ABSTRACT
OF
THE THESIS ENTITLED

A STUDY OF SLANT WEIGHTED
TOEPLITZ OPERATORS

TO BE SUBMITTED
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UNDER
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ABSTRACT

Introduction
Toeplitz operators have diverse applications, particularly in prediction theory, wavelet analysis and solution of differential equations. The discovery of these operators is attributed to O. Toeplitz [Math. Ann. 70(1911), 351–376], who studied matrices which were constant on the diagonals and their relationship with the corresponding one-sided and two-sided infinite matrices. In 1996, M.C. Ho [Indiana University Mathematics Journal, 45(3) (1996), 843–862.] introduced and studied the class of slant Toeplitz operators having the property that the matrices with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrices of the corresponding Laurent operators. However, the underlying spaces in all these studies were the usual Hardy spaces $H^2$ and the $L^2$ spaces. Meanwhile, around the year 1966, the notions of the weighted sequence spaces $H^2(\beta)$ and $L^2(\beta)$ were brought forth by R.L. Kelley [Weighted Shift on Hilbert space, Dissertation Ann. Arbor Mich., 1966] and R. Gellar [Shift Operators in Banach Space, Dissertation, Columbia Univ. 1968]


The aim of the present thesis, entitled “A STUDY OF SLANT WEIGHTED TOEPLITZ OPERATORS”, is to introduce the notion of slant weighted Toeplitz operators, generalized slant weighted Toeplitz operators and study their properties and the properties of their compressions to $H^2(\beta)$.

Preliminaries

Let $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$, $0 < \frac{\beta_n}{\beta_{n+1}} \leq 1$ when $n \geq 0$ and $0 < \frac{\beta_n}{\beta_{n-1}} \leq 1$ when $n \leq 0$. Consider the spaces

$$L^2(\beta) = \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n | a_n \in \mathbb{C} \text{ and } \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

and

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n | a_n \in \mathbb{C} \text{ and } \|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

Let $P : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. The operator $M_z : L^2(\beta) \to L^2(\beta)$ defined as $M_z e_k(z) = w_k e_{k+1}(z)$, where $\left\{ e_k(z) = \frac{z^k}{\beta_k} \right\}_{k \in \mathbb{Z}}$ is the orthonormal basis, is known as a weighted shift. The sequence $\left\{ w_k = \frac{\beta_{k+1}}{\beta_k} \right\}_{k \in \mathbb{Z}}$ is usually called the weight sequence. For a given $\phi$ in

$$L^\infty(\beta) = \left\{ \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n | \phi L^2(\beta) \subseteq L^2(\beta) \text{ and } \exists c \in \mathbb{R} \text{ such that } \|\phi f\|_\beta < c \|f\|_\beta \forall f \in L^2(\beta) \right\},$$

the induced weighted multiplication operator is denoted by $M_\phi$ and
is given by

\[ M_\phi : L^2(\beta) \to L^2(\beta) \text{ such that } M_\phi f = \phi f \ \forall \ f \in L^2(\beta). \]

The weighted Toeplitz operator \( T_\phi \) is the compression of \( M_\phi \) to \( H^2(\beta) \). Thus, \( T_\phi(f) = P(\phi f) \ \forall \ f \in H^2(\beta) \). The slant weighted Toeplitz operator \( A_\phi \) is defined as \( A_\phi = WM_\phi \) where \( W \) is an operator on \( L^2(\beta) \) given by \( We_{2n}(z) = \frac{\beta_n e_n(z)}{\beta_{2n}} \) and \( We_{2n-1}(z) = 0, \ n \in \mathbb{Z} \).

**Summary of the Thesis**

The thesis is divided into six chapters.

Chapter 1 is entitled “**INTRODUCTION**” and it comprises of three sections. Section 1.1 gives the historical background and motivation of the topic of research. Section 1.2 introduces the notations and the terminologies and Section 1.3 presents the chapter-wise summary of the results contained in the thesis.

Chapter 2 is entitled “**WEIGHTED TOEPLITZ OPERATORS**” and it is divided into four sections. In Section 2.1 we discuss the multiplication operator \( M_\phi \) on \( L^2(\beta) \). We also introduce the notion of a weighted Laurent matrix and obtain a characterization of \( M_\phi \) in terms of this matrix. In Section 2.2, we define a Toeplitz operator \( T_\phi \) on \( H^2(\beta) \) and a weighted Toeplitz matrix and characterize \( T_\phi \) in terms of this matrix. A necessary and sufficient condition for an operator on \( H^2(\beta) \) to be a weighted Toeplitz operator is that it commutes with the weighted unilateral shift. The sum and the product of two weighted Toeplitz operators are discussed in
Section 2.3. The only idempotent weighted Toeplitz operators are 0 and $I$. We obtain a condition for $T_\phi$ to be Hermitian. In Section 2.4, we discuss the eigenvalues of some weighted Toeplitz operators.

The third chapter, entitled “SLANT WEIGHTED TOEPLITZ OPERATORS” is divided into four sections. In Section 3.1, we introduce the slant weighted Toeplitz operator $A_\phi = WM_\phi$. We prove that a bounded operator $A$ on the space $L^2(\beta)$ is a slant weighted Toeplitz operator if and only if $M_\phi A = AM_\phi$. In Section 3.2 we prove some basic algebraic properties of $A_\phi$. A slant weighted Toeplitz matrix is introduced in Section 3.3 and is followed by a characterization of $A_\phi$ and several other properties of $A_\phi$. In Section 3.4, we prove that the set of slant weighted Toeplitz operators is closed in the strong operator topology. We also discuss the adjoint of $A_\phi$ and prove that there is no non zero self-adjoint slant weighted Toeplitz operator.

The fourth chapter bears the title “GENERALIZED SLANT WEIGHTED TOEPLITZ OPERATORS” and aims at studying the $k$-th order slant weighted Toeplitz operator. The whole study in the chapter has been classified into three sections. In Section 4.1, we introduce the operator $W_k$ and the operator $U_\phi = W_k M_\phi$. We evaluate the matrix of $U_\phi$ with respect to the orthonormal basis and also observe that the projection $P$ commutes with the operator $W_k$. In Section 4.2, it is proved that if $h(z)$ is an $L^2(\beta)$ function then $h(z^k)$ is also an $L^2(\beta)$ function which leads to establishing $M_z U_\phi = U_\phi M_z$. The $k$-th order slant weighted Toeplitz matrix
is defined in Section 4.3 and is followed by a couple of theorems characterizing $U_{\phi}$.

The fifth chapter with title “COMPRESSIONS OF SLANT WEIGHTED TOEPLITZ OPERATORS” deals with the compressions of $A_{\phi}$ and $U_{\phi}$ to $H^2(\beta)$ and contains two sections. In Section 5.1, we obtain the matrix of the operator $B_{\phi} = A_{\phi}|_{H^2(\beta)}$, the compression of the operator $A_{\phi}$ to the space $H^2(\beta)$ and prove some more basic properties of $B_{\phi}$. We also show that for a bounded weight sequence, $B$ is the compression of a slant weighted Toeplitz operator if and only if $B = S^*BTz^2$ with $S$ defined as $Se_j = \frac{1}{w_j}e_{j+1}$. Further, $B_{\phi}$ is compact if and only if $\phi = 0$. We also show that the set of all the compressions of slant weighted Toeplitz operators is weakly closed. Section 5.2 contains the discussion of the compression of a $k$-th order slant weighted Toeplitz operator $U_{\phi}$ to the space $H^2(\beta)$. We denote this operator by $V_{\phi}$, obtain its matrix and prove some of its properties.

The sixth chapter entitled “FURTHER SCOPE OF STUDY” brings forth a few problems with the objective of proposing further study of the slant weighted Toeplitz operators. Here, we talk about the spectral theory of weighted Toeplitz operators and slant weighted Toeplitz operators. We propose the notions of weighted Hankel operators, slant weighted Hankel operators and essentially slant weighted Toeplitz operators which could be defined and discussed probably in the same manner.
The work contained in the thesis is based upon the following research papers written by the author (jointly with Prof. S.C. Arora):


[7] On $k$-th order slant weighted Toeplitz operator (*communicated*).