CHAPTER 4

GENERALIZED SLANT WEIGHTED TOEPLITZ OPERATORS

We began our study of slant weighted Toeplitz operators with the introduction of the operator \( W \) on \( L^2(\beta) \) defined as \( We_{2n}(z) = \frac{\beta_n}{\beta_{2n}} e_n(z) \) and \( We_{2n-1}(z) = 0, n \in \mathbb{Z} \). We pause for a while to question whether we can generalize this operator \( W \) in some way which would help us to generalize the notion of a slant weighted Toeplitz operator? The main objective of the present chapter centers around this question.

In this chapter, we generalize the operator \( W \) to \( W_k \) for a fixed integer \( k \geq 2 \). We further introduce and study the notion of a \( k \)-th order slant weighted Toeplitz operator which we denote by \( U_\phi \). \( A_\phi \) is a particular case of \( U_\phi \) when \( k = 2 \). The chapter is divided into three sections.

In Section 4.1 we introduce the operator \( W_k \) and a \( k \)-th order slant weighted Toeplitz operator \( U_\phi \) on the space \( L^2(\beta) \), obtain its matrix and study some properties of \( W_k \). Section 4.2 pertains to the study of basic algebraic properties of \( U_\phi \) including that a bounded operator \( U \) on \( L^2(\beta) \) is a \( k \)-th order slant weighted Toeplitz operator if and only if \( M_z U = U M_z^k \). We would like to mention that this study is motivated by the work of S.C. Arora and R. Batra [4] made in the year 2004.
In Section 4.3, we define a \( k \)-th order slant weighted Toeplitz matrix; obtain a characterization of \( U_\phi \) in terms of this matrix and prove some more properties of \( U_\phi \). Throughout the chapter we assume that for a fixed integer \( k \geq 2 \), \( \frac{\beta_{kn}}{\beta_n} \leq M < \infty \) for all \( n \in \mathbb{Z} \), for some \( M \).

4.1 \( k \)-th Order Slant Weighted Toeplitz Operator \( U_\phi \)

Let \( k \geq 2 \) be a fixed integer. We introduce the following:

**Definition 4.1.1** ([15]). Let \( W_k : L^2(\beta) \to L^2(\beta) \) be defined as

\[
W_k e_n(z) = \begin{cases} 
\frac{\beta_{n/ke_{n/k}(z)}}{\beta_n} & \text{if } n \text{ is divisible by } k \\
0 & \text{otherwise.}
\end{cases}
\]

The adjoint of \( W_k \) is given by

\[
W_k^* e_n(z) = \frac{\beta_n}{\beta_{kn}} e_{kn}(z) \quad \text{for all } n \in \mathbb{Z}
\]

Also, the matrix of \( W_k \) is

\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\beta_0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \beta_1 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \beta_k & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \beta_2 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

Therefore, \( \|W_k\| = \sup_n \frac{\beta_n}{\beta_{kn}} \leq 1 \).

**Definition 4.1.2** ([15]). For an integer \( k \geq 2 \), we define the \( k \)-th order slant weighted Toeplitz operator \( U_\phi : L^2(\beta) \to L^2(\beta) \) as \( U_\phi (f) = W_k M_\phi (f) \) for all \( f \in L^2(\beta) \). The effect of \( U_\phi \) on the orthonormal basis \( \{e_i(z) = \frac{z^i}{\beta_i}\}_{i \in \mathbb{Z}} \) can be
4.1 $k$-th Order Slant Weighted Toeplitz Operator $U_{\phi}$

given by: $U_{\phi}e_i(z) = \frac{1}{\beta_i} \sum_{n=-\infty}^{\infty} a_{kn-i} \beta_n e_n(z)$. Also, the $(i,j)$th element of the matrix of $U_{\phi}$ is given by

$$\langle U_{\phi} e_j, e_i \rangle = \left\langle \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{kn-j} \beta_n e_n(z), e_i(z) \right\rangle = a_{i-k-j} \frac{\beta_i}{\beta_j}$$

Therefore the matrix of $U_{\phi}$ with respect to this basis is as follows:

$$\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \frac{\beta_0}{\beta_0} & a_{-1} \frac{\beta_0}{\beta_1} & a_{-2} \frac{\beta_0}{\beta_2} & \vdots \\
\vdots & \frac{\beta_1}{\beta_0} & a_{k} \frac{\beta_1}{\beta_1} & a_{k-2} \frac{\beta_1}{\beta_2} & \vdots \\
\vdots & \frac{\beta_2}{\beta_0} & a_{2k} \frac{\beta_2}{\beta_1} & a_{2k-2} \frac{\beta_2}{\beta_2} & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix}$$

The adjoint of $U_{\phi}$, denoted by $U_{\phi}^*$ is given by

$$\langle U_{\phi}^* e_j, e_i \rangle = \langle e_j, U_{\phi} e_i \rangle = \overline{a}_{i-k-j} \frac{\beta_j}{\beta_i}.$$  

The matrix of $U_{\phi}^*$ is therefore obtained as follows:

$$\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \overline{a}_{0} \frac{\beta_0}{\beta_0} & \overline{a}_{k-1} \frac{\beta_0}{\beta_1} & \overline{a}_{2k} \frac{\beta_0}{\beta_2} & \vdots \\
\vdots & \overline{a}_{-1} \frac{\beta_0}{\beta_1} & \overline{a}_{k-2} \frac{\beta_0}{\beta_1} & \overline{a}_{2k-1} \frac{\beta_0}{\beta_2} & \vdots \\
\vdots & \overline{a}_{-2} \frac{\beta_0}{\beta_2} & \overline{a}_{k-3} \frac{\beta_0}{\beta_2} & \overline{a}_{2k-2} \frac{\beta_0}{\beta_2} & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix}$$

We observe that for $k = 2$, $U_{\phi} = A_{\phi}$. We also make the following observations.
Theorem 4.1.3 ([18]). The mapping $\phi \rightarrow U_\phi$ is linear and one-one.

**Proof.** For linearity, consider

$$U_{(\alpha \phi + \beta \psi)}(f) = W_k M_{(\alpha \phi + \beta \psi)}(f)$$

$$= W_k (M_{\alpha \phi} + M_{\beta \psi})(f)$$

$$= (W_k M_{\alpha \phi} + W_k M_{\beta \psi})(f)$$

$$= \alpha W_k M_\phi (f) + \beta W_k M_\psi (f)$$

$$= \alpha U_\phi (f) + \beta U_\psi (f)$$

For one-one ness, let $U_\phi = U_\psi$.

Then, $U_\phi - U_\psi = 0$. By linearity we get

$$U_{\phi - \psi} = 0$$

Hence

$$U_{\phi - \psi} e_n(z) = 0 \quad \text{for all } n \in \mathbb{Z}$$

$$\Rightarrow \quad W_k M_{\phi - \psi} e_n(z) = 0 \quad \text{for all } n \in \mathbb{Z}$$

$$\Rightarrow \quad W_k (\phi - \psi) e_n(z) = 0 \quad \text{for all } n \in \mathbb{Z}.$$  

This implies that either $\phi - \psi = 0$ or that the degree of $(\phi - \psi) e_n(z)$ is not divisible by $k$. But since this is true for all $n \in \mathbb{Z}$, the second possibility is ruled out. Hence $\phi - \psi = 0$ or $\phi = \psi$. \hfill $\square$

To further investigate the properties of the operator $U_\phi$, we need to study the operator $W_k$, since $W_k$ is a special case of $U_\phi$ as shown below:
**Theorem 4.1.4** ([15]).

(i) \( W_k = U_1 \)

(ii) \( \mathcal{U}_\phi \) is bounded

(iii) \( P \) reduces \( W_k \)

(iv) \( W_k M_\nu W_k^* = 0 \) for \( p = 1, 2, \ldots, k-1 \).

**Proof.** (i) Take \( \phi = 1 \) in \( \mathcal{U}_\phi = W_k M_\phi \).

(ii) \[ \| \mathcal{U}_\phi \| = \| W_k M_\phi \| \]
\[ \leq \| W_k \| \| M_\phi \| \]
\[ \leq \| M_\phi \| \]
\[ = \| \phi \|_\infty \]

since
\[ \| M_\phi \| = \| \phi \|_\infty \] as shown by Shields [70].

(iii) Case (a): \( i = kn, \ n \in \mathbb{N} \cup \{0\} \).

\[ PW_k e_i(z) = PW_k e_{kn}(z) = \frac{p \beta_n}{\beta_{kn}} e_n(z) = \frac{\beta_n}{\beta_{kn}} e_n(z) = \frac{\beta_n}{\beta_{kn}} e_n(z) = W_k e_{kn}(z) = W_k Pe_{kn}(z) = W_k Pe_i(z). \]
Case (b): $i = kn$, $n$ a negative integer.

Then $PW_k e_i(z) = P \frac{\beta_n}{\beta_{kn}} e_n(z) = 0 = W_k Pe_i(z)$

Case (c): $i$ is not a multiple of $k$.

$PW_k e_i(z) = 0 = W_k Pe_i(z)$ for all $i \in \mathbb{Z}$

Hence we conclude that

$PW_k = W_k P$

Hence $P$ reduces $W_k$.

(iv) Consider $f \in L^2(\beta)$. Then $W_k^* f$ lies in the closed span of $\{e_{kn}(z) : n \in \mathbb{Z}\}$. Therefore $M_z(W_k^* f)$ belongs to closed span of $\{e_{kn+1}(z) : n \in \mathbb{Z}\}$.

Also, $M_z(W_k^* f)$ belongs to the closed span of $\{e_{kn+2}(z) : n \in \mathbb{Z}\}$ and so on. In fact, for all $p = 1, 2, \ldots k - 1, M_{zp}(W_k^* f)$ belongs to the closed span of $\{e_{kn+p}(z) : n \in \mathbb{Z}\}$ respectively. Hence $W_k M_{zp} W_k^*(f) = 0$ for all $f \in L^2(\beta)$.

Therefore, $W_k M_{zp} W_k^* = 0$ for all $p = 1, 2, \ldots k - 1$. \qed

Theorem 4.1.5 ([18]). (i) $M_z W_k = W_k M_z$

(ii) $M_{zm} W_k = W_k M_{zm}$, $m \in \mathbb{Z}$.

Proof. (i) It is sufficient to prove that

$M_z W_k e_n(z) = W_k M_z e_n(z)$ for all $n \in \mathbb{Z}$.

Suppose $n$ is not a multiple of $k$. Then

$M_z W_k e_n(z) = M_z 0 = 0 = W_k M_z e_n(z)$.

Now, when $n = pk$ (multiple of $k$),

$M_z W_k e_n(z) = M_z \frac{\beta_p}{\beta_{pk}} e_p(z) = \frac{\beta_{p+1}}{\beta_{pk}} e_{p+1}(z)$ (4.1)
On the other hand,

\[
W_k M_z e_n(z) = W_k M_z e_{pk}(z)
\]

\[
= W_k \frac{\beta_{(p+1)k}}{\beta_{pk}} e_{(p+1)k}(z)
\]

\[
= \frac{\beta_{p+1}}{\beta_{pk}} e_{(p+1)}(z)
\]  \hspace{1cm} (4.2)

Hence from Equations (4.1) and (4.2) we get that \( M_z W_k e_n(z) = W_k M_z e_n(z) \) whenever \( n \) is a multiple of \( k \).

We may therefore conclude that for all \( n \in \mathbb{Z} \),

\[
M_z W_k e_n(z) = W_k M_z e_n(z), \quad \text{or equivalently,}
\]

\[
M_z W_k = W_k M_z
\]

(ii) From (i) and on applying induction, the result is true for all \( m \in \mathbb{Z}^+ \).

Now, when \( m = -1 \) and \( n \) is not a multiple of \( k \), we have

\[
M_z^{-1} W_k e_n(z) = M_z^{-1} 0 = 0 = W_k M_z^{-1} e_n(z)
\]  \hspace{1cm} (4.3)

For \( m = -1 \) and \( n = pk \) (a multiple of \( k \)),

\[
M_z^{-1} W_k e_n(z) = M_z^{-1} W_k e_{pk}(z)
\]

\[
= M_z^{-1} \frac{\beta_p}{\beta_{pk}} e_p(z)
\]

\[
= \frac{\beta_p}{\beta_{pk}} M_z^{-1} e_p(z)
\]

\[
= \frac{\beta_p}{\beta_{pk}} \frac{\beta_{p-1}}{\beta_p} e_{p-1}(z)
\]

\[
= \frac{\beta_{p-1}}{\beta_{pk}} e_{p-1}(z)
\]  \hspace{1cm} (4.4)
On the other hand,

\[ W_k M_z e_n(z) = W_k M_z^{-k} e_{pk}(z) \]

\[ = W_k z^{-k} \frac{z^{pk}}{\beta_{pk}} \]

\[ = W_k \frac{z^{(p-1)k}}{\beta_{(p-1)k}} \frac{\beta_{(p-1)k}}{\beta_{pk}} \]

\[ = \frac{\beta_{(p-1)k}}{\beta_{pk}} \cdot W_k e_{(p-1)k}(z) \]

\[ = \frac{\beta_{(p-1)k}}{\beta_{pk}} \cdot \frac{\beta_{p-1}}{\beta_{(p-1)k}} e_{p-1}(z) \]

\[ = \frac{\beta_{p-1}}{\beta_{pk}} e_{p-1}(z). \quad (4.5) \]

From Equations (4.3), (4.4) and (4.5) we conclude that \( M_{z^{-1}} W_k = W_k M_{z^{-k}} \).

This shows that the result is true for \( m = -1 \) also. Therefore by using induction, we can prove it for all negative integers \( m \). The case when \( m = 0 \) is trivially true. Hence the theorem. \( \square \)

**Corollary 4.1.6 ([18]).** Let \( \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta) \). Then

\[ M_{\phi(z)} W_k = W_k M_{\phi(z^k)}. \]

### 4.2 Properties of \( U_\phi \)

In this section, our aim is to prove the basic algebraic properties of the \( k \)-th order slant weighted Toeplitz operator \( U_\phi \). For the purpose, we first obtain a few results on the operator \( W_k \). We observe that the operator \( W_k \) defined on the space \( L^2(\beta) \) is not a co-isometry here. This fact is well illustrated in
some of the results obtained in this section subsequently used to obtain a characterization of $U_{\phi}$, which further helps us in studying various properties of $U_{\phi}$.

**Lemma 4.2.1** ([15]). If $h(z)$ is an $L^2(\beta)$ function, then $h(z^k)$ is also an $L^2(\beta)$ function.

**Proof.** Let $h(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^n$ be an $L^2(\beta)$ function.

Then

$$||h(z)||_{\beta}^2 = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 < \infty$$  \hspace{1cm} (4.6)

Also, then

$$h(z^k) = \sum_{n=-\infty}^{\infty} \alpha_n z^{kn} = \sum_{n=-\infty}^{\infty} \alpha_n \beta_{kn} e_{kn}(z)$$

Hence

$$\sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_{kn}^2 = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 \times \frac{\beta_{kn}^2}{\beta_n^2}$$

$$\leq M^2 \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \beta_n^2 < \infty$$

Therefore $h(z^k)$ is also an $L^2(\beta)$ function. \hfill \square

**Theorem 4.2.2** ([15]). Let $f(z) = \sum a_n z^n$ be in $L^2(\beta)$. Then

(i) $W_k^* f = f_k(z^k)$ where $f_k(z) = \sum \frac{\beta_n^2}{\beta_{kn}^2} a_n z^n$.

(ii) $W_k f (z) = g(z)$ where $g(z) = \sum a_{km} z^m$. 
Proof. (i) Let \( f(z) = \sum a_n z^n \) be in \( L^2(\beta) \). Then

\[
W_k^* f(z) = W_k^* \sum a_n z^n \\
= W_k^* \sum a_n \beta_n e_n(z) \\
= \sum a_n \beta_n W_k^* e_n(z) \\
= \sum a_n \beta_n \frac{\beta_n}{\beta_{kn}} e_{kn}(z) \\
= \sum a_n z^{kn} \frac{\beta_n^2}{\beta_{kn}^2} \\
= f_k(z^k)
\]

where \( f_k(z) = \sum a_n \frac{\beta_n^2}{\beta_{kn}^2} z^n \).

(ii) Let \( f = \sum a_n z^n \in L^2(\beta) \). Then

\[
W_k(f) = W_k \sum a_n z^n \\
= \sum a_n W_k \beta_n e_n(z).
\]

Since \( W_k \) eliminates all other terms, we consider only those terms for which \( n \) is a multiple of \( k \). That is \( n = km \) (say). Then

\[
W_k(f) = \sum_{m=-\infty}^{\infty} a_{km} \beta_m e_m(z) \\
= \sum_{m=-\infty}^{\infty} a_n z^m \\
= g(z).
\]
Theorem 4.2.3 ([15]).

\[ W_k(f(z^k)) = f(z) \quad \text{for all } f \in L^2(\beta). \]

**Proof.** Let \( f(z) = \sum a_n z^n \) be in \( L^2(\beta) \). Then

\[
W_k(f(z^k)) = W_k \sum a_n z^{kn} \\
= \sum a_n W_k(\beta_{kn} e_{kn}(z)) \\
= \sum a_n \beta_{kn} e_n(z) \\
= \sum a_n z^n = f(z). \]

\[\square\]

Corollary 4.2.4 ([15]).

\[ W_k \phi(z^k) = \phi(z) \quad \text{for all } \phi \in L^\infty(\beta). \]

Corollary 4.2.5 ([15]). (i) \( W_k W_k^* f(z) = f_k(z) \) where \( f_k(z) = \sum a_n \frac{\beta_n^2}{\beta_{kn}^2} z^n \).

(ii) \( W_k^* W_k f(z) = h(z^k) \) where \( h(z) = \sum a_{kn} \frac{\beta_n^2}{\beta_{kn}^2} z^n, \ n \in \mathbb{Z}. \)

**Proof.** (i) \( W_k W_k^* f(z) = W_k f_k(z^k) = f_k(z). \)

(ii)

\[
W_k^* W_k f(z) = W_k^* \sum a_{kn} z^n \\
= \sum a_{kn} W_k^* \beta_n e_n(z) \\
= \sum a_{kn} \beta_n \frac{\beta_n}{\beta_{kn}} e_{kn}(z) \\
= \sum a_{kn} \frac{\beta_n^2}{\beta_{kn}^2} z^{kn} \]
Thus $W_k^*W_k f(z) = h(z^k)$ where

$$h(z) = \sum a_{kn} \frac{\beta_n^2}{\beta_{kn}^2} z^n, \ n \in \mathbb{Z}. \quad \square$$

**Note.** We observe that $W_k^*W_k \neq I$ as in the ordinary space $L^2$ [43].

**Theorem 4.2.6 ([15]).** A bounded operator $U$ on $L^2(\beta)$ is a $k$-th order slant weighted Toeplitz operator if and only if $M_z U = UM_{z^k}$.

**Proof.** For necessity, let $U = U_\phi$ be a slant weighted Toeplitz operator. Then

$$M_z U e_i(z) = M_z U_\phi e_i(z)$$

$$= M_z \left( \sum_{n=-\infty}^{\infty} a_{kn-i} \frac{\beta_n e_n(z)}{\beta_i} \right)$$

$$= \sum_{n=-\infty}^{\infty} a_{kn-i} \frac{\beta_{n+1} e_{n+1}(z)}{\beta_i}$$

$$= \sum_{n=-\infty}^{\infty} a_{k(n-1)-i} \frac{\beta_n e_n(z)}{\beta_i}. \quad (4.7)$$

On the other hand,

$$UM_{z^k} e_i(z) = U_\phi M_{z^k} e_i(z)$$

$$= U_\phi \left( \frac{\beta_{i+k}}{\beta_i} e_{i+k}(z) \right) = \frac{\beta_{i+k}}{\beta_i} U_\phi e_{i+k}(z)$$

$$= \beta_{i+k} \frac{1}{\beta_i} \beta_{i+k} \sum_{n=-\infty}^{\infty} a_{kn-i-k} \beta_n e_n(z). \quad (4.8)$$

Hence, from Equations (4.7) and (4.8) we conclude that

$$M_z U = UM_{z^k}.$$
For sufficiency, suppose that $U$ is a bounded operator on $L^2(\beta)$ which satisfies $M_z U = U M_z$ and let $f(z) = \sum a_n z^n$ be in $L^2(\beta)$. Then
\[
U(f(z^k)) = U \sum a_n z^{kn}
= \sum a_n U M_z^{kn} \cdot 1
= \sum a_n M_z^n U \cdot 1
= \sum a_n z^n U = f(z) U1
\]
By Lemma 4.2.1,
\[
\|f(z) U1\|_\beta = \|U f(z^k)\|_\beta
\leq \|U\| \|f(z^k)\|_\beta
\leq M \|U\| \|f(z)\|_\beta < \infty.
\]
Now, take $U1 = \phi_0(z)$. Then $\phi_0(z)$ is bounded.
Similarly, we can show that
\[
U(z f(z^k)) = f(z) U z
\]
\[
U(z^2 f(z^k)) = f(z) U z^2
\]
\[
\vdots
\]
\[
U(z^{k-1} f(z^k)) = f(z) U z^{k-1}
\]
Also, then, on taking $\phi_1(z) = Uz$, $\phi_2(z) = U z^2$, $\ldots$, $\phi_{k-1}(z) = U z^{k-1}$ we get that $\phi_1(z), \phi_2(z), \phi_{k-1}(z)$ are all bounded. So by Lemma 4.2.1, each $\phi_j(z^k)$ is bounded for $j = 0, 1, 2, \ldots k - 1$. Hence the function
\[
\phi(z) = \phi_0(z^k) + \bar{z} \phi_1(z^k) + \ldots + \bar{z}^{k-1} \phi_{k-1}(z^k)
\]
is also bounded. Next we show that $\phi$ is fact the inducing function for the $k$-th order slant weighted Toeplitz operator $U_\phi = U$. Or, in other words, we show that $U = W_k M_\phi$.

Since $f \in L^2(\beta)$, we can write

$$f(z) = f_0(z^k) + zf_1(z^k) + \ldots + z^{k-1}f_{k-1}(z^k),$$

where $f_0, f_1, \ldots, f_{k-1}$ are all in $L^2(\beta)$. Then

$$W_k M_\phi(f) = W_k[\phi f]$$

$$= W_k[[\phi_0(z^k) + \bar{z}\phi_1(z^k) + \ldots + z^{k-1}\phi_{k-1}(z^k)]$$

$$\times [f_0(z^k) + zf_1(z^k) + \ldots + z^{k-1}f_{k-1}(z^k)]]$$

Now, as $W_k$ eliminates all other terms, we get

$$W_k M_\phi(f) = W_k[\phi_0(z^k)f_0(z^k)] + W_k[\phi_1(z^k)f_1(z^k)] + \ldots + W_k[\phi_{k-1}(z^k)f_{k-1}(z^k)]$$

$$= \phi_0(z)f_0(z) + \phi_1(z)f_1(z) + \ldots + \phi_{k-1}(z)f_{k-1}(z)$$

$$= f_0(z)\phi_0(z) + f_1(z)\phi_1(z) + \ldots + f_{k-1}(z)\phi_{k-1}(z)$$

$$= f_0(z)U1 + f_1(z)Uz + \ldots + f_{k-1}(z)Uz^{k-1}$$

$$= Uf_0(z^k) + Uzf_1(z^k) + \ldots + Uz^{k-1}f_{k-1}(z^{k-1})$$

$$= Uf.$$

4.3 Matrix Characterization of $U_\phi$

An obvious question that strikes at this stage is that whether we can characterize a $k$-th order slant weighted Toeplitz operator $U_\phi$ in terms
of some matrix? If so, what would be that matrix like? We answer these questions in the present section. To begin, we introduce the notion of a $k$-th order slant weighted Toeplitz matrix for a given fixed integer $k \geq 2$ as follows:

**Definition 4.3.1 ([18])**. Let $w_n = \frac{\beta_{n+1}}{\beta_n}$ for all $n \in \mathbb{Z}$. Then the $k$-th order slant weighted Toeplitz matrix corresponding to the weight sequence $\{w_n\}$ is a bilaterally infinite matrix $\langle \lambda_{ij} \rangle$ such that

$$\lambda_{i+1,j+k} = \frac{w_i}{w_jw_{j+1}\ldots w_{j+k-1}}\lambda_{i,j}$$

Using Theorem 4.2.6 we now obtain a characterization of the operator $U_{\phi}$ on the space $L^2(\beta)$ in terms of a $k$-th order slant weighted Toeplitz matrix. The technique used here is suggested by T. Zegeye, S.C. Arora and M.P. Singh [87].

**Theorem 4.3.2 ([18])**. A necessary and sufficient condition that an operator $U$ on $L^2(\beta)$ be a $k$-th order slant weighted Toeplitz operator is that its matrix with respect to the orthonormal basis $\left\{ e_i(z) = \frac{z^i}{\beta_i} \right\}_{i \in \mathbb{Z}}$ is a $k$-th order slant weighted Toeplitz matrix.

**Proof.** Let $U_{\phi}$ be a $k$-th order slant weighted Toeplitz operator for some $\phi \in L^\infty(\beta)$. Then the matrix $\langle \lambda_{ij} \rangle$ of $U_{\phi}$ is given by

$$\lambda_{i,j} = \langle U_{\phi}e_j, e_i \rangle = a_{ki-j} \frac{\beta_i}{\beta_j}$$

Now,

$$\lambda_{i+1,j+k} = a_{ki+k-j-k} \frac{\beta_{i+1}}{\beta_{j+k}}$$
\[ a_{k_{i-j}} = \frac{\beta_{i+1}}{\beta_{j+k}} \]
\[ = \frac{\beta_{i+1} \beta_j}{\beta_i \beta_{j+k}} \lambda_{i,j} \]
\[ = \frac{w_i}{w_j w_{j+1} \ldots w_{j+k-1}} \lambda_{i,j} \quad \text{(4.9)} \]

where
\[ w_n = \frac{\beta_{n+1}}{\beta_n} \quad \text{for every } n \in \mathbb{Z}. \]

Hence the matrix of \( U_\phi \) is a \( k \)-th order slant weighted Toeplitz matrix.

Conversely, suppose that the matrix \( \langle \lambda_{ij} \rangle \) of an operator \( U \) on \( L^2(\beta) \) is a \( k \)-th order slant weighted Toeplitz matrix. Then we have
\[
\langle U e_j(z), e_i(z) \rangle = \lambda_{i,j} = \frac{w_j w_{j+1} \ldots w_{j+k-1}}{w_i} \lambda_{i+1,j+k}
\]
\[
= \frac{w_j w_{j+1} \ldots w_{j+k-1}}{w_i} \langle U e_{j+k}(z), e_{i+1}(z) \rangle,
\]
\[
i, j = 0, \pm 1, \pm 2, \ldots .
\]

However,
\[
\langle M_z U e_j(z), e_i(z) \rangle = \langle U e_j(z), M_z^* e_i(z) \rangle
\]
\[
= \langle U e_j(z), w_{i-1} e_{i-1}(z) \rangle
\]
\[
= w_{i-1} \langle U e_j(z), e_{i-1}(z) \rangle
\]
\[
= \frac{w_{i-1}}{w_i} w_j w_{j+1} \ldots w_{j+k-1} \langle U e_{j+k}(z), e_i(z) \rangle
\]
\[
= \langle UM_z e_j(z), e_i(z) \rangle, \quad i, j = 0, \pm 1, \pm 2, \ldots .
\]
Hence we get that $M_z U = U M_z k$.

Therefore by Theorem 4.2.6, $U$ is a $k$-th order slant weighted Toeplitz operator. \qed

The above characterization simplifies our task to study some more properties of $U\phi$. We obtain the following:

**Corollary 4.3.3** ([18]). A bounded operator $U$ on $L^2(\beta)$ is a $k$-th order slant weighted Toeplitz operator if and only if $U = M_z^{-1} U M_z^k$ where $M_z$ and $M_z^k$ are the weighted multiplication operators induced by $z$ and $z^k$ respectively.

**Corollary 4.3.4** ([18]). $M_{\phi U \psi}$ is a $k$-th order slant weighted Toeplitz operator and in that case $M_{\phi(z) U \psi(z)} = U_{\phi(z^k) \psi(z)}$.

**Proof.** For the first part, consider

$$M_z M_{\phi U \psi} = M_{\phi} M_z U \psi$$

$$= M_{\phi} U \psi M_z^k$$

Thus $M_{\phi U \psi}$ is $k$-th order slant weighted Toeplitz operator.

Next,

$$M_{\phi(z) U \psi(z)} = M_{\phi(z)} W_k M_{\psi(z)} = W_k M_{\phi(z^k)} M_{\psi(z)} = U_{\phi(z^k) \psi(z)} \text{.} \quad \square$$

Before we finish the present chapter, we make yet another use of the matrix characterization of $U\phi$. Our objective is to investigate the closure of the set of all $k$-th order slant weighted Toeplitz operators. We present the following:

**Theorem 4.3.5** ([18]). For a fixed integer $k \geq 2$, the set of all $k$-th order slant weighted Toeplitz operators is weakly closed and hence strongly closed.
Proof. We assume that for each positive integer \( n \), \( U_n \) is a \( k \)-th order slant weighted Toeplitz operator and let \( U_n \to U \) weakly. Then for any \( f, g \in L^2(\beta) \) we get that

\[
\langle U_n f, g \rangle \to \langle U f, g \rangle
\]

From Corollary 4.3.3 we get that \( U_n = M_{z}^{-1}U_{n}M_{z}^{k} \) for each positive integer \( n \). Therefore,

\[
\langle M_{z}^{-1}U_{n}M_{z}^{k}f, g \rangle = \langle U_{n}M_{z}^{k}f, (M_{z}^{-1})^{*}g \rangle = \langle Uz^{k}f, (M_{z}^{-1})^{*}g \rangle \to \langle Uz^{k}f, (M_{z}^{-1})^{*}g \rangle = \langle M_{z}^{-1}UM_{z}^{k}f, g \rangle
\]

Thus \( U_n \to M_{z}^{-1}UM_{z}^{k} \) weakly. But \( U_n \to U \) weakly by assumption. Therefore \( U = M_{z}^{-1}UM_{z}^{k} \). Hence \( U \) is a \( k \)-th order slant weighted Toeplitz operator. \( \square \)