CHAPTER 4
INTERESTING DYNAMIC BEHAVIOR IN POPULATION EVOLUTION OF SOME DISCRETE SYSTEMS

4.1 INTRODUCTION

Real systems are mostly nonlinear. While evolving real systems display very interesting, and in some cases, peculiar behaviour. With appropriate computer programming one can explore interesting and very exciting results. In this regard studies relating to the dynamics of real life population explosion, characteristics of their fluctuations, their extinction, coexistence, survival in different environmental situations etc. are interesting.

The modelling exercise of populations is of great importance in ecological systems. In this regard, from different predator-prey mathematical models and their competition and interactions phenomena, one learns to predict the dynamics behind environmental changes. Some population models describe the change of the number of species due to birth, death, and movement from position to position (in space) or from stage to stage (age, size, etc.). Mathematical models, available for real life populations are may be of one, two or multidimensional. Also, such models may be in discrete or continuous form. One has to use in proper manner an appropriate principle to study a particular model.

A lot of recent articles on population dynamics suggest, successful prediction of population’s response to a fluctuating environment is rare, even in the simplest realm of the laboratory. This is because real systems are of complex nature. But one has to find certain mathematical model which may be close to the actual system. Controlled laboratory experimental have addressed the effect of time variant factors for few models which are mostly autonomous.

Robert May has initiated pioneer investigation of population evolution, May, (1974-1977), followed by a numerous research works for different types of populations, e.g. Neubert (1997), Nisbet and Gurney (1981), Petratis and Latham (1999), Renshaw (1991), Shaffer (1981). Appearance of chaos in various models have been observed through bifurcation phenomena and published in a number of recent literature speak
interesting analysis of systems related to the real world. There are number of articles published in this regard and some of these noteworthy such as that of Osinga (2010, 2011). Statistical measures in dynamical system show significant developments for evolutionary analysis. In this direction the work of Alsed and Costa (2008) can be taken to be very significant. Also, the recent article by Henson et al.(1999) provides a detail study of evolutionary behaviour in insect populations. Chaos has been widely found in nonlinear population models.

Certain important factors to be noted are that the insect populations get regulated by parasitoids, predators and other mortality factors. Also, the insect populations grow rapidly; because of its female members usually produce large number of eggs. Therefore, within a short period the population may grow to many folds. But this does not happen in nature due to various reasons. Some of them die at different stages of their life and so, in nature, we do not observe any type of population explosion.

The objective of this work is to observe dynamic behaviour of certain insect population, (e.g. larvae), through different stages of their changes. Towards this, here, we have considered some population models and first examined the stability criteria of their steady state solutions and then use appropriate simulation work to find their bifurcation diagram. This leads to understand their evolutionary properties and also indicates the parameter domain within which the population may evolve regularly or chaotically. For each of these models, we have obtained the plots of their bifurcation diagrams, Lyapunov exponents and also calculated correlation dimensions. Our aim is to establish certain meaningful explanation of evolutionary behaviour of the species under various conditions. In this chapter, we review some mathematical results for deterministic and discrete one and two dimensional population models.

We use recursion, (iteration), procedure to investigate evolution and bifurcation in discrete maps. A general way to write such recursion is given by

\[ x_{n+1} = f(x_n) = f^{n+1}(x_0) \]

To study the dynamics of larvae population behaviour, the following models have been considered in our present study:
First we consider the Ricker-type map described by following equation, Henson et al. (1999),

\[ x_{n+1} = f_0(x_n) = bx_n e^{-cx_n} + (1 - \mu)x_n \] (4.1)

Here, \( x_{n+1} \) represents the density of individuals in a population at census \( n + 1 \) for a given density of individuals \( x_n \) at census \( n \). The parameter \( b > 0 \) stands for the inherent per capita recruitment rate per census interval at small population sizes and \( e^{-cx_n} \) represents the fractional reduction of recruitment due to density-dependent effects. The parameter \( \mu, 0 \leq \mu \leq 1 \), represents the fraction of individuals expected to die during one census period.

There are two fixed points for system (4.1), \( x_{1}^* = 0 \) and \( x_{2}^* = \frac{1}{c} \ln \left( \frac{b}{\mu} \right) \). The point origin, \( x_{1}^* = 0 \), is stable in the range \(-2 \leq b - \mu \leq 0\) and within this one can observe the first cycle; outside this range this fixed point is unstable. The other fixed point, \( x_{2}^* = \frac{1}{c} \ln \left( \frac{b}{\mu} \right) \), is stable when \( 0 \leq \mu \ln \left( \frac{b}{\mu} \right) \leq 2 \) and \( \mu > \frac{b}{\mu} \) this point is no more stable. Thus, the orbit originating nearby this fixed point, \( x_{2}^* \), would show attracting behaviour as long as the parameters \( b \) and \( \mu \) takes values such that the above stability condition holds. The bifurcation diagram would produce a one cycle in such a case.

For two cycles, we have to obtain fixed points by solving equation

\[ x_2 = f(x_1) \]

and proceed to find stability of fixed points by repeating the process of first cycle and in the bifurcation diagram we observe the second cycle. Analytically, one may repeat this for consecutive cycles. After several bifurcations, finally one observes chaos.

If we introduce a certain period-forcing such that the birth rate oscillates with relative amplitude \( 0 < \alpha < 1 \) and average \( b \), then we obtain another model modified from equation (4.1) as:

\[ x_{n+1} = f_\alpha(t, x_n) = b[1 + \alpha(-1)^n]x_n e^{-cx_n} + (1 - \mu)x_n \] (4.2)
For this map, the fixed points when $n$ is even are, $0, \frac{1}{c} \ln \left( \frac{b(1+\alpha)}{\mu} \right)$ and when $n$ is odd these are $0, \frac{1}{c} \ln \left( \frac{b(1-\alpha)}{\mu} \right)$. One can proceed for stability analysis similar to the first map (4.1). The parameter values are assigned while computing the bifurcation diagram where one gets the clear ideas of various cycles and then chaos. Emergence of chaos through bifurcations is now quite familiar in literature since Feigenbaum (1978).

A two-species population model described by discrete system

\[
\begin{align*}
    x_{n+1} &= ax_n(1-x_n-y_n) \\
    y_{n+1} &= bx_ny_n
\end{align*}
\]

There are three fixed points for this two dimensional map; these are $(0,0), \left(1-\frac{1}{a},0\right)$ and \(\left(\frac{1}{b},1-\frac{1}{a}-\frac{1}{b}\right)\). Stability of each of these fixed points can be discussed as above cases and cycles appearing in the bifurcation diagram can be analyzed. All these are depending on the parameters $a$ and $b$.

There can be various discrete models but, here we have selected only these few models. Our objective is to see the change in behavioural dynamics of the population with regard to the change of control parameters by observing the phenomena of bifurcation. Besides drawing bifurcations for above models, we have numerically evaluated the Lyapunov exponents and correlation dimensions to justify regular and chaotic evolutions. Correlation dimension gives the measure of complexity whenever a system evolves chaotically.

4.2. BIFURCATION PHENOMENA

Bifurcation in ordinary sense is splitting into two. In dynamical system it is sudden change in behaviour due to sudden change of set of parameter values according to certain rule. During changing process of the parameters a critical set of values be obtained where we observe a sudden change in behaviour of the system. Such a point
is known as the bifurcation point. We witness bifurcation of systems at certain bifurcation points within the parameter domain.

During numerical exploration, we observe bifurcations of above described maps and explained the dynamic behaviour through various graphics. Emergence of chaos can be easily visualize by observing these diagrams and obtain dynamic behaviour. Certain periodic windows appearing in figures have very specific significance for nonlinear systems emerging to chaos.

**Bifurcation in one dimensional model represented by eqn. (4.1)**

For the model described by equation (4.1), with $c = 1.0$, $\mu = 0.93$ fixed and varying $b$ from $b = 0$ to $b = 250$, we observe the appearance of one cycle up to the value $b = 7.988$. Then, one cycle suddenly changes into two cycles when $b$ exceeds 7.99 up to some higher value. The system again bifurcates and shows four cycles around the value $b = 110$ and the process of bifurcation continues. Then, finally, It becomes chaotic when $b$ takes the value $b = 195$ and onwards. The bifurcation scenario of this system, when we vary $b$, is shown in Fig.4.1.

![Fig.4.1: Bifurcation of model described by eqn. (4.1). Here, $c = 1.0$, $\mu = 0.93$ and $b$ is varied from $0 \leq b \leq 280$.](image)

Then, we fixed $c = 1.0$, $b = 60$ and varied $\mu$ from $\mu = 0$ to $\mu = 1.2$. We see up to the values $\mu = 0.44$, only one cycle appears. One cycle bifurcates into two cycles at about the parameter values $\mu = 0.45$. Then system see four cycles around the value $\mu = 0.6$. Bifurcation phenomena continue until the parameter $\mu$ reaches to the value
\( \mu = 0.7 \) where it is highly chaotic. We can see this result by the following bifurcation diagram shown in Fig.4.2.

![Bifurcation Diagram](image)

**Fig.4.2:** Bifurcation of model described by eqn. (4.1). Here, \( c = 1.0, b = 60 \) and \( \mu \) is varied from \( 0 \leq \mu \leq 1.2 \).

**Bifurcation in one dimensional model represented by eqn. (4.2):**

To explain the bifurcation scenario of the model described by equation (4.2), which is obtained when the relative amplitude is increased slightly from zero, (e.g., \( \alpha = 0.01 \)), we have considered two cases for \( n \) even and for \( n \) odd. Fixing \( c = 1.0, \mu = 0.93 \) and changing \( b \) from \( b = 0 \) to \( b = 250 \), we have obtained the bifurcation diagram shown in Fig.4.3 (a) when \( n \) is even and in Fig.4.3 (b) when \( n \) is odd. For first case, we see from Fig.4.3, up to the values \( b = 7.90 \), only one cycle appears. From one cycle the system suddenly emerges into two cycle at about the parameter value \( b = 7.91 \). Then to four cycle around \( b = 107 \) and so on. The bifurcation diagram shows chaos after the value \( b = 195 \).

![Bifurcation Diagram](image)

**Fig.4.3:** Bifurcation of model described by eqn. (4.2) (a) when \( n \) is even and (b) when \( n \) is odd. Here, \( \alpha = 0.01, c = 1.0, \mu = 0.93 \) and \( b \) is varied \( 0 \leq b \leq 270 \).
Again fixing $c = 1.0$, $b = 60$, and varying the death rate $\mu$ from $\mu = 0$ to $\mu = 1$, we have again drawn the bifurcation diagrams for both the cases. We observe that up to values $\mu = 0.39$ only one cycle. From one cycle the system suddenly emerges into two cycle at the parameter values $\mu = 0.4$. The system emerges to four cycle around the value $\mu = 0.6$. Then, it becomes highly chaotic after the value $\mu = 0.7$. The results are demonstrated by the following bifurcation diagram for equation (4.2).

![Bifurcation diagram for eqn. (4.2)](image)

**Fig.4.4:** Bifurcation diagram of model described by eqn. (4.2), (a) when $n$ is even and (b) when $n$ is odd. Here, $\alpha = 0.01$, $c = 1.0$, $b = 60$ and $\mu$ is varied $0 \leq \mu \leq 1.0$.

**Bifurcation in two dimensional model represented by eqn. (4.3)**

In the case of model (4.3), keeping $b = 0.5$ fixed and changing the other parameter $a$ from 0 to 4.0, we have obtained the bifurcation diagram. From Fig.4.5, we find that up to the value $a = 2.98$, only one cycle appears. As we proceed further at $a = 2.99$ the system suddenly changes into two cycle and around $a = 3.46$ it bifurcates and changes into four cycles. Proceeding further in similar fashion we can observe that the system become chaotic around the value $a = 3.6$.

![Bifurcation diagram for eqn. (4.3)](image)

**Fig.4.5:** Bifurcation of model described by eqn. (4.3). Here, $b = 0.5$ and $a$ is varied $0 \leq a \leq 4.0$. 

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4.3 CALCULATION OF LCE AND LYAPUNOV NUMBER

We have plotted the LCEs curve for above described models, discussed in section 4.1, by using Mathematica software, Sandri (1996). LCEs curves corresponding to these models are shown through Fig. 4.6 – Fig. 4.12.

Fig 4.6 : LCE curve of Equation (4.1) for the parameter values $c = 1.0$, $\mu = 0.93$ and $b$ is varied (i) $7.8 \leq b \leq 250$ (ii) $180 \leq b \leq 225$

Fig.4.7: LCE curve of Equation (4.1) for the parameter values $c = 1.0$, $b = 60$ and $\mu$ is varied (i) $0.1 \leq \mu \leq 1.0$. (ii) $0.6 \leq \mu \leq 0.9$

Fig.4.8: LCE curve of Equation (4.2) for the parameter values $c = 1.0$, $\mu = 0.93$, $\alpha = 0.01$, and $b$ is varied (i) $7.8 \leq b \leq 250$ (ii) $180 \leq b \leq 280$ and $n$ is even.
For the two dimensional map of two species problem model (4.3) we have the following Lyapunov characteristic exponent map.

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Fig. 4.12: LCE curve of Equation (4.3) for the parameter values $b = 0.5$, $a = 4.0$.

4.4 TOPOLOGICAL ENTROPY

For different values of parameter $b$ and $\mu$, we have calculated the topological entropies of model (4.1) and (4.2) displayed in Fig. 4.13–Fig. 4.15.

Fig. 4.13: Topological entropy of Equation (4.1) for the parameter values (i) $c = 1.0$, $\mu = 0.93$ and $b$ is varied $180 \leq b \leq 225$ (ii) $c = 1.0$, $b = 60$ and $\mu$ is varied from $0.6 \leq \mu \leq 0.9$.

Fig. 4.14: Topological entropy of map (4.2) for the parameter values (i) $\alpha = 0.01$, $c = 1.0$, $\mu = 0.93$ and $b$ is varied $180 \leq b \leq 225$ (ii) $\alpha = 0.01$, $c = 1.0$, $b = 60$ and $\mu$ is varied from $0.6 \leq \mu \leq 0.9$ and $n$ is even.
Fig. 4.15: Topological entropy of map (4.2) for the parameter values (i) $a=0.01$, $c = 1.0$, $\mu = 0.93$ and $b$ is varied $180 \leq b \leq 225$ (ii) $a=0.01$, $c = 1.0$, $b = 60$ and $\mu$ is varied from $0.6 \leq \mu \leq 0.9$ and $n$ is odd.

4.5. CORRELATION DIMENSION

We have calculated the correlation curves for models (4.1), (4.2) and (4.3) for regular and chaotic motions and demonstrated through graphics Fig.4.16 – Fig.4.19 below.

Fig.4.16: Correlation dimension curves of the model (4.1) for regular case. (i) when $\mu$ is constant and $b$ is varying & (ii) when $b$ is constant and $\mu$ is varying

Fig.4.17: Correlation dimension curves of the model (4.2) for regular case (a) when $n$ is odd, (i) $\mu$ is constant and $b$ is varying (ii) $b$ is constant and $\mu$ is varying (b) when $n$ is even, (i) $\mu$ is constant and $b$ is varying (ii) $b$ is constant and $\mu$ is varying
Correlation dimensions of the model (4.3) for parameters $a = 4.0$, $b = 0.5$ are calculated and it is plotted. From the graph, Fig. 4.20 and by linear fitting of the data obtained for the graph correlation dimension be calculated as $1.15427$. Similar procedure is also applied for other models and their correlation dimension are obtained for each models.
4.5. RESULTS AND DISCUSSION

Evolutions of certain insects represented through models (4.1), (4.2) and (4.3) are explained by drawing bifurcation diagrams of these models for variation of certain parameter for their certain ranges. Significant revelation obtained through bifurcation graphics indicates how the stable solutions change into chaotic when parameters varied. Also the bifurcation scenarios are different, in many ways, for different models. For model (4.1), one observes bifurcation figures Fig.4.1 and Fig.4.2 are completely different when \( b \) and \( \mu \) are made to vary. Same is the case for model (4.2), which is obtained after certain modification of earlier model. From Fig.4.3 and Fig.4.4. We observe clear period doubling type bifurcation phenomena for model (4.3) in Fig.4.5 when \( a \) is varied. Lyapunov characteristic exponents (LCE), have been calculated for above explained models to see the parameter ranges showing regularity and chaotic evolution. The positive LCE indicates the chaotic regions whereas its negative value indicates the regular regions of evolution. Figures, Fig.4.6 – Fig.4.12, are drawn for LCE for models (4.1) – (4.3). For different values of parameter \( b \) and \( \mu \), we have calculated the topological entropies of model (4.1) and (4.2) in Fig. 4.13–Fig.4.15. We have also used meaningful statistical measures to justify the results obtained through this study. The correlation curves for each model shown through figures, Fig.4.16-Fig. 4.20. We conclude that the environmental situations and certain social conditions for coexistence have significant role for regular and chaotic evolution of the insect population.