(x)$. Also, we note that

$$H_n(2x, -1) = H_n(x),$$  \hspace{1cm} (1.5.3)

$$H_n\left(x, -\frac{1}{2}\right) = He_n(x).$$  \hspace{1cm} (1.5.4)

Also, the 2VHKdFP $H_n(x,y)$ satisfy the property

$$t^n H_n(x,y) = H_n(xt, yt^2).$$  \hspace{1cm} (1.5.5)

The relationship of the 2VHKdFP $H_n(x,y)$ with the heat problem is well known. The heat problem is defined in the half-plane $y > 0$ as follows:

$$\frac{\partial S}{\partial y} = \frac{\partial^2 S}{\partial x^2},$$  \hspace{1cm} (1.5.6)

$$S(x,0) = f(x),$$  \hspace{1cm} (1.5.7)

where $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is an analytic function.

The heat problem (1.5.6), (1.5.7) admits the formal solution

$$S(x,y) = \exp(y\hat{D}_x^2)\{f(x)\}.$$  \hspace{1cm} (1.5.8)

The solution of the heat equation can also be represented by the Gauss-Weierstrass transform [126]

$$S(x,y) = \frac{1}{2^{\frac{n}{2}}\sqrt{\pi y}} \int_{-\infty}^{+\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4y}} d\xi.$$  \hspace{1cm} (1.5.9)

Consequently from equations (1.5.8) and (1.5.9), the integral representation of the exponential operator of the second-order derivative follows:

$$\exp(y\hat{D}_x^2)\{f(x)\} = \frac{1}{2^{\frac{n}{2}}\sqrt{\pi y}} \int_{-\infty}^{+\infty} f(\xi) e^{-\frac{(x-\xi)^2}{4y}} d\xi.$$  \hspace{1cm} (1.5.10)
Expanding $f(x)$ in series and recalling

$$\exp(y\hat{D}^2_x)\{f(x)\} = \sum_{n=0}^{\infty} a_n H_n(x,y),$$

(1.5.11)

the Gauss-Weierstrass transform representation of the 2VHKdFP $H_n(x,y)$ follows:

$$H_n(x,y) = \frac{1}{2\sqrt{\pi}y} \int_{-\infty}^{+\infty} \xi^n e^{-\frac{(x-\xi)^2}{4y}} d\xi.$$  (1.5.12)

This is to emphasize the importance of the 2VHKdFP $H_n(x,y)$, which are, for any integral $n$, particular solutions of the heat problem, by means of which solution (1.5.11) can be constructed for any analytic function $f$ such that the corresponding Gauss-Weierstrass transform exists. Conditions under which the solution exists are deeply studied in [126].

Now, we show that the properties of the 2VHKdFP $H_n(x,y)$ can be derived by using the concepts and the formalism associated with monomiality principle.

We recall that the 2VHKdFP $H_n(x,y)$ are quasi-monomial with respect to the following multiplicative and derivative operators $[22]:$

$$\hat{M}_H := x + 2y \frac{\partial}{\partial x}$$

(1.5.13a)

and

$$\hat{P}_H := \frac{\partial}{\partial x},$$

(1.5.13b)

respectively.

Using the expression for $\hat{M}_H$ and $\hat{P}_H$ in monomiality principle equations (1.4.24a) and (1.4.24b), we find

$$\left(x + 2y \frac{\partial}{\partial x}\right) H_n(x,y) = H_{n+1}(x,y)$$

(1.5.14a)

and

$$\frac{\partial}{\partial x} H_n(x,y) = nH_{n-1}(x,y),$$

(1.5.14b)

respectively.

The above equations give the following recurrence relation satisfied by $H_n(x,y)$:

$$H_{n+1}(x,y) = xH_n(x,y) + 2nyH_{n-1}(x,y).$$

(1.5.15)
It is easily verified that $\hat{M}_H$ and $\hat{P}_H$ satisfy the commutation relation identical to (1.4.25), i.e., we have

$$[\hat{P}_H, \hat{M}_H] = \hat{1}. \quad (1.5.16)$$

Further, since $\hat{M}_H$ and $\hat{P}_H$ have differential realization, therefore using expressions for $\hat{M}_H$ and $\hat{P}_H$ given in equations (1.5.13a) and (1.5.13b) in the monomiality principle equation (1.4.26), we find

$$\left( x + 2y \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial x} \right) \{ H_n(x, y) \} = n \, H_n(x, y),$$

which yields the following differential equation satisfied by $H_n(x, y)$:

$$\left( 2y \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - n \right) H_n(x, y) = 0. \quad (1.5.17)$$

Again, since $H_0(x, y) = 1$, therefore from equation (1.4.27), we have

$$H_n(x, y) = \hat{M}_H^n \{ 1 \} = \left( x + 2y \frac{\partial}{\partial x} \right)^n \{ 1 \}, \quad (1.5.18)$$

which on simplifying yields series definition (1.5.1) of the 2VHKdFP $H_n(x, y)$.

Finally, using the expression for $\hat{M}_H$ in equation (1.4.28), we find

$$\exp \left( xt + 2yt \frac{\partial}{\partial x} \right) \{ 1 \} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}, \quad (1.5.19)$$

which on using the Weyl identity (1.4.7) to decouple the exponential operator in the l.h.s. gives the generating function for $H_n(x, y)$ as:

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}, \quad (|t| < \infty). \quad (1.5.20)$$

Also, since the 2VHKdFP $H_n(x, y)$ are solutions of the heat equation

$$\frac{\partial}{\partial y} H_n(x, y) = \frac{\partial^2}{\partial x^2} H_n(x, y), \quad (1.5.21)$$

$$H_n(x, 0) = x^n, \quad (1.5.22)$$

we find that the 2VHKdFP $H_n(x, y)$ are defined by the following operational rule:

$$H_n(x, y) = \exp \left( y \frac{\partial^2}{\partial x^2} \right) \{ x^n \}. \quad (1.5.23)$$
We note the following link between the $2VHKdFP H_n(x, y)$ and the hypergeometric function $\text{$_2F_1$}[.]$:

$$H_n(x, y) = x^n \text{$_2F_0$} \left[ -\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}; -\frac{4y}{x^2} \right].$$

(1.5.24)

**2-variable Laguerre polynomials**

Dattoli and Torre [47,48] introduced and discussed a theory of $2VLP L_n(x, y)$. The reason of interest for this family of Laguerre polynomials is due to their intrinsic mathematical importance and also due to their applications in physics. These polynomials are shown to be natural solutions of a particular set of partial differential equations which often appear in the treatment of radiation physics problems such as the electromagnetic wave propagation and quantum beam life-time in storage rings, the details are given in [127].

The $2VLP L_n(x, y)$ are defined by the series [23, p. 121]

$$L_n(x, y) = n! \sum_{k=0}^{n} \frac{(-1)^k x^k y^{n-k}}{(k!)^2(n-k)!}.$$  

(1.5.25)

These polynomials are linked to the classical Laguerre polynomials $L_n(x)$ by the relation

$$L_n(x, y) = y^n L_n \left( \frac{x}{y} \right),$$

(1.5.26)

also, we note that

$$L_n(x, 1) = L_n(x),$$

(1.5.27)

$$L_n(0, y) = y^n.$$  

(1.5.28)

The possibility of solving Schrödinger and Fokker-Planck-type equations, by exploiting methods based on the evolution operator technique and on generalized transforms, has been discussed by Dattoli et al. [38,40,52]. Within such a context, it has been shown that the so-called fractional Fourier transform [99] and Hankel transform [102] can be viewed as particular cases of an appropriate evolution operator [52] and that the method of the Gauss transform can be extended to a large class of diffusion problems, providing generalized forms of the heat equation [38,40]. The point of view
of Dattoli et al. [38,40,52] is further used to treat propagation or diffusion problems often encountered in charged beam transport and in optics.

We consider the Fokker-Plank equation [127]:
\[
\frac{\partial}{\partial y} f(x,y) = - \left( \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \right) f(x,y),
\]
(1.5.29)

which is used to study the beam life-time due to quantum fluctuation in storage rings [127]. In the case of storage-rings, equation (1.5.29) is solved with the initial condition
\[
f(x,0) = g(x)
\]
(1.5.30)

and a boundary condition
\[
f(1,y) = 0,
\]
(1.5.31)

which takes into account the effect of the vacuum pipe walls. The solution of equation (1.5.29) is thus formally written as:
\[
f(x,y) = \exp \left( -y \left( \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \right) \right) \{g(x)\}.
\]
(1.5.32)

If \(g(x)\) is just an exponential function, the solution is readily found in the form
\[
f(x,y) = \frac{1}{(1-yt)} \exp \left( \frac{-xt}{1-yt} \right),
\]
(1.5.33)

which is obtained by using the Glaisher’s operational rule (1.4.18). The r.h.s. of equation (1.5.33) is recognized as the generating function of the 2VLP \(L_n(x,y)\) which will be mentioned later.

Next, we recall that the 2VLP \(L_n(x,y)\) are quasi-monomial under the action of the following multiplicative and derivative operators [22]
\[
\hat{M}_L := y - \hat{D}_x^{-1}
\]
(1.5.34a)

and
\[
\hat{P}_L := -\hat{D}_x x \hat{D}_x = - \frac{\partial}{\partial x} x \frac{\partial}{\partial x},
\]
(1.5.34b)

respectively, where \(\hat{D}_x^{-1}\) denotes the inverse of the derivative operator \(\hat{D}_x := \frac{\partial}{\partial x}\) defined by equations (1.4.21) and (1.4.23). Now, we use the concepts associated with monomiality principle to derive certain properties of the 2VLP \(L_n(x,y)\).
Using the expressions for $\hat{M}_L$ and $\hat{P}_L$ in equations (1.4.24a) and (1.4.24b), we find

\[
\left( y - \hat{D}_x^{-1} \right) L_n(x, y) = L_{n+1}(x, y)
\]

(1.5.35a)

and

\[
\left( -\hat{D}_x x \hat{D}_x \right) L_n(x, y) = \left( -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} \right) L_n(x, y) = n L_{n-1}(x, y),
\]

(1.5.35b)

respectively.

Simplifying equations (1.5.35a) and (1.5.35b), we get the following pure and differential recurrence relations:

\[
L_{n+1}(x, y) = \frac{1}{(n+1)} \left( ((2n+1)y - x)L_n(x, y) - ny^2L_{n-1}(x, y) \right),
\]

(1.5.36)

\[
\frac{\partial}{\partial x} L_n(x, y) = \frac{n}{x} L_n(x, y) - \frac{ny}{x} L_{n-1}(x, y).
\]

(1.5.37)

It is easily verified that $\hat{M}_L$ and $\hat{P}_L$ satisfy the commutation relation identical to (1.4.25), i.e., we have

\[
[\hat{P}_L, \hat{M}_L] = \hat{1}.
\]

(1.5.38)

In order to derive the differential equation satisfied by $L_n(x, y)$, we substitute the expressions for $\hat{M}_L$ and $\hat{P}_L$ in equation (1.4.26), so that we have

\[
\left( y - \hat{D}_x^{-1} \right) \left( -\frac{\partial}{\partial x} x \frac{\partial}{\partial x} \right) \left\{ L_n(x, y) \right\} = n L_n(x, y),
\]

which gives

\[
\left( -y \frac{\partial}{\partial x} \left( x \frac{\partial}{\partial x} \right) L_n(x, y) + x \frac{\partial}{\partial x} L_n(x, y) \right) = n L_n(x, y).
\]

Simplifying the above equation, we get the following differential equation satisfied by $L_n(x, y)$:

\[
\left( xy \frac{\partial^2}{\partial x^2} - (x - y) \frac{\partial}{\partial x} + n \right) L_n(x, y) = 0,
\]

(1.5.39a)

or, equivalently

\[
\left( y \frac{\partial^2}{\partial x \partial y} - \left( n \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \right) L_n(x, y) = 0.
\]

(1.5.39b)
Further, since $L_0(x,y) = 1$, using expression of $\hat{M}_L$ in equation (1.4.27), we find

$$L_n(x,y) = \hat{M}^n_L \{1\} = (y - \hat{D}^{-1}_x)^n \{1\}, \quad (1.5.40)$$

which on using the binomial theorem in the r.h.s. and then using equation (1.4.23) yields series definition (1.5.25) of the 2VLP $L_n(x,y)$.

Finally, using the expression for $\hat{M}_L$ in equation (1.4.28), we find

$$\exp \left( t(y - \hat{D}^{-1}_x) \right) = \sum_{n=0}^{\infty} L_n(x,y) \frac{t^n}{n!}, \quad (1.5.41)$$

which on using the Weyl identity (1.4.7) in the l.h.s. and then using equation (1.4.23) and definition (1.3.19) gives the following generating function of the 2VLP $L_n(x,y)$:

$$\exp(yt) C_0(x t) = \sum_{n=0}^{\infty} L_n(x,y) \frac{t^n}{n!}, \quad (1.5.42a)$$

or, equivalently

$$\frac{1}{(1 - yt)} \exp \left( \frac{-xt}{1 - yt} \right) = \sum_{n=0}^{\infty} L_n(x,y) t^n \quad (|yt| < 1), \quad (1.5.42b)$$

where $C_0(x)$ denotes the 0th order Tricomi function. The $n^{th}$ order Tricomi functions $C_n(x)$ are defined by equation (1.3.19).

To find the operational rule for the 2VLP $L_n(x,y)$, we note that since

$$\frac{\partial}{\partial y} L_n(x,y) = n \, L_{n-1}(x,y). \quad (1.5.43)$$

Therefore, in view of equation (1.5.35b), it follows that the 2VLP $L_n(x,y)$ are the natural solutions of the equation

$$\frac{\partial}{\partial y} L_n(x,y) = -\left( \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \right) L_n(x,y), \quad (1.5.44)$$

with the initial condition

$$L_n(x,0) = \frac{(-x)^n}{n!}, \quad (1.5.45)$$

which is kind of heat diffusion equation of type (1.5.29). As a consequence of equations (1.5.44) and (1.5.45), the 2VLP $L_n(x,y)$ are defined by the following operational rule:

$$L_n(x,y) = \exp \left( -y \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \right) \left\{ \frac{(-x)^n}{n!} \right\}. \quad (1.5.46)$$
In view of equation (1.5.40) and using the shift identity (1.4.2), we have the following equivalent form of operational rule (1.5.46):

\[ L_n(x, y) = \exp \left( -\hat{D}_x^{-1} \frac{\partial}{\partial y} \right) \{ y^n \}. \] (1.5.47)

We note the following link between the 2VLP \( L_n(x, y) \) and the confluent hypergeometric function \( _1F_1[..] \):

\[ L_n(x, y) = y^n \, _1F_1 \left[ -n; 1; \frac{x}{y} \right]. \] (1.5.48)

Although the 2VHKdFP \( H_n(x, y) \) and the 2VLP \( L_n(x, y) \) are essentially the classical Hermite and Laguerre polynomials \( H_n(x) \) and \( L_n(x) \) respectively being linked to them by obvious relations (1.5.2) and (1.5.26) respectively, but these are regarded as two-variable polynomials because of a further “degree of freedom” which allows the derivation of a number of useful identities in a fairly straightforward way and provide the introduction of new families of Hermite and Laguerre based special polynomials.