CHAPTER 5

COMPRESSIONS OF SLANT WEIGHTED TOEPLITZ OPERATORS

The weighted Hardy space $H^2(\beta)$ is quite favourite amongst operator theorists [34], [59], [82]. In view of this it is quite worthwhile to focus our ongoing study on this space. In this chapter we study the compressions of slant weighted Toeplitz operators to the space $H^2(\beta)$. It is known that a weighted Toeplitz operator $T_\phi$ is the compression of the weighted multiplication operator $M_\phi$. Also the compressions of slant Toeplitz operators to the usual Hardy space $H^2$ have been studied [86], [5]. These considerations motivate us to study the compressions of the slant weighted Toeplitz operators.

The chapter is divided into two sections. In the first section we introduce the notion of the compression of a slant weighted Toeplitz operator $A_\phi$ to the space $H^2(\beta)$. Denoting this operator by $B_\phi$, we obtain the matrix of $B_\phi$ and study some of its basic algebraic properties. We also show that the set of all compressions of slant weighted Toeplitz operators is weakly closed. In the second section, we introduce and study the notion of the compression of a $k$-th order slant weighted Toeplitz operator $U_\phi$ to the space $H^2(\beta)$. The main objective behind this to generalize the operator $B_\phi$ defined in the first section.
5.1 Compression of $A_{\phi}$ to $H^2(\beta)$

The notion of the compression of a slant Toeplitz operator to the space $H^2$ was introduced by Ho [43] and many mathematicians studied its algebraic and spectral properties [84], [85], [86], [5]. Motivated by these developments, we now introduce the following:

**Definition 5.1.1** ([17]). Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^2(\beta)$ be given. We denote by $B_{\phi}$, the compression of $A_{\phi}$ defined on $L^2(\beta)$ to the space $H^2(\beta)$.

Thus $B_{\phi} : H^2(\beta) \rightarrow H^2(\beta)$ is given by

$$B_{\phi} = PA_{\phi} |_{H^2(\beta)} = P WM_{\phi} |_{H^2(\beta)}$$

$$= W PM_{\phi} |_{H^2(\beta)} \quad \text{(since $P$ reduces $W$)}$$

$$= WT_{\phi},$$

where $T_{\phi}$ is the Toeplitz operator on $H^2(\beta)$ induced by $\phi$. The effect of $B_{\phi}$ on the orthonormal basis $\{e_k(z)\}_{k=0}^{\infty}$ is given by

$$B_{\phi} e_k(z) = \frac{1}{\beta_k} \sum_{n=0}^{\infty} a_{2n-k} \beta_n e_n(z).$$

Also,

$$\langle B_{\phi} e_j, e_i \rangle = \langle PA_{\phi} e_j, e_i \rangle$$

$$= a_{2i-j} \frac{\beta_i}{\beta_j} \quad i, j = 0, 1, 2, \ldots.$$
Hence the matrix of $B\phi$ is obtained as follows:

$$
\begin{bmatrix}
  & & & & & & \\
  & a_0\beta_0 & a_{-1}\beta_0 & a_{-2}\beta_0 & \ldots & & \\
  & \beta_0 & \beta_0 & \beta_0 & & & \\
  & a_1\beta_1 & a_0\beta_1 & \beta_1 & & & \\
  & \beta_0 & \beta_1 & \beta_0 & & & \\
  & a_2\beta_2 & a_1\beta_2 & a_0\beta_2 & \beta_2 & & \\
  & \beta_0 & \beta_1 & \beta_0 & \beta_1 & \beta_0 & \\
  & a_3\beta_3 & a_2\beta_3 & a_1\beta_3 & a_0\beta_3 & & \\
  & \beta_0 & \beta_1 & \beta_0 & \beta_1 & \beta_0 & \beta_0 & \\
  & \ldots & \ldots & \ldots & & \ldots & & \\
\end{bmatrix}
$$

This shows that the inducing function $\phi$ can be recaptured from the matrix of $B\phi$. The non-negative Fourier coefficients of $\phi$ are the entries of the main diagonal respectively, whereas the non-positive Fourier coefficients of $\phi$ can be obtained by multiplying respectively the entries of the 0-th row by $\beta_k$, $k = 0, 1, 2, \ldots$. The adjoint of $B\phi$ denoted by $B^*_\phi$ is such that

$$
\langle B^*_\phi e_j, e_i \rangle = \langle e_j, B\phi e_i \rangle = \overline{a_{2j-i}} \frac{\beta_j}{\beta_i}, \quad i, j = 0, 1, 2, \ldots.
$$

Further,

$$
B^*_\phi e_k(z) = \beta_k \sum_{n=0}^{\infty} \frac{e_n(z)}{\beta_n}.
$$

To begin with, we observe the following:

**Theorem 5.1.2 ([17]).** The mapping $\phi \rightarrow B\phi$ is linear and one-one.

**Proof.** Since the mapping $\phi \rightarrow T\phi$ is linear and $B\phi = WT\phi$, we get that the mapping $\phi \rightarrow B\phi$ is linear. To show that $\phi \rightarrow B\phi$ is one-one, we assume that $B\phi = B\psi$. Then

$$
B\phi - B\psi = 0
$$

$$
\Rightarrow \quad \langle B\phi - B\psi e_j, e_i \rangle = 0 \quad \text{for } i, j = 0, 1, 2, 3\ldots
$$
\[ \Rightarrow \langle W T_{\phi - \psi} e_j, e_i \rangle = 0 \]
\[ \Rightarrow \langle T_{\phi - \psi} e_j, W^* e_i \rangle = 0 \]
\[ \Rightarrow \frac{\beta_i}{\beta_{2i}} \langle T_{\phi - \psi} e_j, e_{2i} \rangle = 0, \quad i, j = 0, 1, 2, \ldots \]

Since \( \beta_n \)'s are all positive, we finally get that
\[ \langle (\phi - \psi), e_{2i-j} \rangle = 0 \quad i, j = 0, 1, \ldots \]
\[ \Rightarrow \phi - \psi = 0. \quad \square \]

We now study some more properties of \( B_\phi \).

Let \( S \) denote the shift operator on \( H^2(\beta) \) such that \( Se_j = \frac{1}{w_j} e_{j+1} \). Then \( S \) is bounded since \( \left\langle \frac{1}{w_n} \right\rangle \) is positive and bounded.

This operator \( S \) is well defined as for any \( f = \sum_{n=0}^{\infty} a_n z^n \) in \( H^2(\beta) \),
\[
S(f) = S \sum_{n=0}^{\infty} a_n \beta_n e_n(z)
\]
\[
= \sum_{n=0}^{\infty} a_n \frac{\beta_n}{w_n} e_{n+1}(z)
\]
\[
= \sum_{n=0}^{\infty} a_n \frac{\beta_n^2}{\beta_{n+1}} e_{n+1}(z)
\]
\[
= \sum_{n=0}^{\infty} a_n \left( \frac{\beta_n}{\beta_{n+1}} \right)^2 z^{n+1}
\]
\[
= \sum_{n=1}^{\infty} a_{n-1} \left( \frac{\beta_{n-1}}{\beta_n} \right)^2 z^{n}
\]
Now
\[
\|S(f)\|_{\beta}^2 = \sum_{n=1}^{\infty} |a_{n-1}|^2 \left( \frac{\beta_{n-1}}{\beta_n} \right)^4 \beta_n^2 \\
\leq \sum_{n=1}^{\infty} |a_{n-1}|^2 \beta_n^2 < \infty.
\]

Also, we observe that \( S^* e_j = \frac{1}{w_{j-1}} e_{j-1}, \) \( j \geq 1 \) and \( S^* e_0 = 0. \)

We now obtain a characterization for the compression of a slant weighted Toeplitz operator to the space \( H^2(\beta) \). We would like to mention here that the technique used in the proof is motivated by that of Halmos [25].

**Theorem 5.1.3** ([17]). A bounded linear operator \( B \) on \( H^2(\beta) \) is the compression of a slant weighted Toeplitz operator on \( L^2(\beta) \) to \( H^2(\beta) \) if and only if \( B = S^* BT_{z^2} \) where \( T_{z^2} \) is the weighted Toeplitz operator on \( H^2(\beta) \) induced by \( z^2 \).

**Proof.** Let \( B \) be the compression of a slant weighted Toeplitz operator \( A_\phi \) induced by some \( \phi \in L^\infty(\beta) \) to the space \( H^2(\beta) \). Then, for \( i, j = 0, 1, \ldots \) we have
\[
\langle Be_j, e_i \rangle = \langle PA_\phi |_{H^2(\beta)} e_j, e_i \rangle \\
= \frac{w_j w_{j+1}}{w_i} \langle PA_\phi |_{H^2(\beta)} e_{j+2}, e_{i+1} \rangle \\
= \frac{w_j w_{j+1}}{w_i} \langle Be_{j+2}, e_{i+1} \rangle \\
= \left( Bw_j w_{j+1} e_{j+2}, \frac{1}{w_i} e_{i+1} \right) \\
= \langle BT_{z^2} e_j, Se_i \rangle
\]
\[ = \langle S^*BT_{z^2}e_j, e_i \rangle, \quad i, j = 0, 1, 2, \ldots \]

Hence \( B = S^*BT_{z^2} \).

Conversely, suppose that \( B \) is a bounded operator on \( H^2(\beta) \) and \( B = S^*BT_{z^2} \). Then for all \( i, j = 0, 1, 2, \ldots \)

We have

\[ \langle Be_j, e_i \rangle = \langle S^*BT_{z^2}e_j, e_i \rangle \]

\[ = \frac{w_jw_{j+1}}{w_i} \langle Be_{j+2}, e_{i+1} \rangle. \]

Now, for each non-negative integer \( n \), consider the operator \( B_n : L^2(\beta) \rightarrow L^2(\beta) \) such that

\[ B_n = S^*BP T_{z^2}^n \]

Then

\[ \langle B_ne_j, e_i \rangle = \langle S^*BP T_{z^2}^n e_j, e_i \rangle \]

\[ = \langle S^*(n-1)BP T_{z^2}^{(n-1)}e_j, Se_i \rangle \]

\[ = \frac{w_jw_{j+1}}{w_i} \langle S^*(n-1)BP T_{z^2}^{(n-1)}e_{j+2}, e_{i+1} \rangle \]

\[ = \frac{w_jw_{j+1}}{w_i} \langle B_{n-1}e_{j+2}, e_{i+1} \rangle \]

\[ = \prod_{k=0}^{n-1} \frac{w_{j+k}w_{j+k+1}}{w_{i+k}} \langle B_0e_{j+2n}, e_{i+n} \rangle. \quad (5.1) \]

But \( B_0 = B \). Therefore,

\[ \langle B_0e_{j+2n}, e_{i+n} \rangle = \langle Be_{j+2n}, e_{i+n} \rangle \]

\[ = \prod_{k=0}^{n-1} \frac{w_{i+k}}{w_{j+k}w_{j+k+1}} \langle Be_j, e_i \rangle \]

\[ = \prod_{k=0}^{n-1} \frac{w_{i+k}}{w_{j+k}w_{j+k+1}} \langle Be_j, e_i \rangle \quad (5.2) \]
On substituting from (5.2) in (5.1) we get

\[ \langle B_n e_j, e_i \rangle = \langle Be, e_i \rangle. \]

Also, if \( i \) or \( j \) is negative, then for sufficiently large values of \( n \), both \( i + n \) and \( j + 2n \) are positive, and from then onwards, \( \langle BPe_{j+2n}, e_{i+n} \rangle \) is independent of \( n \). Further, if \( p \) and \( q \) are trigonometric polynomials, then the sequence \( \{\langle B_n p, q \rangle\} \) is convergent. So, for given \( f, g \in L^2(\beta) \), the sequence \( \{\langle B_n f, g \rangle\} \) is Cauchy and hence convergent. We therefore conclude that the sequence \( \{B_n\} \) converges weakly to a bounded operator \( B_\infty \) on \( L^2(\beta) \). Now for each \( i \) and \( j \) in \( \mathbb{Z} \), we have

\[ \langle B_\infty e_j, e_i \rangle = \lim_{n \to \infty} \langle B_n e_j, e_i \rangle \]

\[ = \lim_{n \to \infty} \langle S^n BPT_z^{2n} e_j, e_i \rangle \]

\[ = \lim_{n \to \infty} \langle S^n(BPT_z^{2n+1}) e_j, e_i \rangle \]

\[ = \lim_{n \to \infty} \langle S^n BPT_z^{2n} T_z e_j, Se_i \rangle \]

\[ = \lim_{n \to \infty} \frac{w_j w_{j+1}}{w_i} \langle S^n BPT_z^{2n} e_{j+2}, e_{i+1} \rangle \]

\[ = \lim_{n \to \infty} \frac{w_j w_{j+1}}{w_i} \langle B_n e_{j+2}, e_{i+1} \rangle \]

\[ = \frac{w_j w_{j+1}}{w_i} \langle B_\infty e_{j+2}, e_{i+1} \rangle. \]

It follows therefore that the operator \( B_\infty \) is a slant weighted Toeplitz operator on \( L^2(\beta) \). Next suppose \( f, g \in H^2(\beta) \). Then

\[ \langle PB_\infty f, g \rangle = \lim_{n \to \infty} \langle PB_n f, g \rangle \quad (5.3) \]
But we have seen that \( \langle B_n e_j, e_i \rangle = \langle Be_j, e_i \rangle \) for all \( i, j \geq 0 \). Hence for all \( f \) and \( g \) in \( H^2(\beta) \),

\[
\lim_{n \to \infty} \langle PB_n f, g \rangle = \langle B f, g \rangle
\]  

(5.4)

From (5.3) and (5.4) we get that

\[
\langle PB_\infty f, g \rangle = \langle B f, g \rangle \quad \text{for all } f \text{ and } g \text{ in } H^2(\beta).
\]

This implies that \( PB_\infty f = B f \) for all \( f \in H^2(\beta) \).

In other words, \( B \) is the compression of \( B_\infty \) on \( H^2(\beta) \). \( \square \)

Ho [43] proved that the only compact slant Toeplitz operator is zero.

We prove the following:

**Theorem 5.1.4 ([17]).** \( B_\phi \) is compact if and only if \( \phi = 0 \).

**Proof.** Suppose \( \phi = 0 \). Then \( B_\phi = A_\phi |_{H^2(\beta)} = 0 \) is compact.

Conversely, let \( B_\phi \) be compact. Then

\[
WT_\phi = T_{W(\phi)} \quad \text{is compact}
\]

Hence

\[
W(\phi) = 0
\]

Therefore,

\[
\langle W(\phi), e_n \rangle = 0 \quad \forall \ n \in Z
\]

This means that

\[
\langle \phi, W^* e_n \rangle = 0 \quad \forall \ n \in Z
\]
5.1 Compression of $A_{\phi}$ to $H^2(\beta)$

$$\Rightarrow \left\langle \phi, \frac{\beta_n}{\beta_{2n}} e_{2n} \right\rangle = 0 \quad \forall \ n \in Z$$

$$\Rightarrow a_{2n} = 0 \quad n \in Z, \text{ where } \phi = \sum a_n z^n. \quad (5.5)$$

On the other hand, $B_{\phi}$ is compact, then $B_{\phi} T$ is compact

$$\Rightarrow T W(\phi(z) \cdot z) \text{ is compact (because } B_{\phi} T_z = T W(\phi(z) \cdot z))$$

$$\Rightarrow W(\phi(z) \cdot z) = 0$$

$$\Rightarrow \left\langle W(\phi(z) \cdot z), e_n(z) \right\rangle = 0 \quad \forall \ n \in Z$$

$$\Rightarrow \left\langle \phi(z) \cdot z, \frac{\beta_n}{\beta_{2n}} e_n(z) \right\rangle = 0 \quad \forall \ n \in Z$$

$$\Rightarrow a_{2n+1} = 0 \quad n \in Z \quad (5.6)$$

from Equations (5.5) and (5.6) we get that $\phi = 0$. \qed

Before we end this section, we question the closure of the set of all compressions of slant weighted Toeplitz operators. We use the characterization proved in Theorem 5.1.3 to prove the following:

**Theorem 5.1.5** ([17]). The set of all compressions of slant weighted Toeplitz operators is weakly closed.

**Proof.** Let $\langle B_n f, g \rangle \to \langle B f, g \rangle$ for all $f, g$ in $H^2(\beta)$, where each $B_n$ is the compression of a slant weighted Toeplitz operator. Then

$$B_n = S^* B_n T z^2 \text{ for all } n.$$

Therefore as $n \to \infty$, we have

$$\langle B_n f, g \rangle = \langle B_n T_{z^2} f, S g \rangle \to \langle B T_{z^2} f, S g \rangle$$

$$= \langle S^* B T_{z^2} f, g \rangle$$

So,

$$S^* B_n T_{z^2} \to S^* B T_{z^2} \text{ weakly.}$$

Hence,

$$B = S^* B T_{z^2}.$$  

Thus $B$ is the compression of a slant weighted Toeplitz operator by Theorem 5.1.3.

The notion of the compression of a slant weighted Toeplitz operator and its study inspires us to question whether we can generalize it? This motivates us to introduce the compression of a $k$-th order slant weighted Toeplitz operator defined on $L^2(\beta)$ to the space $H^2(\beta)$. We now present the following study which is again motivated by the study made in [5].

### 5.2 Compression of $U_{\varphi}$ to $H^2(\beta)$

We recall that for any integer $k$, the $k$-th order slant weighted Toeplitz operator $U_{\varphi}$ was defined in Section 4.1 as $U_{\varphi}(f) = W_k M_{\varphi}(f)$ for all $f \in L^2(\beta)$, where $W_k : L^2(\beta) \to L^2(\beta)$ is the operator given by

$$W_k e_n(z) = \begin{cases} 
\frac{\beta_{n/k}}{\beta_n} e_{n/k}(z) & \text{if } n \text{ is divisible by } k \\
0 & \text{otherwise.}
\end{cases}$$

We define the compression of $U_{\varphi}$ to the weighted Hardy space $H^2(\beta)$ as follows.
5.2 Compression of $U_\phi$ to $H^2(\beta)$

**Definition 5.2.1** ([18]). We denote the compression of $k$-th order slant weighted Toeplitz operator $U_\phi$ on $L^2(\beta)$ to the space $H^2(\beta)$ by $V_\phi$. Thus $V_\phi : H^2(\beta) \to H^2(\beta)$ is defined as

$$V_\phi e_j(z) = \frac{1}{\beta_j} \sum_{n=0}^{\infty} a_{kn-j} \beta_n e_n(z), \quad j = 0, 1, 2, \ldots$$

Now, for all $i, j = 0, 1, 2, \ldots$

$$\langle V_\phi e_j, e_i \rangle = \langle PU_\phi e_j, e_i \rangle = a_{ik-j} \frac{\beta_i}{\beta_j}.$$

Therefore, the matrix of $V_\phi$ is unilaterally infinite and has the form

$$\begin{bmatrix}
a_{00} \frac{\beta_0}{\beta_0} & a_{10} \frac{\beta_0}{\beta_1} & a_{20} \frac{\beta_0}{\beta_2} & \cdots \\
a_{01} \frac{\beta_1}{\beta_0} & a_{11} \frac{\beta_1}{\beta_1} & a_{21} \frac{\beta_1}{\beta_2} & \cdots \\
a_{02} \frac{\beta_2}{\beta_0} & a_{12} \frac{\beta_2}{\beta_1} & a_{22} \frac{\beta_2}{\beta_2} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

for a given $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

Also, the adjoint of $V_\phi$, denoted by $V_\phi^*$ has the property that for $i, j = 0, 1, 2, \ldots$,

$$\langle V_\phi^* e_j, e_i \rangle = \langle e_j, V_\phi e_i \rangle = \frac{\beta_i}{\beta_j} \overline{a_{kj-i}}.$$

Hence for each $j \geq 0$,

$$V_\phi^* e_j(z) = \beta_j \sum_{n=0}^{\infty} \overline{a_{kj-n}} \frac{e_n(z)}{\beta_n}.$$
We observe the following:

**Theorem 5.2.2 ([18]).** If $\phi$ is in $L^2(\beta)$, then

(i) $V_\phi = W_k T_\phi$.

(ii) $T_\phi W_k = T_{\phi(z^k)}$

**Proof.** (i) It is proved in Theorem 4.4.1(iii) that $P$ reduces $W_k$. It follows that

\[
V_\phi = PU_\phi|_{H^2(\beta)} = PW_k M_\phi|_{H^2(\beta)} = W_k P M_\phi|_{H^2(\beta)} = W_k T_\phi
\]

where $T_\phi$ is the weighted Toeplitz operator on $H^2(\beta)$.

(ii) $T_\phi W_k = P M_\phi W_k = PW_k M_\phi(z^k) = PU_{\phi(z^k)} = V_{\phi(z^k)}$. \hfill $\square$

The above theorem connects a weighted Toeplitz operator on $H^2(\beta)$ and the compression of a $k$-th order slant weighted Toeplitz operator to $H^1(\beta)$.

**Theorem 5.2.3 ([18]).** For a given $\phi$ in $L^\infty(\beta)$,

(i) The mapping $\phi \to V_\phi$ is linear and one-one.

(ii) $\|V_\phi\| \leq \|\phi\|_\infty$.

**Proof.** Since the mapping $\phi \to T_\phi$ is linear as shown in Theorem 2.3.1(ii) and $V_\phi = W_k T_\phi$, we get that the mapping $\phi \to V_\phi$ is linear. To show that
the correspondence $\phi \to V_\phi$ is one-one, we suppose that $V_\phi = V_\psi$ for some $\psi$ in $L^\infty(\beta)$. Then by linearity, $V_{\phi - \psi} = 0$. Therefore for $i, j = 0, 1, 2, \ldots$

$$\langle V_{\phi - \psi} e_j, e_i \rangle = 0.$$  

Now, let $(\phi - \psi)(z) = \sum_{n=-\infty}^{\infty} b_n z^n$ where $b_n$’s are suitably chosen. Then we get that

$$\langle V_{\phi - \psi} e_j, e_i \rangle = \frac{\beta_i}{\beta_j} b_{ki - j} = 0 \text{ for all } i, j = 0, 1, 2, \ldots$$

Therefore $\phi - \psi \equiv 0$.

(ii) We have $\|V_\phi\| = \|W_k T_\phi\| \leq \|W_k\| \|T_\phi\| \leq \|\phi\|_\infty$.

The following theorem shows that $V_\phi$ does not commute with a weighted Toeplitz operator in general.

**Theorem 5.2.4 ([18]).** If either $\phi$ is co-analytic or $\psi$ is analytic, then

(i) $V_\phi T_\psi = V_\phi \psi$

(ii) $T_\psi V_\phi = V_{\psi(z^+)\phi}$.

**Proof.** (i) $V_\phi T_\psi = W_k T_\phi T_\psi = W_k T_\phi \psi = V_\phi \psi$

(ii) $T_\psi V_\phi = T_\psi W_k T_\phi = W_k T_\psi(z^+) T_\phi = W_k T_{\psi(z^+)\phi} = V_{\psi(z^+)\phi}$.

**Remark.** Before we conclude this chapter, we would like to point out that the operator $B_\phi$ is a particular case of the operator $V_\phi$ when $k = 2$.

Therefore by adopting the technique suggested by PR. Halmos [25] and taking the motivation from Theorem 5.1.3, we can obtain the following characterization for the compression of a $k$-th order slant weighted Toeplitz operator to the space $H^2(\beta)$:


A bounded linear operator $V$ on $H^2(\beta)$ is the compression of a $k$-th order slant weighted Toeplitz operator on $L^2(\beta)$ if and only if $V = S^*VT_{z^k}$ where $T_{z^k}$ is the weighted Toeplitz operator on $H^2(\beta)$ induced by $z^k$, and $S$ is the unilateral shift given in Section 5.1.

It is natural to ask about the closure of the set of all compressions of $k$-th order slant weighted Toeplitz operators and we can observe that the weak closure of this set can be deduced from the above characterization. Likewise, other similar properties of this operator $V_\phi$ on the space $H^2(\beta)$ can be examined.