WEIGHTED TOEPLITZ OPERATORS

A weighted Toeplitz operator $T_\phi$ is the compression of a weighted multiplication operator $M_\phi$ on $L^2(\beta)$ to the space $H^2(\beta)$ for a given symbol $\phi$ in $L^\infty(\beta)$. In this chapter we initiate the study of these operators on these spaces and discussed some of the algebraic properties of weighted Toeplitz operators.

The chapter is divided into four sections. In Section 2.1, we introduce the weighted Laurent matrix. We also obtain a matrix characterization of the weighted multiplication operator $M_\phi$ on the space $L^2(\beta)$. In Section 2.2, we introduce the weighted Toeplitz matrix and obtain the corresponding characterization for a weighted Toeplitz operator $T_\phi$ on $H^2(\beta)$. We also discuss the commutativity of a weighted Toeplitz operator with the (unilateral) weighted shift. Section 2.3 is devoted to the study of algebraic properties of the weighted Toeplitz operators. Here we discuss the boundedness of $T_\phi$, the sum and the product of two weighted Toeplitz operators and the adjoint of $T_\phi$. Section 2.4, the last section of this chapter, deals with a few examples of weighted Toeplitz operators and their eigenvalues.

2.1 Multiplication Operator on $L^2(\beta)$

The multiplication operator on $L^2(\beta)$, induced by $\phi$ in $L^\infty(\beta)$, is denoted by $M_\phi$ and is defined as $M_\phi : L^2(\beta) \to L^2(\beta)$ such that $M_\phi f = \phi f$
for all \( f \in L^2(\beta) \). If the Fourier expansion of \( \phi \) is \( \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \), then the effect of \( M_\phi \) on the orthonormal basis \( \langle e_k \rangle_{k=-\infty}^{\infty} \) is expressed by
\[
M_\phi e_k(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_{n-k} \beta_n e_k(z).
\]
Putting \( k = 0, \pm 1, \pm 2, \ldots \) we can obtain the matrix of \( M_\phi \) as a bilaterally infinite matrix given by
\[
\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\ldots & a_0 \beta_{-2} & a_{-1} \beta_{-2} & a_{-2} \beta_{-2} & \ldots \\
\ldots & a_1 \beta_{-1} & a_0 \beta_{-1} & a_{-1} \beta_{-1} & \ldots \\
\ldots & a_1 \beta_{-1} & a_0 \beta_{-1} & a_{-1} \beta_{-1} & \ldots \\
\ldots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}
\]

A second look at the above matrix tells us that the entries along the diagonals are related in a uniform way. This motivates us to introduce the following:

**Definition 2.1.1 ([16])**. Let \( w = \langle w_n \rangle_{n \in \mathbb{Z}} \) be a sequence of positive real numbers i.e. \( 0 < w_n < \infty \) for each \( n \). The weighted Laurent matrix corresponding to \( w \) is a bilaterally infinite matrix \( \langle \lambda_{ij} \rangle \) such that
\[
\lambda_{i+1,j+1} = \frac{w_i}{w_j} \lambda_{i,j}, \quad i, j = 0, \pm 1, \pm 2, \ldots.
\]

**Theorem 2.1.2 ([16])**. A necessary and sufficient condition that an operator on \( L^2(\beta) \) is a weighted multiplication operator on the space is that its matrix with respect to the orthonormal basis \( \{ e_k(z) = \frac{z^k}{\beta_k} \}_{k \in \mathbb{Z}} \) is a weighted Laurent matrix corresponding to the weight sequence \( w = \left\{ w_k = \frac{\beta_{k+1}}{\beta_k} \right\}_{k \in \mathbb{Z}} \).
Proof. For necessity, let $M_\phi$ be a weighted multiplication operator on $L^2(\beta)$. Then,

$$\lambda_{i,j} = \langle M_\phi e_j, e_i \rangle$$

$$= \langle \phi e_j, e_i \rangle$$

$$= \left\langle \sum a_n \frac{z^j}{\beta_j}, e_i \right\rangle$$

$$= \left\langle \sum a_n \frac{\beta_{n+j}}{\beta_j} e_{n+j}, e_i \right\rangle$$

$$= a_{i-j} \frac{\beta_i}{\beta_j}, \quad i, j = 0, \pm 1, \pm 2 \ldots$$

Further,

$$\lambda_{i+1,j+1} = a_{i-j} \frac{\beta_{i+1}}{\beta_{j+1}}$$

$$= \frac{w_i}{w_j} \lambda_{i,j}, \quad \text{where} \quad w_n = \frac{\beta_{n+1}}{\beta_n}, \quad n \in \mathbb{Z}$$

Hence from Equation (2.1), the matrix of $M_\phi$ is a weighted Laurent matrix corresponding to the weight sequence $w = \langle w_n \rangle_{n \in \mathbb{Z}}$

For sufficiency, let $A$ be an operator on $L^2(\beta)$ with its matrix as the weighted Laurent matrix corresponding to the weight sequence $w = \langle w_k \rangle_{k \in \mathbb{Z}}$ given by $w_k = \frac{\beta_{k+1}}{\beta_k}$. Now, since [70] an operator on $L^2(\beta)$ that commutes with the weighted shift operator $M_z$ is a multiplication operator $M_\phi$ for some $\phi \in L^\infty(\beta)$, hence it is enough to show that $A$ commutes with $M_z$. The proof is immediate. Given that

$$\langle Ae_{j+1}, e_{i+1} \rangle = \frac{w_i}{w_j} \langle Ae_j, e_i \rangle, \quad i, j = 0, \pm 1, \pm 2, \ldots$$
Now,

\[
\langle AM_z e_j, e_i \rangle = \langle Aw_j e_{j+1}, e_i \rangle \\
= w_j \langle Ae_{j+1}, e_i \rangle \\
= w_j \frac{w_{i-1}}{w_j} \langle Ae_j, e_{i-1} \rangle \\
= \langle Ae_j, w_{i-1} e_{i-1} \rangle \\
= \langle Ae_j, M_z^i e_i \rangle \\
= \langle M_z Ae_j, e_i \rangle, \quad i, j = 0, \pm 1, \pm 2, \ldots
\]

Thus \( AM_z = M_z A \). \qed

Example 2.1.3. Let \( \beta_0 = 1 \) and \( \beta_n = n \) for \( n > 0 \), \( \beta_n = -n \) for \( n < 0 \). Also, let \( az^2 + bz + c = \phi(z) \). Then the matrix of \( M_\phi \) on \( L^2(\beta) \) is the following matrix

\[
\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & c & 0 & 0 & \cdots \\
\cdots & \frac{b}{2} & c & 0 & \cdots \\
\cdots & \frac{a}{2} & b & c & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

since \( a_2 = a, \ a_1 = b, \ a_0 = c \) and \( a_n = 0 \) for all other \( n \).

2.2 Toeplitz Operators on \( H^2(\beta) \)

The study of Toeplitz operators and matrices has travelled great distances in different directions. However, still the interest in these operators and
the corresponding matrices has not diminished because of their striking applications [60], [42]. The idea, that a weighted shift operator could be viewed as “multiplication by \( z \)” on a Hilbert space of formal power series, inspired Shields [70] to study the weighted shift operators on the space \( L^2(\beta) \). This further motivated the idea of weighted Toeplitz operators. Lauric [52] focussed on a particular Toeplitz operator induced by \( \phi(z) = az + \frac{b}{z} \) defined on the space \( H^2(\beta) \) and obtained its commutant. This consideration motivates us to study some of the basic algebraic properties of \( T_\phi \) for a general inducing function \( \phi \) in \( L^\infty(\beta) \).

A weighted Toeplitz operator on the space \( H^2(\beta) \) is defined to be the orthogonal projection on \( H^2(\beta) \) of a weighted multiplication operator \( M_\phi \) on the space \( L^2(\beta) \). Hence

\[
T_\phi : H^2(\beta) \to H^2(\beta)
\]

such that for all \( f \in H^2(\beta) \)

\[
T_\phi(f) = P(\phi f).
\]

This mapping is well defined since if \( f \in H^2(\beta) \subset L^2(\beta) \), then \( \phi f \in L^2(\beta) \) and hence \( P(\phi f) \in H^2(\beta) \). Further,

\[
T_\phi e_k(z) = P(\phi e_k(z))
\]

\[
= P\left( \sum_{n=-\infty}^{\infty} a_n z^n \frac{z^k}{\beta_k} \right)
\]

\[
= P\left( \sum_{n=-\infty}^{\infty} a_{n-k} z^n \frac{z^k}{\beta_k} \right)
\]

\[
= \frac{1}{\beta_k} \sum_{n=0}^{\infty} a_{n-k} \beta_n e_n(z).
\]
Hence the matrix of $T_\phi$ is a unilaterally infinite matrix $\langle \lambda_{ij} \rangle_{i,j=0}^\infty$ given by
\[
\begin{bmatrix}
  a_0 \frac{\beta_0}{\beta_0} & a_{-1} \frac{\beta_0}{\beta_1} & a_{-2} \frac{\beta_0}{\beta_2} & \cdots \\
  a_1 \frac{\beta_1}{\beta_0} & a_0 \frac{\beta_1}{\beta_1} & a_{-1} \frac{\beta_1}{\beta_2} & \cdots \\
  a_2 \frac{\beta_2}{\beta_0} & a_1 \frac{\beta_2}{\beta_1} & a_0 \frac{\beta_2}{\beta_2} & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

It is interesting to note that the inducing function $\phi$ can be recaptured from the matrix of $T_\phi$. The non-positive Fourier coefficients of $\phi$ can be obtained from the matrix of $T_\phi$ by multiplying the entries in the 0-th row by $1, \frac{\beta_1}{\beta_0}, \frac{\beta_2}{\beta_0}, \ldots$, respectively whereas the non-negative Fourier coefficients can be obtained by multiplying the entries in the 0-th column by $1, \frac{\beta_0}{\beta_1}, \frac{\beta_0}{\beta_2}, \ldots$, respectively. These observations inspire us to introduce the following:

**Definition 2.2.1** ([16]). Let $w = (w_0, w_1, w_2, \ldots)$ be a sequence of positive numbers and $0 < w_n < \infty$ for all non-negative integers $n$. The weighted Toeplitz matrix corresponding to the weight sequence $w$ is a unilaterally infinite matrix $\langle \lambda_{ij} \rangle$ such that
\[
\lambda_{i+1,j+1} = \frac{w_i}{w_j} \lambda_{i,j}, \quad i, j = 0, 1, 2, \ldots
\]

We now obtain a characterization of the weighted Toeplitz operator in terms of the corresponding weighted Toeplitz matrix. We mention here that the method of proof is adopted from that given by Halmos [41].

**Theorem 2.2.2** ([16]). A necessary and sufficient condition that an operator on $H^2(\beta)$ be a weighted Toeplitz operator $T_\phi$ is that its matrix $\langle \lambda_{ij} \rangle$ with respect to the orthonormal basis $\left\{ e_k(z) = \frac{z^k}{\beta_k} \right\}_{k \in \mathbb{Z}^+ \cup \{0\}}$ is a weighted Toeplitz
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matrix corresponding to the weight sequence $w = \langle w_n \rangle$ given by $w_n = \frac{\beta_{n+1}}{\beta_n}$, $n \in \mathbb{Z}^+ \cup \{0\}$.

**Proof.** For necessity, let $T_\phi$ be a weighted Toeplitz operator on $H^2(\beta)$. Then

$$
\lambda_{i+1,j+1} = \langle T_\phi e_{j+1}, e_{i+1} \rangle \\
= \langle PM_\phi e_{j+1}, e_{i+1} \rangle \\
= \langle M_\phi e_{j+1}, P^* e_{i+1} \rangle \\
= \langle M_\phi e_{j+1}, e_{i+1} \rangle \\
= \frac{w_i}{w_j} \lambda_{i,j}, \quad i,j = 0,1,2,\ldots
$$

Thus the matrix of $T_\phi$ is a weighted Toeplitz matrix.

For sufficiency, let $A$ be an operator on $H^2(\beta)$ such that $\langle Ae_{j+1}, e_{i+1} \rangle = \frac{w_i}{w_j} \langle Ae_j, e_i \rangle$ where

$$
w_k = \frac{\beta_{k+1}}{\beta_k} \quad \text{and} \quad i,j,k = 0,1,2,\ldots
$$

We now prove that $A$ is a weighted Toeplitz operator on $H^2(\beta)$. Let $N : L^2(\beta) \to L^2(\beta)$ be an operator given by $Ne_j = \frac{1}{w_j} e_{j+1}$. Also, let $M_x$ be denoted by $M$. For each non-negative integer $n$, consider the operator on $L^2(\beta)$ given by

$$
A_n = N^*nAPM^n.
$$
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Case (i): If \( i,j \geq 0 \) then,

\[
\langle A_n e_j, e_i \rangle = \langle N^*A P M^n e_j, e_i \rangle \\
= \langle N^*A P M^{n-1} M e_j, N e_i \rangle \\
= \frac{w_j}{w_i} \langle A_{n-1} e_{j+1}, e_{i+1} \rangle \\
= \prod_{k=0}^{n-1} \left( \frac{w_{j+k}}{w_{i+k}} \right) \langle A_0 e_{j+n}, e_{i+n} \rangle \\
= \prod_{k=0}^{n-1} \left( \frac{w_{j+k}}{w_{i+k}} \right) \langle A e_{j+n}, e_{i+n} \rangle \tag{2.2}
\]

On the other hand,

\[
\langle A e_{j+n}, e_{i+n} \rangle = \prod_{k=0}^{n-1} \left( \frac{w_{i+k}}{w_{j+k}} \right) \langle A e_j, e_i \rangle \tag{2.3}
\]

From (2.2) and (2.3) we get that for \( i, j \geq 0 \)

\[
\langle A_n e_j, e_i \rangle = \langle A e_j, e_i \rangle
\]

Case (ii): If either \( i < 0 \) or \( j < 0 \) or if both \( i \) and \( j \) are negative, then for sufficiently large values of \( n, j+n \) and \( i+n \) are positive, so that the sequence \( \{\langle A_n e_j, e_i \rangle\} \) is convergent.

Thus if \( p \) and \( q \) are trigonometric polynomials (finite linear combinations of the \( e_i \)'s, \( i = 0, \pm 1, \pm 2, \ldots \)), then the sequence \( \{\langle A_n p, q \rangle\} \) is convergent.

Also, \( \|A_n\| = \|N^*A P M^n\| \leq \|N^*\| \|A\| \|P\| \|M^n\| = \|A\| \). Next, we show that \( \lim_{n \to \infty} \langle A_n f, g \rangle \) exists for all \( f, g \in L^2(\beta) \).

Let \( f, g \in L^2(\beta) \). By Weierstrass Approximation theorem, every continuous function can be approximated by polynomials. Hence, for each
$\epsilon > 0$, there exist trigonometric polynomials $p$ and $q$ such that [41],

$$\|g - q\| < \frac{\epsilon}{4(\|A\| + 1)(\|f\| + 1)}$$

and

$$\|f - p\| < \frac{\epsilon}{8(\|A\| + 1)(\|q\| + 1)}$$

Now for $n \geq m$, consider

\[
|\langle A_n f, g \rangle - \langle A_m f, g \rangle| = |\langle A_n f, g \rangle - \langle A_n f, q \rangle + \langle A_n f, q \rangle - \langle A_m f, q \rangle + \langle A_m f, q \rangle - \langle A_m f, g \rangle| \\
= |\|A_n\| \|f\| \|g - q\| + |\langle A_n f, q \rangle - \langle A_m f, q \rangle| + \|A_m\| \|f\| \|g - q\| | \\
\leq 2\|A\| \|f\| \|g - q\| + |\langle A_n f, q \rangle - \langle A_m f, q \rangle| \\
\leq 2\|A\| \|f\| \|g - q\| + \frac{\epsilon}{4} \\
< \frac{\epsilon}{4} + \frac{\epsilon}{4}.
\]

(2.4)

Consider

\[
|\langle A_n f, q \rangle - \langle A_m f, q \rangle| \\
\leq |\langle A_n f, q \rangle - \langle A_n p, q \rangle + \langle A_n p, q \rangle - \langle A_m p, q \rangle + \langle A_m p, q \rangle - \langle A_m f, q \rangle| \\
\leq \|A_n\| \|f - p\| \|q\| + \|A_m\| \|f - p\| \|q\| + |\langle A_n p, q \rangle - \langle A_m p, q \rangle| \\
\leq 2\|A\| \|f - p\| \|q\| + \frac{\epsilon}{4} \\
\leq \frac{\epsilon}{4} + \frac{\epsilon}{4}.
\]

(2.5)

Putting from (2.5) in (2.4),

\[
|\langle A_n f, g \rangle - \langle A_m f, g \rangle| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
Therefore $\langle A_n f, g \rangle$ is Cauchy in $\mathbb{C}$. Hence $\langle A_n f, g \rangle$ is convergent. Now let us define

$$\Phi : L^2(\beta) \times L^2(\beta) \to \mathbb{C}$$

as

$$\Phi(f, g) = \lim_{n \to \infty} \langle A_n f, g \rangle$$

$\Phi$, with addition and scalar multiplication satisfies the following:

(i) $\Phi(f_1 + f_2, g) = \Phi(f_1, g) + \Phi(f_2, g)$

(ii) $\Phi(\alpha f, g) = \alpha \Phi(f, g)$

(iii) $\Phi(f, g_1 + g_2) = \Phi(f, g_1) + \Phi(f, g_2)$

(iv) $\Phi(f, \alpha g) = \bar{\alpha} \Phi(f, g)$

Also, $|\Phi(f, g)| \leq ||A|| ||f|| ||g||$. Thus $\Phi$ is a bounded sesquilinear function on $L^2(\beta) \times L^2(\beta)$. Hence there exists a unique bounded linear operator $A_\infty$ on $L^2(\beta)$ such that $\Phi(f, g) = \langle A_\infty f, g \rangle$ for all $f, g \in L^2(\beta)$. Therefore for all $f, g \in L^2(\beta)$,

$$\lim_{n \to \infty} \langle A_n f, g \rangle = \langle A_\infty f, g \rangle.$$  

Hence, the sequence $\{A_n\}$ of operators is weakly convergent to an operator $A_\infty$ on $L^2(\beta)$. Further for all $i$ and $j$,

$$\langle A_\infty e_j, e_i \rangle = \lim_{n \to \infty} \langle N^n APM^n e_j, e_i \rangle$$

$$= \lim_{n \to \infty} \langle N^n APM^{n+1} e_j, e_i \rangle$$

$$= \lim_{n \to \infty} \langle N^n APM^n Me_j, Ne_i \rangle$$
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\[ = \frac{w_j}{w_i} \lim_{n \to \infty} \langle N^n AP M^n e_{j+1}, e_{i+1} \rangle \]

\[ = \frac{w_j}{w_i} \langle A_\infty e_{j+1}, e_{i+1} \rangle \]

Hence $\langle A_\infty e_{j+1}, e_{i+1} \rangle = \frac{w_i}{w_j} \langle A_\infty e_j, e_i \rangle$. Thus, $A_\infty$ is a weighted Laurent operator (weighted multiplication operator) on $L^2(\beta)$. Now, for $f, g \in H^2(\beta)$,

\[ \langle PA_\infty f, g \rangle = \langle A_\infty f, Pg \rangle \]

\[ = \langle A_\infty f, g \rangle \]

\[ = \lim_{n \to \infty} \langle A_n f, g \rangle \]

Thus, $A_n$ maps $H^2(\beta)$ to $H^2(\beta)$. Therefore, $A_n e_j \in H^2(\beta)$ for all $j \geq 0$. Also, for all $i, j \geq 0$

\[ \langle A_n e_j, e_i \rangle = \langle Ae_j, e_i \rangle. \]

Hence $A_n e_j = Ae_j$. This is true for all $j$. Thus $A_n = A$ on $H^2(\beta)$.

Hence for all $f, g \in H^2(\beta)$

\[ \langle PA_\infty f, g \rangle = \lim_{n \to \infty} \langle A_n f, g \rangle \]

\[ = \langle Af, g \rangle \]

This gives $PA_\infty f = Af$ for all $f$. Thus $A$ is the compression of $A_\infty$ on $H^2(\beta)$. Therefore, $A$ is a weighted Toeplitz operator. \[ \square \]

If the weight sequence $w_n = \frac{\beta_{n+1}}{\beta_n}$ is known, the Fourier coefficients of $\phi$ can be obtained from the matrix of $T_\phi = \langle \lambda_{ij} \rangle$, $i, j = 0, 1, 2, \ldots$ by
the following set of equations.

\[
\begin{align*}
  a_0 &= \lambda_{0,0} \\
  a_k &= \lambda_{k,0} \frac{\beta_0}{\beta_k} = \lambda_{k,0} \frac{\beta_0}{\beta_k} \\
  a_{-k} &= \lambda_{0,k} \frac{\beta_k}{\beta_0} = \lambda_{0,k} \frac{\beta_k}{\beta_0}
\end{align*}
\]

\[\text{(2.6)}\]

Let the compression of the bilateral weighted shift operator \( M \) on \( H^2(\beta) \) be denoted by \( U \). Then \( U : H^2(\beta) \to H^2(\beta) \) satisfies

\[
U e_j = w_j e_{j+1}, \quad j = 0, 1, 2, \ldots
\]

So that \( U^* e_j = w_{j-1} e_{j-1}, \quad j \geq 1 \) and \( U^* e_0 = 0 \).

Brown and Halmos [25] obtained a necessary and sufficient condition for the commutativity of a bounded linear operator on \( H^2 \) with the unilateral shift. We prove a characterization for a weighted Toeplitz operator.

**Theorem 2.2.3** ([16]). A necessary and sufficient condition that an operator \( T \) on \( H^2(\beta) \) be a weighted Toeplitz operator is that \( TU = UT \); that is it commutes with the unilateral weighted shift \( U \).

**Proof.** Let \( T \) be a weighted Toeplitz operator on \( H^2(\beta) \).

Then \( \langle Te_{j+1}, e_{i+1} \rangle = \frac{w_i}{w_j} \langle Te_j, e_i \rangle \).

Now,

\[
\langle TU e_j, e_i \rangle = \langle Tw_j e_{j+1}, e_i \rangle
\]

\[
= w_j \langle Te_{j+1}, e_i \rangle
\]

\[
= w_j w_{i-1} \frac{w_i}{w_j} \langle Te_j, e_{i-1} \rangle
\]


\[ = w_{i-1} \langle Te_j, e_{i-1} \rangle \]
\[ = \langle Te_j, U^* e_i \rangle = \langle UTe_j, e_i \rangle. \]

Thus \( TU = UT \).

Conversely, let \( TU = UT \).

Then
\[ \langle TUE_j, e_i \rangle = \langle UTe_j, e_i \rangle \]
\[ \Rightarrow \langle Ue_j, T^* e_i \rangle = \langle Te_j, U^* e_i \rangle \]
\[ \Rightarrow \langle w_je_{j+1}, T^* e_i \rangle = \langle Te_j, w_{i-1} e_{i-1} \rangle \]
\[ \Rightarrow \langle Te_{j+1}, e_i \rangle = \frac{w_{i-1}}{w_j} \langle Te_j, e_{i-1} \rangle. \]

Changing \( i \) to \( i + 1 \) on both the sides we get
\[ \langle Te_{j+1}, e_{i+1} \rangle = \frac{w_i}{w_j} \langle Te_j, e_i \rangle. \]

Thus the matrix of \( T \) is a weighted Toeplitz matrix. Hence by the above theorem, \( T \) is a weighted Toeplitz operator. \( \square \)

### 2.3 Properties of Weighted Toeplitz Operators

Weighted Toeplitz operators and their different versions have attracted many mathematicians \([58], [59], [63]\) in the last decade. However the basic algebraic properties of the operator \( T_\phi \) defined on the space \( H^2(\beta) \) need to be discussed yet. We present here some of these properties for the sequence space that we have considered.
We begin with the following:

**Theorem 2.3.1** ([13]). Let $\phi, \psi \in L^\infty(\beta)$. Then

(i) $T_\phi^*$ is a weighted Toeplitz operator if and only if $\langle w_n \rangle$ is constant.

(ii) $T_\phi + T_\psi = T_{\phi+\psi}$

**Proof.** (i) The matrix of $T_\phi$ is

\[
\begin{bmatrix}
  a_0 \beta_0 & a_{-1} \beta_1 & a_{-2} \beta_2 & a_{-3} \beta_3 & \cdots \\
  a_1 \beta_0 & a_0 \beta_1 & a_{-1} \beta_2 & a_{-2} \beta_3 & \cdots \\
  a_2 \beta_0 & a_1 \beta_1 & a_0 \beta_2 & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\]

Therefore the matrix of $T_\phi^*$ will be

\[
\begin{bmatrix}
  \overline{a}_0 \beta_0 & \overline{a}_{-1} \beta_1 & \overline{a}_{-2} \beta_2 & \overline{a}_{-3} \beta_3 & \cdots \\
  \overline{a}_1 \beta_0 & \overline{a}_0 \beta_1 & \overline{a}_{-1} \beta_2 & \overline{a}_{-2} \beta_3 & \cdots \\
  \overline{a}_2 \beta_0 & \overline{a}_1 \beta_1 & \overline{a}_0 \beta_2 & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}
\]

If $\langle \lambda'_{i,j} \rangle$ denote the entries of the above matrix, then we can see that

\[\lambda'_{i,j} = \overline{a}_{j-1} \beta_j \overline{\beta}_i.\]

So,

\[\lambda'_{i+1,j+1} = \overline{a}_{j-1} \beta_{j+1} \overline{\beta}_{i+1} = \lambda'_{i,j} \frac{w_j}{w_i}.\]
Thus $T^*_\phi$ is not a weighted Toeplitz operator in general and this also gives
that a necessary and sufficient condition for $T^*_\phi$ to be a weighted Toeplitz
operator is that
\[ w_i = w_j \quad \text{for all } i, j. \]
This implies that the weight sequence $w_n = \frac{\beta_{n+1}}{\beta_n}$ is constant.

(ii) \((T_\phi + T_\psi)e_k(z) = (T_\phi e_k + T_\psi e_k)(z) = (T_\phi e_k)(z) + (T_\psi e_k)(z)\)

Now, if $\phi = \sum_{n=-\infty}^{\infty} a_n z^n$ and $\psi = \sum_{n=-\infty}^{\infty} b_n z^n$, let $c_n = a_n + b_n$ for all $n$ and
\[ \xi(z) = \sum_{n=-\infty}^{\infty} c_n z^n. \]
Then $\xi \in L^\infty(\beta)$, and $T_\phi + T_\psi = T_{\phi+\psi} = T_\xi$. \(\square\)

We now obtain some results on the product $T_\phi T_\psi$ motivated by the study
made by Halmos [25] and Douglas [31].

**Theorem 2.3.2** ([13]). A necessary and sufficient condition for the product
$T_\phi T_\psi$ of two weighted Toeplitz operators $T_\phi$ and $T_\psi$ to be a weighted Toeplitz
operator is that either $\phi$ be co-analytic or $\psi$ be analytic.

**Proof.** Let $C = T_\phi T_\psi$.

Let $\langle \gamma_{i,j} \rangle$ be the matrix of $C$. If the respective Fourier expansions of $\phi$
and $\psi$ are $\phi(z) = \sum_i a_i z^i$ and $\psi(z) = \sum_j b_j z^j$ and the respective matrices of
$T_\phi$ and $T_\psi$ are $\langle \lambda_{ij} \rangle$ and $\langle \delta_{ij} \rangle$, then,
\[ \gamma_{ij} = \sum_{k=0}^{\infty} \lambda_{ik} \delta_{kj} \]
\[ = \sum_{k=0}^{\infty} a_{i-k} b_k \frac{\beta_i}{\beta_k} \frac{\beta_k}{\beta_j} \]
\[ = \frac{\beta_i}{\beta_j} \sum_{k=0}^{\infty} a_{i-k} b_{k-j}. \]
Similarly,

\[ \gamma_{i+1,j+1} = \frac{\beta_{i+1}}{\beta_{j+1}} \sum_{k=0}^{\infty} a_{i+1-k} b_{k-j-1} \]

\[ = \frac{\beta_{i+1}}{\beta_{j+1}} a_{i+1} b_{-j-1} + \frac{\beta_{i+1}}{\beta_{j+1}} \sum_{k=1}^{\infty} a_{i+1-k} b_{k-j-1} \]

\[ = \frac{\beta_{i+1}}{\beta_{j+1}} a_{i+1} b_{-j-1} + \frac{\beta_{i+1}}{\beta_{j+1}} \sum_{k=0}^{\infty} a_{i-k} b_{k-j} . \]

Thus,

\[ \gamma_{i+1,j+1} = \frac{\beta_{i+1}}{\beta_{j+1}} a_{i+1} b_{-j-1} + \frac{w_i}{w_j} \gamma_{i,j} . \]

Hence a necessary and sufficient condition for the product \( T_\phi T_\psi \) to be a weighted Toeplitz operator is that

\[ \frac{\beta_{i+1}}{\beta_{j+1}} a_{i+1} b_{-j-1} = 0. \]

Since \( \beta_n \)'s are positive numbers, therefore either

\[ a_{i+1} = 0 \quad \text{for all } i \geq 0 \]

or

\[ b_{-j-1} = 0 \quad \text{for all } j \geq 0. \]

This implies that either \( \phi \) is co-analytic or \( \psi \) is analytic. Hence the proof. \( \square \)

**Corollary 2.3.3** ([13]). If \( \psi \) is analytic, Then \( T_\phi T_\psi = T_\phi \psi \).

**Proof.** Let \( \psi \in L^\infty(\beta) \) be analytic, Then,

\[ T_\phi T_\psi f = T_\phi(\psi \cdot f) = P(\phi \cdot \psi \cdot f) = T_\phi \psi f \]

for all \( f \) in \( H^2(\beta) \). Hence \( T_\phi T_\psi = T_\phi \psi \). \( \square \)
Corollary 2.3.4 ([13]). A necessary and sufficient condition that the product of two weighted Toeplitz operators be zero is that at least one factor is zero.

Proof. Suppose either $\phi$ or $\psi$ is equal to zero. This means that either $\phi$ is co-analytic or $\psi$ is analytic. Hence by Theorem 2.3.2, $T_\phi T_\psi = T_{\phi \psi}$. But $\phi \psi = 0$. This implies that $T_\phi T_\psi = 0$.

Conversely, let $T_\phi T_\psi = 0$. Now, $0 = T_\phi T_\psi$ is a weighted Toeplitz operator. Therefore, again by Theorem 2.3.2, either $\phi$ is co-analytic or $\psi$ is analytic. So $T_\phi T_\psi = T_{\phi \psi} = 0$. Hence $\phi \psi = 0$. Further, if $\phi$ is co-analytic then $\phi \psi = 0$ implies that $\psi = 0$. Alternatively, if $\psi$ is analytic then $\phi \psi = 0$ implies that $\phi = 0$.

Corollary 2.3.5 ([13]). The only idempotent weighted Toeplitz operators are 0 and I.

Proof. If $T_\phi$ is an idempotent Toeplitz operator, then

$$T_\phi^2 = T_\phi \Rightarrow T_\phi (T_\phi - I) = 0$$

Therefore either $T_\phi = 0$ or $T_\phi = I$.

Theorem 2.3.6 ([13]). A necessary and sufficient condition that two weighted Toeplitz operators commute is that either both be analytic or both be co-analytic or one be a linear function of the other.

Proof. We first prove sufficiency:

Let $\langle \gamma_{ij} \rangle$ denote the matrix of $T_\phi T_\psi$ and $\langle \gamma'_{i,j} \rangle$ denote the matrix of $T_\psi T_\phi$. Then

$$\gamma_{ij} = \frac{\beta_i}{\beta_j} \sum_{k=0}^{\infty} a_{i-k} b_{k-j}$$
and

\[
\gamma'_{ij} = \frac{\beta_i}{\beta_j} \sum_{k=0}^{\infty} b_{i-k} a_{k-j}
\]

Since \( \phi \) and \( \psi \) are both analytic,

\[
\begin{align*}
    a_{i-k} &= 0 \text{ whenever } i < k \\
    b_{k-j} &= 0 \text{ whenever } j > k
\end{align*}
\]

(2.7)

Similarly, \( b_{i-k} = 0 \) whenever \( i < k \)

and \( a_{k-j} = 0 \) whenever \( j > k \)

so

\[
\begin{align*}
    \gamma_{ij} &= \frac{\beta_i}{\beta_j} \sum_{j \leq k \leq i} a_{i-k} b_{k-j} \\
    \gamma'_{ij} &= \frac{\beta_i}{\beta_j} \sum_{j \leq k \leq i} b_{i-k} a_{k-j}
\end{align*}
\]

(2.8)

Hence

\[\gamma_{ij} = \gamma'_{ij} \quad \text{for all } i, j = 0, 1, 2, \ldots\]

Similarly, the cases when both are co-analytic or when one is a linear function of the other.

Now for the necessity, suppose that \( T_\phi T_\psi = T_\psi T_\phi \). Then \( \gamma_{ij} = \gamma'_{ij} \) and \( \gamma_{i+1,j+1} = \gamma_{i+1,j+1} \) together imply that

\[
\frac{\beta_{i+1}}{\beta_{j+1}} a_{i+1} b_{j-1} = \frac{\beta_{i+1}}{\beta_{j+1}} b_{i+1} a_{j-1} \quad \text{for all } i, j \geq 0.
\]

That is

\[a_{i+1} b_{j-1} = b_{i+1} a_{j-1} .\]  (2.9)
Now, \( b_{-j-1} = a_{-j-1} = 0 \) for \( j \geq 0 \) if and only if both \( \phi \) and \( \psi \) are analytic. Also, \( a_{i+1} = b_{i+1} = 0 \) for all \( i \geq 0 \) if and only if both \( \phi \) and \( \psi \) are co-analytic. If \( \phi = 0 \) then the case is trivial. In the remaining cases, from Equation (2.9) we get

\[
\frac{b_{i+1}}{a_{i+1}} = \frac{b_{-j-1}}{a_{-j-1}} = \lambda \quad \text{(say)}
\]

Then

\[
b_{i+1} = \lambda(a_{i+1}) \quad \text{for } i \geq 0,
\]

and

\[
b_{-j-1} = \lambda(a_{-j-1}) \quad \text{for } j \geq 0.
\]

Therefore, \( b_k = \lambda a_k \) whenever \( k \neq 0 \). Hence \( \psi = b_0 + \lambda(\phi - a_0) \) or equivalently, \( T_\psi - b_0 = \lambda(T_\phi - a_0) \). Hence the theorem. \( \square \)

**Theorem 2.3.7 ([13]).**

\[
\|T_\phi\| \leq \|\phi\|_\infty.
\]

**Proof.** Let \( f \in H^2(\beta) \). Then

\[
\|T_\phi f\| = \|P(\phi \cdot f)\|
\]

\[
\leq \|P\| \|\phi f\|_\beta
\]

\[
\leq \|\phi\|_\infty \|f\|_\beta
\]

Since this is true for all \( f \) in \( H^2(\beta) \), we have

\[
\|T_\phi\| \leq \|\phi\|_\infty. \quad \square
\]
Theorem 2.3.8 ([13]). $T_\phi$ is Hermitian if and only if $a_0 = a$, $a$ is real and $\beta_k^2 a_k = \beta_0^2 \bar{a}_{-k}$ for all $k \neq 0$.

**Proof.** Let $T_\phi$ be Hermitian. Hence $T_\phi^*$ is also a weighted Toeplitz operator. So from Theorem 2.3.1(i) $(w_n)$ is constant. This means that $(\beta_n)$ is a geometric progression. Further $T_\phi = T_\phi^*$.

Now, from Equations (2.6),

\[
a_k = \langle T_\phi e_0, e_k \rangle \frac{\beta_0}{\beta_k} = \langle e_0, T_\phi e_k \rangle \frac{\beta_0}{\beta_k} = \langle e_0, \sum_{n=0}^{\infty} \frac{\beta_n}{\beta_k} a_{n-k} e_n \rangle \frac{\beta_0}{\beta_k} = \langle e_0, \frac{\beta_0}{\beta_k} a_{-k} e_0 \rangle \frac{\beta_0}{\beta_k}
\]

Therefore

\[
\beta_k^2 a_k = \beta_0^2 \bar{a}_{-k}
\]

Also,

\[
a_0 = \langle T_\phi e_0, e_0 \rangle = \langle e_0, T_\phi e_0 \rangle = \bar{a}_0
\]

Hence $a_0 = a$, some real number.

Conversely, let

\[
a_0 = a, \text{ some real number, and } \left\{ \beta_k^2 a_k = \beta_0^2 \bar{a}_{-k} \text{ for all } k \neq 0 \right\}
\]

(2.10)
We know that the matrix of $T^*_\phi$ is
\[
\begin{bmatrix}
\frac{\bar{a}_0}{\beta_0} & \frac{\beta_1}{\beta_0} & \frac{\bar{a}_2}{\beta_0} & \cdots \\
\frac{\bar{a}_0}{\beta_1} & \frac{\beta_1}{\beta_1} & \frac{\bar{a}_2}{\beta_1} & \cdots \\
\frac{\bar{a}_0}{\beta_2} & \frac{\beta_2}{\beta_2} & \frac{\bar{a}_2}{\beta_2} & \cdots \\
\cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]

Applying Equation (2.10) the above matrix becomes
\[
\begin{bmatrix}
a_0 & \frac{\beta_0}{\beta_0} & a_{-1} & \frac{\beta_0}{\beta_2} & \cdots \\
a_1 & \frac{\beta_1}{\beta_0} & a_{-1} & \frac{\beta_1}{\beta_2} & \cdots \\
a_2 & \frac{\beta_2}{\beta_0} & a_{-1} & \frac{\beta_2}{\beta_2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]

which is the same as that of $T_\phi$, as shown above. Thus $T_\phi = T^*_\phi$. Hence $T_\phi$ is Hermitian.

2.4 Eigenvalues of Some Weighted Toeplitz Operators

To conclude the study we pick up a few inducing functions namely $\phi(z) = \alpha z$ ($\alpha \neq 0$), $\phi(z) = z^k$ and $\phi(z) = az + \frac{b}{z}$ and try to find out the eigenvalues of the corresponding operator $T_\phi$.

**Theorem 2.4.1** ([16]). If $\phi = \alpha z$, $\alpha \neq 0$, then a complex number $\lambda$ is an eigenvalue of $T_\phi$ if and only if it satisfies the relation $a_n = \left(\frac{\alpha}{\lambda}\right)^n a_0$ for all $n$, where $f = \sum_{n=0}^{\infty} a_n z^n$ is the corresponding eigenvector.
Proof. Let $\lambda$ be an eigenvalue of $T_\phi$. Then, for some $0 \neq f \in H^2(\beta)$, we must have

\[ T_\phi f = \lambda f \]

\[ \Rightarrow \alpha z f = \lambda f \]

\[ \Rightarrow \alpha \sum_{n=0}^{\infty} a_n z^{n+1} = \lambda \sum_{n=0}^{\infty} a_n z^n \]

\[ \Rightarrow \alpha \sum_{n=0}^{\infty} a_{n-1} z^n = \lambda \sum_{n=0}^{\infty} a_n z^n \]

\[ \Rightarrow \alpha a_{n-1} = \lambda a_n \text{ for all } n. \quad (2.11) \]

Taking $n = 1, 2, \ldots$ we get

\[ a_1 = \frac{\alpha}{\lambda} a_0, \quad a_2 = \frac{\alpha}{\lambda} a_1 = \left(\frac{\alpha}{\lambda}\right)^2 a_0 \quad \text{and so on.} \]

In general, $a_n = \left(\frac{\alpha}{\lambda}\right)^n a_0$. \hfill \Box

Observation 2.4.2 (16). From Equation (2.11) above, we get that $\lambda = \frac{a_n}{a_{n-1}} \alpha$. Hence the eigenspace of $T_\phi$ consists of all functions $f$ such that $\sum a_n$ is a geometric series.

Observation 2.4.3 (16). For the weighted Toeplitz operator $T_\phi$ induced by $\phi = ax$, ($\alpha \neq 0$), zero cannot be an eigenvalue.

Theorem 2.4.4 (16). Zero cannot be an eigenvalue of a weighted Toeplitz operator induced by $\phi(z) = z^k$. 

Proof. Suppose $\lambda$ is an eigenvalue of $T_\phi$. Then there exists $0 \neq f$ such that $T_\phi f = \lambda f$.

$$
\Rightarrow \quad z^k f = \lambda f
$$

$$
\Rightarrow \quad \sum a_{n-k} \beta_n e_n = \lambda \sum a_n \beta_n e_n
$$

$$
\Rightarrow \quad \lambda = \frac{a_{n-k}}{a_n} \forall n \quad (2.12)
$$

So $\lambda = 0$ gives that $a_n = 0 \forall n$. Hence $f = 0$ which is contradiction. \qed

In [52] Lauric has discussed in detail, the weighted Toeplitz operator induced by the function $\phi(z) = az + \frac{b}{z}$. We now investigate the nature of the eigenvalues of this operator.

**Theorem 2.4.5 ([16]).** If $\phi(z) = az + \frac{b}{z}$, then $\lambda$ is an eigenvalue of $T_\phi$ if it satisfies $aa_{n-1} + ba_{n+1} = \lambda a_n \forall n$.

**Proof.** For a given eigenvalue $\lambda \in \mathbb{C}$, we must have $0 \neq f$ satisfying $T_\phi f = \lambda f$. Therefore

$$
\left( az + \frac{b}{z} \right) \sum a_n z^n = \lambda \sum a_n z^n
$$

$$
\Rightarrow \quad \left( az + \frac{b}{z} \right) \sum a_n \beta_n e_n = \lambda \sum a_n \beta_n e_n
$$

$$
\Rightarrow \quad \sum a_{n-1} \beta_n e_n + \sum b a_{n+1} \beta_n e_n = \lambda \sum a_n \beta_n e_n.
$$

This gives us the relation

$$
aa_{n-1} + ba_{n+1} = \lambda a_n \forall n.
$$

\qed
Observation 2.4.6 ([16]). If $a = b = 1$ then $\phi(z) = z + \frac{1}{z}$ and from Theorem 2.4.5 we get $\lambda = \frac{a_0 + a_2}{a_1}$ and so on. Further, if we choose $\lambda = 2$, then the corresponding eigenvectors constitute the set of all functions $f = \sum a_n z^n$ such that $\langle a_n \rangle$ is an arithmetic progression.