Stable laws also called $\alpha$-stable, stable Paretian or Lévy stable were introduced by Lévy (1925) during his study of sums of independently and identically distributed (i.i.d.) random variables. The general central limit theorem asserts that if a suitably normalized sum of random variables has a limit distribution, only possible limits are the stable distributions (Chapter 6 of Feller (1971)). Stable distributions are a rich class of probability distributions that allow skewness and heavy tails. Stable distribution is a direct generalization of the popular Gaussian distribution, and shares a lot of useful properties. It allows us to describe impulsive processes by means of a small number of parameters. Many physical phenomena are non-Gaussian and if the observed data have frequently occurring extreme values, then the phenomena may be modeled as a random process with an $\alpha$-stable dis-
Chapter 5. q-stable Distribution and Related Distributions with Applications in Modelling Gene Expression Data

Stable density behaves approximately like a Gaussian density near the origin, its tails decay at a lower rate than the Gaussian density tails. Since stable distributions can accommodate the fat tails and asymmetry, they often give a very good fit to empirical data. Samorodnitsky and Taqqu (2004) described the important properties of this distribution. Stable laws have found applications in a variety of fields, including physics, astronomy, finance, biology and electrical engineering. Stable distributions have been proposed as a model for many types of physical and economic systems.

A multivariate extension of the stable distribution is also available in the literature. The multivariate stable laws are only partially accessible due to the lack of closed form expressions for densities, and the possible complexity of the dependence structures. In the bivariate case, there are some methods of computing densities and estimation and extensions to higher dimensions are more complex. In order to handle this problem researchers use polar form, elliptically contoured stable laws etc.

A large-scale experiment which involves monitoring the expression levels of thousands of genes simultaneously under a particular condition is called gene expression analysis. Microarray technology makes this possible and the quantity of data generated from each experiment is enormous, dwarfing the amount of data generated by genome sequencing projects. Microarray technology has become one of the indispensable tools that many biologists use to monitor genome wide expression levels of genes in a given organism. A microarray is typically a glass slide on to which DNA molecules are fixed in an orderly manner at specific locations called spots. A microarray may contain thousands of spots and each spot may contain a few million copies of identical DNA molecules that uniquely correspond to a gene. The DNA in a spot may either be genomic DNA or short stretch of oligo-nucleotide strands that correspond to a gene. The spots are printed on to the glass slide by a robot or are synthesised by the process of photolithography.

During the last few years, a measurement technique called DNA microarrays has become widely used and substantially developed, both on the technical and the analytical
side. DNA microarrays are efficient tools for analyzing gene expression for large sets of genes simultaneously. Microarrays have become an important tool for studying the molecular basis of complex disease traits and fundamental biological processes.

Recently various researchers have used the microarrays to investigate the behavior of thousands of genes simultaneously under various conditions. A great deal of statistical research has focused on eliminating known biases introduced at different stages of the process and then discriminating which genes are differentially expressed across the conditions of interest. One challenging problem in the statistical analysis of microarray data is that the datasets have thousands of dimensions (genes) but only relatively few (biological) repetitions, typically 3 to 10. The challenge is then to take advantage of this structure, using the fact that many genes behave similarly and that many genes are measured on the same array, potentially carrying shared characteristics.

Lönnstedt and Speed (2002) proposed a normal mixture model for the gene expression data and defined a log posterior odd statistic for ranking genes. Smyth (2004) extended their approach to linear models for more general designs. Gottardo et al. (2003) developed an iterative method for estimating the proportion of differentially expressed genes. Kuznetsov (2001) modelled the distribution of gene expression using different classes of skewed probability function such as Poisson, logarithmic series and Pareto-like distribution. The results are shown only for Pareto-like distributions and it is proved that this distribution fits the empirical gene expression distribution better than other distributions.

Hoyle et al. (2002) analyzed a wide range of datasets empirically and the error distribution is approximated by two distributions: a log-normal in the bulk of microarray pot intensities and a power law in the tails. Purdom and Holmes (2005) modelled the gene expression data using the asymmetric Laplace distribution. Bhowmick et al. (2006) introduced a Laplace mixture model as a long-tailed alternative to the normal distribution when identifying differentially expressed genes in microarray experiments, and provide an extension to asymmetric for modelling over or under expressed data. Salas-Gonzalez
et al. (2009) modelled the error distribution for gene expression data using the \( \alpha \)-stable distribution and the distribution is tested for four different datasets.

In the present chapter, we introduce a new generalized Linnik distribution. Autoregressive process with this marginal distribution is developed. A geometric version of the generalized Linnik distribution is also introduced and its properties are studied. We study stable distribution and some of its important properties. The q-distributions and its applications are studied and a q-stable distribution is introduced. A numerical illustration is done on a microarray data. A symmetric q-stable autoregressive process is developed. The multivariate stable distribution is studied and a q-extension of the multivariate stable distribution is introduced. In the limiting case, the new generalized Linnik distribution and the q-stable distribution tend to a stable distribution.

### 5.1 Linnik or \( \alpha \)-Laplace distribution

The characteristic function (c.f) of a univariate Linnik distribution is

\[
\varphi(t) = \frac{1}{1 + |t|^\alpha}, \quad 0 < \alpha \leq 2.
\]  

(5.1.1)

When \( \alpha = 2 \), we get symmetric Laplace distribution. Therefore it is also known as \( \alpha \)-Laplace distribution, see Pillai (1985). Pakes (1998) introduced generalized Linnik distribution with c.f.

\[
\varphi(t) = \frac{1}{(1 + |t|^\alpha)^\nu}, \quad 0 < \alpha \leq 2, \ \nu > 0.
\]  

(5.1.2)

This distribution is called Pakes generalized Linnik distribution. Seethalekshmi and Jose (2006) introduced the geometric Pakes generalized Linnik with c.f.

\[
\varphi(t) = \frac{1}{\nu \ln(1 + |t|^\alpha)}, \quad 0 < \alpha \leq 2, \ \nu > 0.
\]  

(5.1.3)
5.2 A new class of generalized Linnik distribution

Now we can define a new class of the generalized Linnik distribution with c.f.

\[
\varphi(t) = \frac{1}{(1 + |\gamma t|^\alpha \omega(t, \alpha, \beta))^\nu}, \quad 0 < \alpha \leq 2, \quad -1 \leq \beta \leq 1, \quad \nu > 0,
\]

(5.2.1)

where

\[
\omega(t, \alpha, \beta) = \begin{cases} 
\exp(-i\beta \Phi(\alpha) \text{sign}(t)), & \alpha \neq 1; \\
\frac{\pi}{2} + i\beta \ln|t| \text{sign}(t), & \alpha = 1.
\end{cases}
\]

(5.2.2)

and

\[
\Phi(\alpha) = \begin{cases} 
\alpha \pi/2, & \alpha < 1; \\
(\alpha - 2)\pi/2, & \alpha > 1.
\end{cases}
\]

(5.2.3)

The c.f given in (5.1.1) does not contain the skewness parameter. The skewness parameter \( \beta \) can be introduced in Pakes generalized Linnik distribution and we can obtain the c.f in (5.2.1). The c.f. of a stable distribution is \( \exp(-|\gamma t|^\alpha \omega(t, \alpha, \beta)) \) (see Uchaikin and Zolotarev, 1999). If \( \nu = 1 \), then (5.2.1) gives geometric stable (GS) distribution with parameters \( \alpha \) and \( \beta \). The concept of geometric infinite divisibility was introduced by Klebanov et al. (1984). Mittnik and Rachev (1993) showed the one-to-one correspondence between c.f.’s of GS and stable random distributions: \( Y \) is GS if and only if its c.f. \( \psi \) has the form

\[
\psi(t) = \frac{1}{1 - \ln(g(t))}, \quad t \in \mathbb{R},
\]

(5.2.4)

where \( g(t) \) is the c.f. of stable distribution.

**Theorem 5.2.1.** If \( V \) and \( Y \) are independent random variables such that \( V \) has gamma distribution with Laplace-Stieltjes transform \( (1 + t)^{-\nu} \) and \( Y \) has stable distribution having c.f. \( g(t) = \exp(-|\gamma t|^\alpha \omega(t, \alpha, \beta)) \), then \( V^{1/\alpha} Y \overset{d}{=} W(\alpha, \beta, \nu) \), where \( W(\alpha, \beta, \nu) \) denotes a random variable having the c.f. (5.2.1).
Proof. Consider $W \overset{d}{=} V^{1/\alpha}Y$

$$\varphi_W(t) = E(\exp(itV^{1/\alpha}Y))$$
$$= E\left[E(\exp(itV^{1/\alpha}Y|Y))\right]$$
$$= E[\exp(-|\gamma t|^{\alpha}\omega(t, \alpha, \beta)V)]$$
$$= \frac{1}{(1 + |\gamma t|^{\alpha}\omega(t, \alpha, \beta))^{\nu}}.$$  

This representation can be used for simulating the new generalized Linnik law. When $\beta = 0$, this mixture representation defines the generalized symmetric Linnik law.

Remark 5.2.1. Stable laws can be regarded as the limit distribution of the new class of generalized Linnik distribution.

Proof. Stable laws can be regarded as the limit distribution of $S_n/n$, where $S_n = X_1 + \cdots + X_n$, where $X_i$'s are i.i.d. as generalized Linnik random variables. Using the c.f., we can easily prove the result.

5.3 Generalized Linnik first order autoregressive process

Consider the first order autoregressive model given by

$$X_n = aX_{n-1} + \varepsilon_n, \quad 0 < a < 1, \quad n = 0, \pm 1, \pm 2, \ldots,$$  

Rewriting in terms of c.f. on both sides, we get

$$\varphi_{X_n}(t) = \varphi_{X_n}(at)\varphi_{\varepsilon_n}(t).$$

Assuming stationarity, this yields

$$\varphi_{\varepsilon}(t) = \frac{\varphi_X(t)}{\varphi_X(at)}$$
We get the c.f. of $\varepsilon$ as

$$\varphi_\varepsilon(t) = \left[ |a|^\alpha + (1 - |a|^\alpha) \frac{1}{1 + |\gamma t|^\alpha \omega(t, \alpha, \beta)} \right]^\nu.$$ (5.3.2)

From this, it can be seen that the innovation $\{\varepsilon\}$ is distributed as the $\nu$-fold convolutions of random variables $V_n$ given by

$$V_n = \begin{cases} 0, & \text{with probability } |a|\alpha \\ L_n, & \text{with probability } 1 - |a|\alpha \end{cases}$$

where $L_n$’s are independently and identically distributed (i.i.d.) new generalized Linnik random variables.

### 5.4 A new class of geometric generalized Linnik distribution

Here we introduce a new distribution namely, geometric generalized Linnik distribution ($\text{GGLD}(\alpha, \beta, \nu)$). A random variable $X$ is said to follow geometric generalized Linnik distribution if its c.f. is given by

$$\varphi(t) = \frac{1}{1 + \nu \ln(1 + |\gamma t|^\alpha \omega(t, \alpha, \beta))}.$$ (5.4.1)

If $\nu = 1$ and $\beta = 0$, (5.4.1) reduces to geometric $\alpha$-Laplace distribution. If $\beta = 0$, (5.4.1) reduces to geometric Pakes generalized Linnik distribution.

**Theorem 5.4.1.** Let $X_1, X_2, \ldots$ be i.i.d. $\text{GGLD}(\alpha, \beta, \nu)$ random variables and $N$ be geometric with mean $\frac{1}{p}$, such that $P[N = k] = p(1 - p)^{k-1}, k = 1, 2, \ldots, 0 < p < 1$. Then $Y = X_1 + X_2 + \cdots + X_N \overset{d}{=} \text{GGLD}(\alpha, \beta, \nu/p)$.
Proof. The c.f. of \( Y \) is

\[
\phi_Y(t) = \sum_{k=1}^{\infty} [\phi_X(t)]^k p(1-p)^{k-1} = \frac{p/(1 + \nu \ln(1 + |\gamma t|^{\alpha} \omega(t,\alpha,\beta)))}{1 - ((1-p)/(1 + \nu \ln(1 + |\gamma t|^{\alpha} \omega(t,\alpha,\beta)))} = \frac{1}{1 + (\nu/p) \ln(1 + |\gamma t|^{\alpha} \omega(t,\alpha,\beta))}.
\]

Hence \( Y \overset{d}{=} \text{GGLD}(\alpha,\beta,\nu/p). \)

**Theorem 5.4.2.** Geometric generalized Linnik distribution \( \text{GGLD}(\alpha,\beta,\nu) \) is the limit distribution of geometric sum of random variables following generalized Linnik distribution.

**Proof.** We have \( [1 + |\gamma t|^{\alpha} \omega(t,\alpha,\beta)]^{-\nu/n} = \{1 + [1 + |\gamma t|^{\alpha} \omega(t,\alpha,\beta)]^{\nu/n} - 1\}^{-1} \) is the c.f. of a probability distribution since the new generalized Linnik distribution is infinitely divisible. Hence by Lemma 3.2 of Pillai (1990),

\[
\phi_n(t) = \left\{1 + n[(1 + |\gamma t|^{\alpha} \omega(t,\alpha,\beta))^{\nu/n} - 1]\right\}^{-1}
\]

is the c.f. of geometric sum of i.i.d. generalized Linnik random variables. Taking limit as \( n \to \infty \).

\[
\phi(t) = \lim_{n \to \infty} \phi_n(t) = \left\{1 + \lim_{n \to \infty} n[(1 + |\gamma t|^{\alpha} \omega(t,\alpha,\beta))^{\nu/n} - 1]\right\}^{-1} = [1 + \nu \ln(1 + |\gamma t|^{\alpha} \omega(t,\alpha,\beta))]^{-1}.
\]

**Theorem 5.4.3.** If \( X \) and \( Y \) are independent random variables such that \( X \) has geometric gamma distribution with Laplace transform \( 1/(\nu \ln(1 + t)) \) and \( Y \) has a stable distribution having c.f. \( g(t) = \exp(-|\delta t|^{\alpha} \omega(t,\alpha,\beta)) \), then \( X^{1/\alpha}Y \overset{d}{=} Z \) where \( Z \overset{d}{=} \text{GGLD}(\alpha,\beta,\nu). \)
Proof. Consider $Z \overset{d}{=} X^{1/\alpha} Y$.

$$\varphi_Z(t) = E(\exp(itX^{1/\alpha}Y))$$

$$= \int_{0}^{\infty} \varphi_Y(tx^{1/\alpha})dF(x)$$

where $F(.)$ is the distribution function of $X$

$$= \int_{0}^{\infty} \exp(-|t|\alpha \omega(t, \alpha, \beta))dF(x)$$

$$= \frac{1}{1 + \nu \ln(1 + |t|\alpha \omega(t, \alpha, \beta))}$$

5.5 Stable distribution

The stable distribution has characteristic function given by

$$\varphi_X(t) = \begin{cases} 
e^{-|t|\alpha[1-i\text{sign}(\omega)\beta \tan \frac{\pi \alpha}{2}]+i\mu t}; & \alpha \ne 1 \\
e^{-|t|\beta \log(|t|)+i\mu t}; & \alpha = 1 \end{cases}$$

where the characteristic exponent $\alpha$ is the index of stability and can also be interpreted as a shape parameter; $\beta$ is the skewness parameter; $\mu$ is a location parameter; $\gamma > 0$ is the dispersion, a scale parameter; and $t$ is the scale parameter.

There is no general closed form expression for the stable probability density function (p.d.f). It is possible to write the stable p.d.f. only for three particular cases. When $\alpha = 2$, it corresponds to a Gaussian distribution with $\gamma = \sigma/\sqrt{2}$ where $\sigma$ is the standard deviation. The case when $\alpha = 0$ and $\beta = 0$ corresponds to a Cauchy distribution and for $\alpha = 1/2$ and $\beta = 1$, it corresponds to a Pearson distribution. Thus the $\alpha$-stable distribution can be seen as a generalization of the normal distribution, and some features of linear system theory developed for normal distribution can be applied directly to this distribution.

The stable random variables with characteristic exponent $\alpha < 2$ have an infinite second order moment. When $\alpha < 2$, the tails probabilities behave like the power law. The $\alpha$-stable distribution is a family of distributions that presents heavy tails and is also capable
of exhibiting a certain degree of asymmetry. This distribution satisfies the stability property which states that any linear combination of random variables with $\alpha$-stable distribution is again an $\alpha$-stable distribution. It exhibits heavier tails than that of normal distribution. A statistical result of the fat-tails is that not all moments may exist. When a distribution has sufficiently long tails, the first few moments will not characterize the distribution because they diverge.

The density curve for a symmetric $\alpha$-stable distribution with $\beta = 0, \gamma = 1, \mu = 0$ for various values of $\alpha$ is given in Figure 5.1.

Figure 5.2 represents the histogram and empirical density corresponding to $\alpha = 1.9, \beta = 1.3, \gamma = 1, \mu = 0$ as the result of a simulation study.

The sample paths of the simulated random variables is studied by generating 100 observations and are given in the following figures. In Figure 5.3(a) and (b), $\alpha = 1.9, \beta = 1.3, \gamma = 1, \mu = 0$ and $\alpha = .8, \beta = .1, \gamma = 1, \mu = 0$ respectively.
Figure 5.2: The histogram and empirical density for $\alpha = 1.9, \beta = 1.3, \gamma = 1, \mu = 0$

Figure 5.3: (a) The sample path for $\alpha = 1.9, \beta = 1.3, \gamma = 1, \mu = 0$ (b) The sample path for $\alpha = 0.8, \beta = 1, \gamma = 1, \mu = 0$
5.6 q-stable distributions

Concepts related to non-extensive statistical mechanics have found applications in a variety of disciplines including physics, chemistry, biology, mathematics, geography, economics, medicine, informatics, linguistics and others. Probability distributions which emerge from the non-extensive formalism also called q-distributions have been applied to an impressive variety of problems in diverse research areas including the interdisciplinary field of complex systems. The success of q-distributions in describing diverse systems is quiet due to its ability to exhibit heavy-tails and model power law phenomena. In recent years, several q-type distributions have been introduced by various authors in the physics literature. q-distributions can arise when the exponential function of the original distribution is replaced by a q-exponential function. For example, this basic procedure applied in standard exponential, Gaussian and Weibull distributions leads to q-exponential, q-Gaussian and q-Weibull, respectively. These q distributions have been applied in the study of a wide variety of systems in several fields such as information theory, statistical mechanics, reliability modelling etc. This viewpoint suggests the consideration of other q-distributions which could be obtained by simply replacing its exponential function by a q-exponential one. The q-exponential distribution can be viewed as a distribution obtained by maximization of the Tsallis entropy introduced by Tsallis (1988) which is a generalization of Boltzmann-Gibbs entropy measure. The q-exponential function has the form

\[ e_q(x) = [1 + (1 - q)x]^{1/(1-q)} \]

The q-exponential distributions play an important role in non-extensive statistics. By choosing suitable values for q, q-exponential distributions may be used to represent both short and long tailed distributions. The q-exponential distribution can be viewed as a stretched model for exponential distribution so that the exponential form can be reached slowly as \( q \to 1 \). Oikonomou et al. (2008) used q-exponential distributions, which maximize the non-extensive entropy, to study the size distributions of non-coding DNA (includ-
ing introns and intergenic regions) in all human chromosomes. Various studies based on the q-type distributions including q-Weibull, in connection with Tsallis statistics are reported by Wilk and Wlodarczyk (2000) and Tsallis (1988). A study of dielectric breakdown in oxides of electronic devices is described by Costa et al. (2006) and the authors show that a q-Weibull distribution gives a good fit for the data. Picoli et al. (2009) made a brief review of q-exponential, q-Gaussian and q-Weibull distributions focusing some of their basic properties and recent applications. Jose and Naik (2009), Jose et al. (2010a) consider various processes and applications of q-Weibull distributions.

A q-version of the stable distribution can be defined by the characteristic function,

\[
\varphi_{q,X}(t) = \begin{cases} 
1 - (1 - q)\{ -|\gamma t|^{\alpha}[1 - isign(w)\beta \tan \frac{\pi \alpha}{2}] + i\mu t \} \frac{1}{1 - q}; & \alpha \neq 1, q < 1 \\
1 + (q - 1)\{ -|\gamma t|^{\alpha}[1 - isign(w)\beta \tan \frac{\pi \alpha}{2}] + i\mu t \} \frac{1}{1 - q}; & \alpha \neq 1, q > 1 
\end{cases}
\]

(5.6.1)

As \(q \to 1\), the above characteristic function becomes the characteristic function of a stable distribution and therefore it can be viewed as a more general case of the stable distribution. It can be used for modelling microarray data since such data are well fitted by stable distributions. The q-stable distribution will give a more flexible model than the usual stable distribution.

5.7 Data analysis

We consider the data of asynchronous control in the experiment namely alpha factor release sample026. The data can be downloaded from the Stanford Microarray Database at www.smd-ftp.stanford.edu/pub/smd/experiments/1-1000/103.xls.gz. We consider 2000 observations from the data set. Usually the microarray data are fat-tailed and sharp peaked at the origin. We make a logarithmic transformation to the data, \(\log \frac{X_{t+1}}{X_t}\).

The histogram of the log transformed data is given in Figure 5.4 (a) and the Q-Q plot the data and the stable random variables is given in Figure 5.4 (b). We obtain the estimates of the parameters from the data as \(\hat{\alpha} = 0.4985, \hat{\beta} = 0.0002, \hat{\gamma} = 0.9899, \hat{\delta} = 0.0001, \hat{\mu} =\)
0.0002, for details see Fama and Roll (1971), McCulloch (1986). We use the one sample Kolmogorov-Smirnov (K-S) statistic for testing the goodness of fit of the data to a stable distribution with the above parameter values. For that we consider the empirical cdf with respect to the data also. The p-value associated with the K-S test statistic is 0.1349. Therefore we accept the hypothesis that the data fits well to the stable distribution.

5.8 Symmetric q-stable autoregressive processes

When $\mu = 0$, we have

$$
\varphi_{q,X}(t) = \begin{cases} 
1 - (1 - q)\{-|\gamma t|^{\alpha}[1 - isign(w)\beta \tan \frac{\pi \alpha}{2}]\}^{\frac{1}{\alpha - 1}} & ; \alpha \neq 1, q < 1 \\
1 + (q - 1)\{-|\gamma t|^{\alpha}[1 - isign(w)\beta \tan \frac{\pi \alpha}{2}]\}^{\frac{1}{\alpha - 1}} & ; \alpha \neq 1, q > 1.
\end{cases} \tag{5.8.1}
$$

The corresponding q-stable distribution is symmetric. In such situations we can construct autoregressive processes of order 1 if the distribution is self-decomposable.

**Theorem 5.8.1.** Symmetric q-stable c.f. $\varphi$ is self decomposable or the corresponding distribution belongs to the class $\mathcal{L}$. 

Figure 5.4: (a) The histogram of the log transformed data (b) The Q-Q plot for the data
Proof. The c.f. of symmetric q-stable distribution when \( q > 1 \) is

\[
\varphi_{q,X}(t) = \left[ \frac{1}{1 + (q - 1)\{-|\gamma t|^\alpha[1 - isign(w)\beta \tan \frac{\pi \alpha}{2}]\}} \right]^{\frac{1}{q - 1}}
\]

\[
= \left[ \frac{1}{1 + (q - 1)\{-|\gamma at|^\alpha[1 - isign(w)\beta \tan \frac{\pi \alpha}{2}]\}} \right]^{\frac{1}{q - 1}}
\]

\[
\times \left[ \frac{1}{1 + (q - 1)\{-|\gamma t|^\alpha[1 - isign(w)\beta \tan \frac{\pi \alpha}{2}]\}} \right]^{\frac{1}{q - 1}}
\]

\[
= \varphi_{q,X,a}(t)\varphi_{q,X,a}(t),
\]

where

\[
\varphi_{q,X,a}(t) = \left[ \frac{1 + (q - 1)\{-|\gamma at|^\alpha[1 - isign(w)\beta \tan \frac{\pi \alpha}{2}]\}}{1 + (q - 1)\{-|\gamma t|^\alpha[1 - isign(w)\beta \tan \frac{\pi \alpha}{2}]\}} \right]^{\frac{1}{q - 1}}
\]

\[
= \left[ |a|^\alpha + (1 - |a|^\alpha) \frac{1}{1 + (q - 1)\{-|\gamma t|^\alpha[1 - isign(w)\beta \tan \frac{\pi \alpha}{2}]\}} \right]^{\frac{1}{q - 1}},
\]

which is the c.f. obtained as in (5.8.3). Therefore the symmetric q-stable c.f. \( \varphi \) is self decomposable and we can develop a first order autoregressive process. Similarly we can prove in the case of \( q < 1 \) also.

Consider the first order autoregressive model given by

\[
X_n = aX_{n-1} + \varepsilon_n, \quad 0 < a < 1, \quad n = 0, \pm 1, \pm 2, \ldots.
\]

(5.8.2)

Rewriting in terms of characteristic function on both sides, we get

\[
\varphi_{X_n}(t) = \varphi_{X_n}(at)\varphi_{\varepsilon_n}(t).
\]

Assuming stationarity, this yields

\[
\varphi_{\varepsilon}(t) = \frac{\varphi_{X}(t)}{\varphi_{X}(at)}.
\]
Let us consider the case when \( q < 1 \). We get the characteristic function of \( \varepsilon \) as

\[
\varphi_{\varepsilon}(t) = \left[ |a|^{\alpha} + (1 - |a|^{\alpha}) \frac{1}{1 - (1 - q) \{ -|\gamma t|^{\alpha}[1 - isign(w)\beta \tan \frac{\alpha}{2}] \}^{\frac{1}{q - 1}}} \right]^{\frac{1}{q - 1}}.
\]  \( (5.8.3) \)

From this, it can be seen that \( \varepsilon \) is distributed as a convolution of an \( \alpha \)-Laplace tailed random variable, which is given by

\[
ET = \begin{cases} 
0, & \text{with probability } |a|^{\alpha} \\
S_n, & \text{with probability } 1 - |a|^{\alpha},
\end{cases}
\]

where \( S_n \)'s are i.i.d. q-stable random variables with \( q < 1 \). Similarly we can develop the process for the case when \( q > 1 \). For details see Jose et al. (2010b).

**5.9 Multivariate q-stable distribution**

The characteristic function of a multivariate symmetric stable distribution is given by

\[
\psi(t) = e^{-\left\langle t, \Sigma t \right\rangle^{\alpha/2}}, \quad t \in \mathbb{R}^p, \alpha \in (0, 2].
\]  \( (5.9.1) \)

It can be treated as an extension of the multivariate normal distribution. When \( \alpha = 2 \), it is equivalent to the multivariate normal distribution. These distributions are used to generate multivariate Linnik distribution. Multivariate stable distributions have various applications. They can be used to model multivariate noise with heavy tails to use in evaluating the robustness of multivariate statistical methods. They are also capable of using Monte Carlo techniques.

The characteristic function of a multivariate q-stable distribution can be obtained as

\[
\psi_{q,X}(t) = \begin{cases} 
[1 + (q - 1)t'\Sigma t]^{\frac{1}{\alpha(1-q)}}, & q > 1 \\
[1 - (1-q)t'\Sigma t]^{\frac{1}{\alpha(1-q)}}, & q < 1
\end{cases}
\]  \( (5.9.2) \)
As $q \to 1$, equation (5.9.2) becomes $e^{-(t^\Sigma \alpha/2)}$, which is the characteristic function of a multivariate symmetric stable distribution. From the characteristic function, it is clear that it can be viewed as an extension of the multivariate Linnik distributions.

5.10 Conclusion

In this chapter we introduced a new generalized Linnik distribution. We developed the autoregressive process with this marginal distribution. A geometric version of the generalized Linnik distribution is also introduced and discussed its properties. We studied stable distribution and some of its important properties. The q-distributions and its important applications are studied. A q-stable distribution is introduced. A numerical illustration is done using a microarray data. A symmetric q-stable autoregressive process is developed. The multivariate stable distribution is studied and a q-extension of the multivariate stable distribution is introduced. In the limiting case, both the new generalized Linnik distribution and q-stable distribution tend to a stable distribution.

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