CHAPTER 5
APPLICATIONS OF INDICATORS IN DISCRETE MAPS

5.1. INTRODUCTION

Since the discovery of chaos by Poincaré (1913), numerous research articles appeared in journals. A brief review has already been stated in the previous chapters. One central problem in most of the works is how to distinguish regular and chaotic motions in dynamical systems. The traditional indicators for regularity and chaos, such as time series graph, phase plot, Poincaré map, power spectrum etc., though efficient, with absolute certainty may not be sufficient. In this regard we may recall the Chirikov map discussed in chapter three. There we could not draw a definite idea regarding the regular / chaotic evolution by looking at the bifurcation diagram as well as the LCEs plot. Also, there are maps of higher dimension where above indicators may fail to identify clearly regular and chaotic behaviour. This necessitates to search some perfect indicator which would efficiently distinguish regular and chaotic motions in dynamical systems.

Some recent tools, identified as the Fast Lyapunov Indicators (FLI), Froeschle et al. (1997, 2002), the Smaller Alignment Indices (SALI), Skokos et al. (2001, 2004) and the Dynamic Lyapunov Indicator (DLI), Saha and Budhraja (2007), proved to be more efficient to detect chaotic motions especially in higher dynamical systems. These indicators are used in a recent article by Saha and Tehri (2010). Numerical results reveal that DLI gives very clear indication of ordered and chaotic motion whenever applied to an evolving system, Deleanu (2011).

The objective of study in this chapter is to obtain plots for FLI, SALI and DLI for various discrete 2-D maps considered in Chapter 3 and to discuss the efficiency of these indicators for distinguishing regular and chaotic motions. We have already defined FLI, SALI and DLI in Chapter 1 and also discussed how these identify chaos and regularity.

Here, let us again recall the behavioural properties of FLI, SALI and DLI which already mentioned in chapter one. As described, plots of FLI’s should show
exponential increase for chaotic orbits and linear increase for regular orbits. A SALI plots should fluctuates around a non-zero value for ordered orbits while it tends to zero for chaotic orbits. In case of DLI, the largest eigenvalues form a definite pattern for regular motion and for chaotic motion they are distributed randomly, (with no definite pattern).

5.2. THE MODELS

We have applied above mentioned indicators for the models given below:

(a) **Behrens - Feichtinger Model:**

This is a micro economic model of two firms X and Y competing on the same market of goods having asymmetric strategies as defined in the work of Holyst [Holyst.et.al(1996)]. The sales \( x_n \) and \( y_n \) of both firms are measured in discrete time steps and evolve according to the equations.

\[
\begin{align*}
x_{n+1} &= \frac{a}{1 + e^{-(x_n-y_n)}} + (1-\alpha)x_n \\
y_{n+1} &= \frac{b}{1 + e^{-(x_n-y_n)}} + (1-\beta)y_n
\end{align*}
\]

where \( \alpha, \beta \ (0 < \alpha, \beta < 1) \) are the time rates at which the sales of both firm decays in the absence of investments. Parameters \( a, b \) describe the investment effectiveness of both the firms. Parameter \( c \) is an “elasticity” measure of the investment strategies.

For parameter values \( \alpha = 0.46, \beta = 0.7, a = 0.16, b = 0.9, c = 105 \), we have observed the chaotic attractor of this model which has been shown in Fig. 5.1(b).

![Fig.5.1: Phase plots of Behrens – Feichtinger Model: (a) regular case and (b) chaotic case.](image-url)
For this map plots of FLI, SALI and DLI are given in Fig. 5.2 – Fig.5.5 below.

**Fig.5.2:** FLI plots of Behrens – Feichtinger Model: (a) regular case and (b) chaotic case.

**Fig.5.3:** Log [FLI] plots of Behrens – Feichtinger Model: (a) regular case and (b) chaotic case.

**Fig.5.4:** SALI plots of Behrens – Feichtinger Model: (a) regular case and (b) chaotic case.
Fig. 5.5: DLI plots of Behrens – Feichtinger Model: (a) regular case and (b) chaotic case.

(b) **Bouncing Ball model**

A discrete form of equation of motion for bouncing ball is represented by

\[
x_{n+1} = x_n + ay_n \\
y_{n+1} = ky_n + (1 + k)\cos(x_n + ay_n)
\]

This map evolve chaotically when \(a = 20\) and \(k = 0.9\) and displays regularity when \(a = 2\) and \(k = 0.35\), which can be observed by looking at the phase plots displayed in Fig. 5.6. Here we have taken the initial condition as \((0.1, 0.1)\).

Fig. 5.6: Phase plots of Bouncing Ball model: (a) regular case and (b) chaotic case.

For this map FLI, SALI and DLI are computed respectively and displayed through following plots.
Fig. 5.7: Log [FLI] plots of Bouncing Ball model: (a) regular case and (b) chaotic case.

Fig. 5.8: FLI plots of Bouncing Ball model: (a) regular case and (b) chaotic case.

Fig. 5.9: SALI plots of Bouncing Ball model: (a) regular case and (b) chaotic case.
Fig. 5.10: DLI plots of Bouncing Ball model: (a) regular case and (b) chaotic case.

(c) **Chirikovs Map**

The discrete form of Chirikov map can be represented by,

\[
x_{n+1} = x_n - k \sin(y_n) \mod(2\pi)
\]

\[
y_{n+1} = y_n + x_{n+1}
\]

This map evolve chaotically when \( k = -0.5 \) and displays regularity when \( k = 2.5 \), which can be observed by looking at the phase plots displayed in Fig. 5.6. Here we have taken the initial condition as (-0.2, 0.2).

Fig. 5.11: Phase plots of Chirikov’s map: (a) regular case and (b) chaotic case.

For this map FLI, SALI and DLI are computed respectively and displayed through following plots.
Fig.5.12: FLI plots of Chirikov’s map: (a) regular case and (b) chaotic case.

Fig.5.13: Log [FLI] plots of Chirikov’s map: (a) regular case and (b) chaotic case.

Fig.5.14: SALI plots of Chirikov’s map: (a) regular case and (b) chaotic case.
Fig. 5.15: DLI plots of Chirikov’s map: (a) regular case and (b) chaotic case.

(d) Ikeda map

The 2-dimensional Ikeda map is described by a pair of equations:

\[ x_{n+1} = 1 + k (x_n \cos \theta - y_n \sin \theta) \]
\[ y_{n+1} = k (x_n \sin \theta + y_n \cos \theta) \]

where \( \theta = 0.4 - \frac{6}{1 + x_n^2 + y_n^2} \)

This map evolve chaotically when \( k = 0.9 \) and displays regularity when \( k = 0.35 \). Here we have taken the initial condition as (0.3, 0.3). In Fig. 5.16 (b), we have shown the chaotic attractor of the Ikeda map.

Fig. 5.16: Phase plots of Ikeda map: (a) regular case and (b) chaotic case.

In Fig. 5.17, the FLIs for regular and chaotic case of Ikeda map are shown. The FLIs increase to about \( 10^{40} \) in 200 iterations. SALI for the chaotic Ikeda map have been shown in Fig. 5.19. SALI approaches 0 in about 22 iterations. In Fig. 5.20, we have
shown the DLIs for this map, which are randomly distributed, again indicating unpredictability and chaos.

Fig.5.17: FLI plots of Ikeda map: (a) regular case and (b) chaotic case.

Fig.5.18: Log [FLI] plots of Ikeda map: (a) regular case and (b) chaotic case.

Fig.5.19: SALI plots of Ikeda map: (a) regular case and (b) chaotic case.
The Burgers map is a two dimensional map which produces rich dynamic patterns during evolution and is described in discrete form by the equation

\[ x_{n+1} = (1 - a) x_n - y_n^2, \]
\[ y_{n+1} = (1 + b) y_n + x_n y_n, \]

The Burgers map evolve chaotically when \( a = 0.9 \) and \( b = 0.856 \). To control chaotic motion we have used pulsive feedback control technique, Litak et al (2007) by changing the map slightly as

\[ x_{n+1} = (1 - a) x_n - y_n^2 + \varepsilon (x + 0.856) \]
\[ y_{n+1} = (1 + b) y_n + x_n y_n + \varepsilon (y - 0.87772433) \]

Here (- 0.856, 0.87772433) is unstable fixed points of the original Burger’s map. It has been observed that at \( \varepsilon = 0.3 \), above chaotic motion gets controlled and displays regular behaviour. This can be observed through the phase plots given in Fig. 5.21.
For this map FLI, SALI and DLI are computed respectively and displayed through the following plots.

![FLI plots of Burger’s map: (a) regular case and (b) chaotic case.](image1)

Fig.5.22: FLI plots of Burger’s map: (a) regular case and (b) chaotic case.

![Log [FLI] plots of Burger’s map: (a) regular case and (b) chaotic case.](image2)

Fig.5.23: Log [FLI] plots of Burger’s map: (a) regular case and (b) chaotic case.

![SALI plots of Burger’s map: (a) regular case and (b) chaotic case.](image3)

Fig.5.24: SALI plots of Burger’s map: (a) regular case and (b) chaotic case.
Fig. 5.25: DLI plots of Burger’s map: (a) regular case and (b) chaotic case.

(f) **Discrete Duffing Map**

The discrete version of Duffing equation is given by

\[
\begin{align*}
    x_{n+1} &= y_n \\
    y_{n+1} &= -b x_n + a y_n - y_n^3
\end{align*}
\]

where \(a\) and \(b\) are parameters.

For parameter values \(a = 2.77, b = 0.1\), we get the chaotic attractor of Discrete duffing map. It display regular behaviour for \(a = 2.4, b = 0.1\). This can be observed through the phase plots given in Fig. 5.26.

Fig. 5.26: Phase plots of Discrete Duffing map: (a) regular case and (b) chaotic case.

For this map FLI, SALI and DLI, respectively are computed with the parameter values mentioned above and with the initial conditions as \((0.1, 0.1)\) and the results are displayed through the following plots:
Fig. 5.27: Log [FLI] plots of Discrete Duffing map: (a) regular case and (b) chaotic case

Fig. 5.28: FLI plots of Generalized Invertible Henon map: (a) regular case and (b) chaotic case

Fig. 5.29: SALI plots of Discrete Duffing map: (a) regular case and (b) chaotic case
(g) Generalized Invertible Henon map

This map is a 2 dimensional discrete map, having unique kind of dynamic behaviour which is already shown in the bifurcation diagram in Fig. 3.13, in chapter 3 and is given by the equation

\[ x_{n+1} = \left[ x_n + 1 - S(x_n) \right] - S(A\left(1 - \left[S(-x_n)^3\right]\right)) - y_n, \]
\[ y_{n+1} = Bx_n, \]

where \( S(x) = \frac{1}{2} \left[ x + (x^2 + 0.1)^{1/2} \right] \)

where \( A \) and \( B \) are non-zero parameters.

For parameter values \( A = 7.0, B = 0.1 \), we get the chaotic attractor of this map. Such a chaotic motion gets controlled and display regular behaviour for \( A = 10.0, B = 0.1 \). This can be observed through the phase plots given in Fig. 5.31.
For this map FLI, SALI and DLI, respectively are computed with the parameter values mentioned above and with the initial conditions as (0.0, 0.0) and the results are displayed through the following plots:

**Fig. 5.32:** Log [FLI] plots of Generalized Invertible Henon map: (a) regular case and (b) chaotic case

**Fig. 5.33:** FLI plots of Generalized Invertible Henon map: (a) regular case and (b) chaotic case

**Fig. 5.34:** SALI plots of Generalized Invertible Henon map: (a) regular case and (b) chaotic case
5.3 DISCUSSIONS OF THE NUMERICAL RESULTS

As per applications of FLI, SALI and DLI indicators in the various discrete maps discussed in this chapter, we observe the following results:

(i) Regular Cases

FLI plots for different models do not show actual linear increase for regular orbits but demonstrate exponential increase after sufficient number of iterations, in case of Burgers map after 1500 iterations. The SALI plots also do not show results as per the definition of SALI plot. For some models, (e.g. Behrens-Feichtinger, Bouncing Ball, Ikeda, discrete Duffing maps), SALI plots show results as per its definition. However, for other maps, (e.g. Chirikov map, Burgers map, Generalized invertible Henon map), SALI plots show results different than what is described in SALI plot. For all the models DLI plots show results as per its definition for regular evolution.

(ii) Chaotic cases

FLI plots show exponential increase for all the models, except Chirikov map. As per the definition, SALI plots tend to zero for Behrens-Feichtinger model, Chirikov, Ikeda and Burgers maps but not for bouncing ball model, Generalized invertible Henon map and discrete Duffing map. The plots of DLI show results as per its definition for chaotic evolution. With these we may agree that DLI may be considered as more efficient and perfect among indicators of regularity and chaos. Yet one has to verify, whether this is true for other nonlinear discrete systems or not. For this one has to investigate other nonlinear systems and also, those with higher dimensions to see how these indicators work.

Fig. 5.35: DLI plots of Generalized Invertible Henon map: (a) regular case and (b) chaotic case
From above observation it reflects that though SALI and FLI have significant merits, they do not clearly identify regular and chaotic motions for all discrete maps. However, DLI is very consistent in identifying such behaviour. For numerical integrations we have used modern technology and software such as Mathematica, Matlab etc., where the possibility of occurring of round off error is minimum.