The normal-Laplace distribution, which results from the convolution of independent normal and Laplace random variables is introduced by Reed and Jorgensen (2004). It is the distribution of the stopped state of a Brownian motion with normally distributed starting value if the stopping hazard rate is constant. Normal-Laplace distribution is a new distribution which (in its symmetric form) behaves somewhat like the normal distribution in the middle of its range, and like the Laplace distribution in its tails. Reed and Jorgensen (2004) also introduced a generalized normal-Laplace distribution, which is useful in financial applications for obtaining an alternative stochastic process model to Brownian motion for logarithmic prices, in which the increments exhibit fatter tails than the normal distribution. Reed (2007) developed Brownian-Laplace motion for modelling financial asset price returns.

Multivariate normal distribution, a generalization of univariate normal distribution is
studied by various authors. They have applications in statistical inference, image processing etc. Ernst (1998) introduced multivariate extension of symmetric Laplace distributions via an elliptic contouring. Many properties in the univariate laws can be extended to this class of distributions. With an appropriate limit of the parameters of multivariate hyperbolic distributions, one can obtain a multivariate and asymmetric extension of the Laplace laws, see Blaesid (1981).

In the present chapter, we consider the univariate normal-Laplace distribution, introduce multivariate normal-Laplace distribution and study its properties. First order autoregressive processes with multivariate normal-Laplace marginals is developed. We introduce multivariate geometric normal-Laplace distribution. Multivariate generalized normal-Laplace distribution is introduced. We introduce the geometric generalized normal-Laplace distribution and study its properties. We consider the estimation of parameters. Some applications are also discussed.

4.1 Normal-Laplace Distribution and its Properties

The normal-Laplace distribution introduced by Reed and Jorgensen (2004), arises as the convolution of independent normal and asymmetric Laplace densities. A normal-Laplace random variable $X$ with parameters $\mu, \sigma^2, \alpha$ and $\beta$ can be represented as

$$X \overset{d}\equiv Z + W,$$

(4.1.1)

where $Z$ and $W$ are independent random variables with $Z$ following normal distribution with mean $\mu$ and variance $\sigma^2$ and $W$ following an asymmetric Laplace distribution with parameters $\alpha, \beta$. The corresponding normal-Laplace distribution shall be denoted by $\text{NL}(\mu, \sigma^2, \alpha, \beta)$. Various results on normal-Laplace distribution are available in Reed (2006).

The characteristic function (c.f.) of $\text{NL}(\mu, \sigma^2, \alpha, \beta)$ can be obtained as the product of the c.f.’s of its normal and Laplace components and is given by Reed and Jorgensen (2004)
as,
\[ \phi_X(t) = \left[ \exp\left(i\mu t - \frac{\sigma^2}{2} t^2\right) \right] \left[ \alpha \beta \left(\alpha - it\right)\left(\beta + it\right) \right]. \tag{4.1.2} \]

The normal-Laplace distribution is infinitely divisible and is closed under linear transformation. The mean, variance and cumulants exist for the distribution.

### 4.2 Multivariate Normal Laplace Distribution

A multivariate extension of the normal-Laplace distribution of Reed and Jorgensen (2004), namely the multivariate normal-Laplace distribution can be obtained as the convolution of multivariate normal and multivariate symmetric Laplace random vectors. The c.f. of multivariate normal-Laplace distribution is given by

\[ \varphi_X(t) = \left( \exp\left( it'\mu - \frac{1}{2} t'\Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2} t'Vt} \right), \quad t \in \mathbb{R}^p, \Sigma > 0, V > 0. \tag{4.2.1} \]

Some properties

A p-variate normal-Laplace distribution with parameters \( \mu, \Sigma \) and \( V \) can be denoted by \( \text{NL}_p(\mu, \Sigma, V) \). Let \( X \sim \text{NL}_p(\mu, \Sigma, V) \), then \( X \) can be expressed as

\[ X \overset{d}{=} Z + Y, \tag{4.2.2} \]

where \( Z \) and \( Y \) are independent random vectors with \( Z \) following a p-variate normal distribution with mean vector \( \mu \) and dispersion matrix \( \Sigma \) (\( \text{N}_p(\mu, \Sigma) \)) and \( Y \) following a p-variate symmetric Laplace distribution with parameter \( V \) (\( \text{L}_p(V) \)).

Another representation is

\[ X \overset{d}{=} Z + \sqrt{W}Y, \tag{4.2.3} \]

where \( Z \) follows a p-variate normal distribution with mean vector \( \mu \) and dispersion matrix \( \Sigma \), \( W \) is a standard exponential variable and \( Y \) follows p-variate normal distribution with mean vector \( 0 \) and dispersion matrix \( \Sigma \).
Moments. The \( k \)th moment \( M_k(x, \mu) \) of a random vector \( x \) and \( \mu \) is defined by

\[
M_k(x, \mu) = E[(x - \mu) \otimes (x - \mu)' \otimes \cdots \otimes (x - \mu) \otimes (x - \mu)'],
\]

if \( k = 2m - 1 \) and

\[
M_k(x, \mu) = E[(x - \mu) \otimes (x - \mu)' \otimes \cdots \otimes (x - \mu) \otimes (x - \mu)'],
\]

if \( k = 2m \). The Kronecker product of matrices \( A = (a_{ij}) : m \times n \) and \( B : p \times q \) is a \( mp \times nq \) matrix \( A \otimes B = (a_{ij}B) \). When \( \mu = 0 \), we write only \( M_k(x) \). By the differentiation of the c.f. of a random vector, we can obtain,

\[
M_k = \left. \frac{1}{i^k} \frac{\partial^k \phi_X(t)}{\partial t^i} \right|_{t=0}.
\]

The mean vector and variance-covariance matrix of \( \text{NL}_p(\mu, \Sigma, V) \) can be obtained as

\[
E(X) = \mu \quad \text{Cov}(X) = \Sigma + V. \quad (4.2.4)
\]

Cumulants. The \( k \)th cumulant of random vector \( x \) is denoted by \( C_k(x) \). The cumulants are obtained as the matrix derivatives of the function

\[
\phi_X(t) = \ln \phi_X(t)
\]

by

\[
C_k = \left. \frac{1}{i^k} \frac{\partial^k \phi_X(t)}{\partial t^i} \right|_{t=0}.
\]

For multivariate normal-Laplace distribution,

\[
C_1 = \mu \quad C_2 = \Sigma + V.
\]
Skewness and Kurtosis. Let \( \mathbf{x} \) be a random vector with mean vector \( \mathbf{\theta} \) and covariance matrix \( \Sigma \) and \( \mathbf{y} = \Sigma^{-1/2}(\mathbf{x} - \mathbf{\theta}) \) with mean vector \( \mathbf{0} \), covariance matrix \( \mathbf{I}_p \) and with third and fourth moments \( M_3(\mathbf{y}) \), \( M_4(\mathbf{y}) \) and cumulants \( C_3(\mathbf{y}), C_4(\mathbf{y}) \). Then skewness measure \( \beta_{1p}(\mathbf{x}) \) and kurtosis characteristic \( \beta_{2p}(\mathbf{x}) \) are defined by

\[
\beta_{1p}(\mathbf{x}) = \text{tr}[C_3'(\mathbf{y})C_3(\mathbf{y})] = \text{tr}[M_3'(\mathbf{y})M_3(\mathbf{y})]
\]

and

\[
\beta_{2p}(\mathbf{x}) = \text{tr}[M_4(\mathbf{y})] = \text{tr}[C_4(\mathbf{y})] + p^2 + 2p
\]

The class of elliptical distributions. The class of elliptical distributions, introduced by Kelker (1970), is a generalization of multivariate normal distributions. These distributions are symmetric and may not adequately represent the data when some asymmetry is present.

**Definition 4.2.1.** The random vector \( \mathbf{X} \) has a multivariate elliptical distribution, if its c.f. can be expressed as

\[
\varphi_X(t) = \exp(it'\mathbf{\mu})\psi\left(\frac{1}{2}t'\Sigma t\right)
\] (4.2.5)

for some column vector \( \mathbf{\mu} \), \( n \times n \) positive matrix \( \Sigma \) and for some function \( \psi(t) \in \psi_n \), which is called the characteristic generator.

The multivariate normal and symmetric Laplace distributions belong to elliptical family, since the c.f. of multivariate normal and symmetric Laplace distributions can be factorized as (4.2.5). The multivariate normal-Laplace distribution belongs to the class of elliptical distributions. It can be easily followed from the result that the sum of elliptical distributions is elliptical, see Fang et al. (1987). This property is very important when we deal with portfolio of assets represented by sum.

Marginal distributions of elliptical distributions are also elliptical distributions. So the
marginal distributions of multivariate normal-Laplace distributions are also elliptical. Also all odd order moments of an elliptical distribution are zero and hence the result holds for multivariate normal-Laplace distribution.

**Infinite divisibility.** The multivariate normal-Laplace distribution is infinitely divisible. Since the c.f. of $\text{NL}_p(\mu, \Sigma, V)$ can be written as

$$\varphi_X(t) = \left[ e^{\left( t^\prime \frac{n}{n} - \frac{1}{2} t^\prime \frac{n}{n} t \right)} \left( \frac{1}{1 + \frac{1}{2} t^\prime V t} \right)^{\frac{1}{2}} \right]^n,$$

for any integer $n > 0$. The term in brackets is the c.f. of a random vector expressed as $Z + Y$, where $Z \sim N_p(\mu, \Sigma)$ and $Y$ following a multivariate generalized symmetric Laplace distribution with parameters $V, \frac{1}{n}(L_p(V, \frac{1}{n}))$.

**Theorem 4.2.1.** If $X \sim \text{NL}_p(\mu, \Sigma, V)$ and $Y = AX + b$, where $A$ is a $p \times p$ matrix and $b \in \mathbb{R}^p$, then $Y \sim \text{NL}_p(A\mu + b, A\Sigma A', AVA')$.

**Proof.** The c.f. of $Y$ is

$$\varphi_Y(t) = E \left( e^{it'AX + b} \right) = e^{it'b} \varphi_X(A't) = e^{it'(A\mu + b) - \frac{1}{2} t'A\Sigma A'} \frac{1}{1 + \frac{1}{2} t'AVA'|t}.$$

Hence $Y \sim \text{NL}_p(A\mu + b, A\Sigma A', AVA')$.

**Theorem 4.2.2.** If $X \sim \text{NL}_p(\mu, \Sigma, V)$, then $d'X a \in \mathbb{R}^p$ follows a univariate normal-Laplace distribution.

**Proof.** This theorem can also be easily be proved using the c.f.
**Theorem 4.2.3.** Let $X \sim \text{NL}_p(\mu, \Sigma, V)$ and partition $X$, $\mu$, $\Sigma$ and $V$ as

$$
X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad 
\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad 
\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \quad 
V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix},
$$

where $X_1$ and $\mu_1$ are $k \times 1$ vectors and $\Sigma_{11}$ and $V_{11}$ are $k \times k$ matrices. Then $X_1 \sim \text{NL}_p(\mu_1, \Sigma_{11}, V_{11})$ and $X_1 \sim \text{NL}_p(\mu_2, \Sigma_{22}, V_{22})$. When $\Sigma_{12} = 0$, $X_1$ and $X_2$ are independently distributed.

**Proof.** By using the joint c.f., we can easily prove the result.

**Definition 4.2.2.** A random variable $X$ with c.f. $\phi$ is said to be semi-self decomposable, if for some $0 < a < 1$, there exists a c.f. $\phi_a$ such that $\phi(t) = \phi_X(at) \phi_a(t), \forall t \in \mathbb{R}$. If this relation holds for every $0 < a < 1$, then $\phi$ is self-decomposable or the corresponding distribution belongs to $\mathcal{L}$ class.

**Theorem 4.2.4.** Multivariate normal-Laplace c.f. $\varphi$ is self decomposable or the corresponding distribution belongs to the class $\mathcal{L}$.

**Proof.** The c.f. of multivariate normal-Laplace distribution is

$$
\varphi(t) = \left( \exp \left( it'\mu - \frac{1}{2} t'\Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2} t'Vt} \right) \\
= \left( \exp \left( it'\mu - \frac{1}{2} t'\Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2} t'Vt} \right) \\
	imes \left( \exp \left( it'(1-a)\mu - \frac{1}{2} t'(1-a^2)\Sigma t \right) \right) \left[ a^2 + (1-a^2) \frac{1}{1 + \frac{1}{2} t'Vt} \right].
$$

The second term in the expression is also a c.f., which can be seen in (4.3.2)
4.3 Multivariate normal-Laplace processes

Consider the usual linear, additive first order autoregressive model given by

\[ X_n = aX_{n-1} + \varepsilon_n, \quad 0 < a \leq 1, \quad n = 0, \pm 1, \pm 2, \ldots \]  \hspace{1cm} (4.3.1)

where \( X_n \) and innovations \( \varepsilon_n \) are independent \( p \)-variate random vectors.

In terms of c.f., we have

\[ \varphi_X(t) = \varphi_X(at)\varphi_\varepsilon(t). \]

The c.f. of \( \{\varepsilon_n\} \) can be obtained as

\[ \varphi_\varepsilon(t) = \left( \exp \left( it'(1-a)\mu - \frac{1}{2}t'(1-a^2)\Sigma t \right) \right) \left[ a^2 + (1 - a^2) \frac{1}{1 + \frac{1}{2}t'Vt} \right]. \]  \hspace{1cm} (4.3.2)

From this, we can obtain the distribution of the innovation sequence as

\[ \varepsilon \overset{d}{=} Z_1 + L. \]

where \( Z_1 \sim N_p((1 - a)\mu, (1 - a^2)\Sigma) \) and \( L \) can be treated as a sequence of random vectors of the form

\[ L = \begin{cases} 0, & \text{with probability } a^2 \\ L_p, & \text{with probability } (1 - a^2) \end{cases}, \]

where \( L_p \)’s are independently and identically distributed (i.i.d.) symmetric multivariate Laplace random vectors.

**Remark 4.3.1.** The process is stationary with \( NL_p(\mu, \Sigma, V) \) marginals.

**Proof.** We can prove this by the method of induction. We assume that \( X_n \sim NL_p(\mu, \Sigma, V) \).
Then

\[
\varphi_{X_n}(t) = \varphi_{X_{n-1}}(at) \varphi_{\varepsilon_n}(t) = \left( \exp \left( it'\mu - \frac{1}{2} a^2 t' \Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2} a^2 t' V_t} \right)
\times \left( \exp \left( it'(1-a)\mu - \frac{1}{2} t'(1-a^2) \Sigma t \right) \right) \left[ a^2 + (1-a^2) \frac{1}{1 + \frac{1}{2} t' V_t} \right]
\]

which is the c.f. of \( \text{NL}_p(\mu, \Sigma, V) \). Therefore \( \{X_n\} \) is strictly stationary with \( \text{NL}_p(\mu, \Sigma, V) \) marginals.

**Remark 4.3.2.** If \( X_0 \) is distributed arbitrary, then also the process is asymptotically Markovian with multivariate normal-Laplace marginal distribution.

**Proof.** We have

\[
X_n = a X_{n-1} + \varepsilon_n = a^n X_0 + \sum_{k=0}^{n-1} a^k \varepsilon_{n-k}.
\]

In terms of c.f., we get

\[
\varphi_{X_n}(t) = \varphi_{X_0}(a^n t) \prod_{k=0}^{n-1} \varphi_{\varepsilon}(a^k t) = \varphi_{X_0}(a^n t) \prod_{k=0}^{n-1} \frac{\exp \left( it' a^k - \frac{1}{2} a^{2k} t' \Sigma t \right)}{1 + \frac{1}{2} a^{2k} t' V_t} \frac{1}{\exp \left( it' a^{k+1} - \frac{1}{2} a^{2(k+1)} t' \Sigma t \right)} \left[ a^2 + (1-a^2) \frac{1}{1 + \frac{1}{2} t' V_t} \right] \rightarrow \left( \exp \left( it' \mu - \frac{1}{2} t' \Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2} t' V_t} \right) \text{ as } n \rightarrow \infty.
\]

Hence even if \( X_0 \) is arbitrarily distributed, the process is asymptotically stationary Markovian with multivariate normal-Laplace marginals.
4.3.1 Distribution of sums and joint distribution of \((X_n, X_{n+1})\)

Consider a stationary sequence \(\{X_n\}\) satisfying (4.3.1). Then we have

\[ X_{n+j} = a^jX_n + a^{j-1}\varepsilon_{n+1} + a^{j-2}\varepsilon_{n+2} + \cdots + \varepsilon_{n+j}. \]

Hence

\[
T_r = X_n + X_{n+1} + \cdots + X_{n+r-1} = \sum_{j=0}^{r-1} [a^jX_n + a^{j-1}\varepsilon_{n+1} + \cdots + \varepsilon_{n+j}]
\]

\[ = X_n \left(1 - \frac{a^r}{1-a}\right) + \sum_{j=1}^{r-1} \varepsilon_{n+j} \left(\frac{1 - a^{r-j}}{1-a}\right). \]

The c.f. of \(T_r\) is given by

\[
\varphi_{T_r}(t) = \varphi_{X_n} \left(t \frac{1 - a^r}{1-a}\right) \prod_{j=1}^{r-1} \varphi_{\varepsilon} \left(t \frac{1 - a^{r-j}}{1-a}\right)
\]

\[ = \left[\exp \left(\frac{1 - a^r}{1-a} \mu t - \frac{1}{2} \left(\frac{1 - a^r}{1-a}\right)^2 \Sigma t^2\right)\right] \left(\frac{1}{1 + \frac{1}{2} \left(\frac{1 - a^r}{1-a}\right)^2 V t}\right)
\]

\[ \times \prod_{j=1}^{r-1} \left\{ \exp \left(\frac{1}{2} \left(1 - a^{r-j}\right)(1+a)\Sigma t \right) \right\}
\]

\[ \times \left[a^2 + (1-a^2) \frac{1}{1 + \frac{1}{2} \left(\frac{1 - a^{r-j}}{1-a}\right)^2 V t}\right]. \]

The distribution of \(T_r\) can be obtained by inverting the above expression.

The joint distribution of contiguous observation vectors \((X_n, X_{n+1})\) can be given in
terms of c.f. as

\[
\varphi_{\mathbf{X}_n, \mathbf{X}_{n+1}}(t_1, t_2) = E[\exp(it'_1 \mathbf{X}_n + it'_2 \mathbf{X}_{n+1})] \\
= E[\exp(it'_1 \mathbf{X}_n + it'_2 (a \mathbf{X}_n + \varepsilon_{n+1}))] \\
= E[\exp(i(t'_1 + at'_2) \mathbf{X}_n + it'_2 \varepsilon_{n+1})] \\
= \varphi_{\mathbf{X}_n}(t_1 + at_2)\varphi_{\varepsilon_{n+1}}(t_2) \\
= \left( \exp \left( it'_1 + at'_2 \right) \mu - \frac{1}{2}(t'_1 + at'_2) \Sigma (t'_1 + at'_2) \right) \\
\times \left( \frac{1}{1 + \frac{1}{2}(t'_1 + at'_2)V(t'_1 + at'_2)} \right) \\
\times \left[ a^2 + (1 - a^2) \frac{1}{1 + \frac{1}{2}t'_2 Vt'_2} \right].
\]

Here \( \varphi_{\mathbf{X}_n, \mathbf{X}_{n+1}}(t_1, t_2) \neq \varphi_{\mathbf{X}_n, \mathbf{X}_{n+1}}(t_2, t_1) \). Therefore the process is not time-reversible.

### 4.4 Multivariate geometric normal-Laplace distribution

The c.f. of the Multivariate Geometric Normal-Laplace distribution denoted by \( \text{GNL}_{p}(\mu, \Sigma, V) \), is given by

\[
\psi(t) = \frac{1}{1 - it' \mu + \frac{1}{2}t' \Sigma t' + \ln(1 + \frac{1}{2}t' Vt')}. 
\]

The above c.f. can be written in the form \( \varphi(t) = \exp \left( 1 - \frac{1}{\psi(t)} \right) \), where \( \varphi(t) \) is the c.f. of \( \text{NL}_{p}(\mu, \Sigma, V) \). Hence the multivariate geometric normal-Laplace distribution is geometrically infinitely divisible.

### 4.5 Multivariate generalized normal-Laplace distribution

A multivariate generalized normal-Laplace distribution can be defined by introducing an additional parameter. Multivariate generalized normal-Laplace distribution can be obtained as the convolution of multivariate normal and multivariate generalized symmetric Laplace random vectors. The c.f. of multivariate generalized normal-Laplace distribution is given
by

$$\varphi_X(t) = \left[ \exp \left( it'\mu - \frac{1}{2}t'\Sigma t \right) \right] \left( \frac{1}{1 + \frac{1}{2}t'Vt} \right)^\nu, \quad t \in \mathbb{R}^p, \quad \Sigma > 0, \quad V > 0, \quad \nu > 0. \quad (4.5.1)$$

A p-variate generalized normal-Laplace distribution with parameters $\mu, \Sigma, V$ and $\nu$ can be denoted by $\text{NL}_p(\mu, \Sigma, V, \nu)$. Let $X \sim \text{NL}_p(\mu, \Sigma, V, \nu)$, then $X$ can be expressed as

$$X \overset{d}{=} Z + Y \quad (4.5.2)$$

where $Z$ and $Y$ are independent random vectors with $Z$ following a p-variate normal distribution with mean vector $(\nu \mu)$ and dispersion matrix $\nu \Sigma$ ($\text{N}_p(\nu \mu, \nu \Sigma)$) and $Y$ following a p-variate generalized symmetric Laplace distribution with parameters $V, \nu$ ($\text{L}_p(V, \nu)$).

$\text{NL}_p(\mu, \Sigma, V, \nu)$ is infinitely divisible and self-decomposable.

### 4.6 Multivariate geometric generalized normal-Laplace distribution

The c.f. of the Multivariate Geometric Generalized Normal-Laplace distribution denoted by $\text{GGNL}_p(\mu, \Sigma, V, \nu)$, is given by

$$\psi(t) = \frac{1}{1 - it'\nu \mu + \frac{1}{2}t'\nu \Sigma t + \nu \ln(1 + \frac{1}{2}t'Vt)}. \quad$$

**Theorem 4.6.1.** Let $X_1, X_2, \cdots$ are i.i.d. $\text{GGNL}_p(\mu, \Sigma, V, \nu)$ random vectors and $N$ be geometric with mean $\frac{1}{p}$, such that $P[N = k] = p(1 - p)^{k-1}, k = 1, 2, \ldots, 0 < p < 1$. This establishes that $Y \sim \text{GGNL}_p(\mu, \Sigma, V, \nu/p)$.

**Proof.** The c.f. of $Y$ is
\[ \psi_Y(t) = \sum_{k=1}^{\infty} \left[ \psi_X(t) \right]^k p(1 - p)^{k-1} \]

\[ = \frac{p/(1 - it' \nu \mu + \frac{1}{2} \nu t' \Sigma t' + \nu \ln(1 + \frac{1}{2} t' V t))}{1 - ((1 - p)/(1 - it' \nu \mu + \frac{1}{2} \nu t' \Sigma t' + \nu \ln(1 + \frac{1}{2} t' V t))} \]

\[ = \frac{1}{1 - it' \nu \mu + \frac{1}{2} \nu t' \Sigma t' + \nu \ln(1 + \frac{1}{2} t' V t)}. \]

Hence \(Y\) is distributed as \(\text{GGNL}_p(\mu, \Sigma, V, \nu/p)\).

**Theorem 4.6.2.** Multivariate geometric generalized normal-Laplace distribution \(\text{GGNL}_p(\mu, \Sigma, V, \nu)\) is the limiting distribution of multivariate generalized normal-Laplace distribution.

**Proof.** We have

\[ \left[ \left( \exp \left( it' \mu - \frac{1}{2} t' \Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2} t' V t} \right) \right]^{\nu/n} \]

\[ = \left\{ 1 + \left[ \left( \exp \left( it' \mu - \frac{1}{2} t' \Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2} t' V t} \right) \right]^{\nu/n} - 1 \right\}^{-1} \]

is the c.f. of a probability distribution since Multivariate generalized normal-Laplace distribution is infinitely divisible. Hence by Lemma 3.2 of Pillai (1990).

\[ \psi_n(t) = \left\{ 1 + n \left[ \left( \exp \left( it' \mu - \frac{1}{2} t' \Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2} t' V t} \right) \right]^{\nu/n} - 1 \right\}^{-1} \]

is the c.f. of geometric sum of independently and identically distributed multivariate gener-
alized normal-Laplace random vectors. Taking limit as \( n \to \infty \).

\[
\psi(t) = \lim_{n \to \infty} \psi_n(t) = \left\{ 1 + \lim_{n \to \infty} \left\{ n \left[ \left( \exp \left( it'\mu - \frac{1}{2}t'\Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2}t'Vt} \right) \right]^{\nu/n} \right\} - 1 \right\}^{-1}
\]

\[
= \frac{1}{1 - it'\nu \mu + \frac{1}{2}it'\Sigma t' + \nu \ln(1 + \frac{1}{2}t'Vt)}.
\]

Hence GGNL\(_p(\mu, \Sigma, V, \nu)\) is the limiting distribution of multivariate generalized normal-Laplace distribution.

### 4.7 Estimation of parameters

Kollo and Srivastava (2004) discussed the estimation of Multivariate Laplace distribution. We can use the method of moments to estimate the mean, covariance matrix and the skewness and kurtosis measures of multivariate normal-Laplace distribution. Let \( x_1, x_2, \ldots, x_n \) are i.i.d. as NL\(_p(\mu, \Sigma, V)\). Then the estimates of mean vector and covariance matrix are obtained as

\[
\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i
\]

and

\[
S = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})'
\]

respectively. Skewness and kurtosis measures can be estimated using the random vector defined by

\[
y_i = S^{-1/2}(x_i - \bar{x}), \quad i = 1, \ldots, n,
\]

where \( S^{-1/2} \) is any square root of \( S^{-1} \) such that \( S^{-1} = S^{-1/2}(S^{-1/2})' \). Then the estimates of third and fourth moments are given by

\[
\hat{M}_3 = \frac{1}{n} \sum_{i=1}^{n} (y_i \otimes y_i' \otimes y_i)
\]
and
\[
\overline{M}_4 = \frac{n + 1}{n - 1} \sum_{i=1}^{n} (y_i \otimes y'_i \otimes y'_i \otimes y'_i).
\]

These estimates are unbiased estimators, see Mardia (1970). The estimates of skewness and kurtosis measures are given by
\[
\hat{\beta}_{1p} = \text{tr} [\hat{M}_3 \hat{M}_3]
\]
and
\[
\hat{\beta}_{2p} = \text{tr} [\hat{M}_4].
\]

### 4.8 Multivariate slash normal-Laplace distribution

Now let us define the slash version of the multivariate Normal-Laplace distribution.

**Definition 4.8.1.** A random vector \( X \in \mathbb{R}^p \) has a \( p \)-variate slash Normal-Laplace (SNL\(_p\)) distribution with location parameter \( \mu \), positive definite scale matrix parameter \( \Sigma \) and tail parameter \( q > 0 \), denoted by \( X \sim \text{SNL}_p(\mu, \Sigma, q) \), if
\[
X = \frac{Y}{U^q} + \mu,
\]
where \( Y \) is Normal-Laplace random vector with characteristic function given by
\[
\varphi_X(t) = \left( \exp \left( i t^\prime \mu - \frac{1}{2} t^\prime \Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2} t^\prime \Sigma^{-1} t} \right), \quad t \in \mathbb{R}^p, \ \Sigma > 0, \ V > 0 \text{ and } U \sim U(0,1), \text{ which is independent of } Y.
\]

The stochastic representations of the random vector \( Y \) can be considered for simulation study. Let \( Y \sim \text{NL}_p(\mu, \Sigma, V) \), then \( Y \) can be expressed as
\[
Y \overset{d}{=} Z + L \tag{4.8.1}
\]
where \( Z \) and \( L \) are independent random vectors with \( Z \) following a \( p \)-variate normal distribution with mean vector \( \mu \) and dispersion matrix \( \Sigma \) \((\mathcal{N}_p(\mu, \Sigma))\) and \( L \) following a \( p \)-variate symmetric Laplace distribution with parameter \( V \) \((\mathcal{L}_p(V))\).

Another representation is
\[
Y \overset{d}{=} Z + \sqrt{E}Z_1 \tag{4.8.2}
\]
where \( Z \) follows a \( p \)-variate normal distribution with mean vector \( \mu \) and dispersion matrix \( \Sigma \), \( E \) is a standard exponential variable and \( Z_1 \) follows \( p \)-variate normal distribution with mean vector \( 0 \) and dispersion matrix \( \Sigma \).

The characteristic function of multivariate slash Normal-Laplace random vector \( X \) can be obtained as

\[
\phi_X(t) = E \left( \exp(it'X) \right) = \int_0^1 \phi_Y(tu^{-1/q}) du
\]

where \( \phi_Y(t) \) is the characteristic function of multivariate Normal-Laplace distribution given by

\[
\phi_Y(t) = \left( \exp \left( it'\mu - \frac{1}{2} t'\Sigma t \right) \right) \left( \frac{1}{1 + \frac{1}{2} t'V t} \right).
\]  

(4.8.3)

As \( q \to \infty \), the multivariate slash Normal-Laplace distribution tends to a multivariate Normal-Laplace distribution.

4.9 Applications

In insurance and financial markets, there is significant need for the development of a standard framework for the risk measurement. Landsman and Valdez (2003) derives explicit formulas for computing tail conditional expectations for elliptical distributions and extends to multivariate case. Multivariate elliptical distributions are useful to model combinations of correlated risks. Since multivariate normal-Laplace distribution comes in the class of elliptical distributions and it also convolutes both Gaussian and non-Gaussian distributions, it also has a very important role in risk analysis. It seems to provide an attractive tool for actuarial and financial risk management because it allows a multivariate portfolio of risks to have the property of regular variation in the marginal tails.

The importance of normal-Laplace model lies in the fact that it is the first attempt to combine Gaussian and non-Gaussian marginals to model time series data, see Jose et al (2008) and Lishamol and Jose (2009, 2010, 2011). Applications of normal-Laplace distribution are wide spread in areas like financial modelling, Levy process, Brownian motion etc.
Reed (2006) showed that it is the distribution of the stopped state of a Brownian motion with normally distributed starting value if the stopping hazard rate is constant. In financial modelling, a normal-Laplace model is a more realistic alternative for Gaussian models as logarithmic price returns do not follow exactly a normal distribution. But it is more realistic to consider multivariate data where several variables are discussed. The models developed in this chapter can be used for modelling multivariate time series data.

4.10 Conclusion

In this chapter the univariate normal-Laplace distribution is considered. We introduced a multivariate normal-Laplace distribution and studied its properties. First order autoregressive processes with multivariate normal-Laplace marginals is developed. We introduced a multivariate geometric normal-Laplace distribution. Multivariate generalized normal-Laplace distribution is also introduced. We introduced the geometric generalized normal-Laplace distribution and studied its properties. We considered the estimation of parameters. Some applications are also discussed.

References


