Chapter 2

Strongly and Perfectly Continuous Multifunctions

2.1 Introduction

In this chapter we extend the notion of strong continuity of Levine[53] and perfect continuity due to Noiri[63] to the framework of multifunctions. We study their basic properties and elaborate on their place in the hierarchy of strong variants of continuity of multifunctions that already exist in the mathematical literature. In the process we extend several known results in the literature including those of singh[71], Ekici[12], and others to the realm of multifunctions.

The chapter is organised as follows. In Section 2.1 we introduce the notions of strongly continuous multifunctions, upper and lower (almost) perfectly continuous multifunctions and discuss the interrelations that exist among them and other strong variants of continuity of multifunctions that already exist in the literature. Examples are included to reflect upon the distinctiveness of the notions so introduced from the ones that already exist in the mathematical lit-
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It turns out that the class of upper(lower)perfectly continuous multifunctions properly includes the class of strongly continuous multifunctions and the class of upper(lower) perfectly continuous multifunctions is strictly contained in the class of upper(lower) cl-supercontinuous multifunctions. Section 2.2 deals with characterizations and basic properties of strongly continuous multifunctions. Section 2.3 is devoted to characterizations and properties of upper perfectly continuous multifunctions. Lower perfectly continuous multifunctions are studied in Section 2.4. Section 2.5 deals with upper almost perfectly continuous multifunctions and Section 2.6 is devoted to the study of lower almost perfectly continuous multifunctions.

**Definition 2.1.1.** (Akdağ [6])\(^1\): A multifunction \(\varphi: X \rightarrow Y\) from a topological space \(X\) into a topological space \(Y\) is said to be

(a) **upper strongly continuous**\([6]\) if \(\varphi^{-1}(V)\) is clopen in \(X\) for every subset \(V\) of \(Y\);

(b) **lower strongly continuous**\([6]\) if \(\varphi^{-1}(V)\) is clopen in \(X\) for every subset \(V\) of \(Y\); and

(c) **strongly continuous**\([6]\) if it is upper strongly continuous and lower strongly continuous.

**Definition 2.1.2.** A multifunction \(\varphi: X \rightarrow Y\) from a topological space \(X\) into a topological space \(Y\) is said to be

(i) **strongly continuous**\([27]\) if \(\varphi^{-1}(U)\) is clopen in \(X\), equivalently \(\varphi^{-1}(U)\) is clopen in \(X\) for every subset \(U\) of \(Y\);

(ii) **upper perfectly continuous**\([27]\) if \(\varphi^{-1}(U)\) is clopen in \(X\) for every open subset \(U\) of \(Y\);

(iii) **lower perfectly continuous**\([27]\) if \(\varphi^{-1}(U)\) is clopen in \(X\) for every open subset \(U\) of \(Y\);

\(^{1}\)It remained unnoticed in Akdağ[6] that all the three notions of upper strong continuity, lower strong continuity and strong continuity are identical notions for multifunctions. However this fact is inherent in the Definition 2.1.2 of strong continuity of multifunctions (Kohli and Arya [27]) and dealt with in this chapter.
(iv) **upper almost perfectly continuous**\[27\] if $\varphi^{-1}(U)$ is clopen in $X$ for every regular open subset $U$ of $Y$; and

(v) **lower almost perfectly continuous**\[27\] if $\varphi^{-1}(U)$ is clopen in $X$ for every regular open subset $U$ of $Y$.

The following diagram well illustrates the interrelations that exist among various strong variants of continuity of multifunctions defined in Definitions 1.2.4 and 2.1.2.

![Diagram](image)

However, none of the above implications is reversible as is well illustrated by the examples in the sequel.

**Example 2.1.3.** Let $X = \{a, b, c\}$ be endowed with the topology $\mathcal{T}_X = \{\emptyset, X, \{a\}, \{b, c\}, \{c\}, \{a, c\}\}$ and let $Y = \{x, y\}$ be equipped with the topology $\mathcal{T}_Y = \{\emptyset, Y, \{x\}\}$. Define a multifunction $\varphi : X \rightarrow Y$ by $\varphi(a) = \{x\}, \varphi(b) = \{y\}, \varphi(c) = \{x, y\}$. Clearly $\varphi$ is upper perfectly continuous. But $\varphi^{-1}(\{y\}) = \{b\}$ is not clopen, and so the multifunction $\varphi$ is not strongly continuous.
Example 2.1.4. Let $X = \{a, b, c\}$ be equipped with the topology $\mathcal{S}_X = \{\emptyset, X, \{b\}, \{b, c\}, \{c\}, \{a, c\}\}$ and let $Y = \{x, y\}$ be endowed with the topology $\mathcal{S}_Y = \{\emptyset, Y, \{x\}\}$. Define a multifunction $\varphi : X \to Y$ by $\varphi(a) = \{x\}$, $\varphi(b) = \{y\}$, $\varphi(c) = \{x, y\}$. Clearly $\varphi$ is lower perfectly continuous, $\varphi^{-1}(\{y\}) = \{b, c\}$ is not clopen, and so the multifunction $\varphi$ is not strongly continuous.

Example 2.1.5. Let $X = Y = R$, the set of real numbers and let $\mathcal{S}$ be the upper limit topology [19] on $X$ and let $U$ be the usual topology on $Y$. Define a multifunction $\varphi : X \to Y$ by $\varphi(x) = \{x\}$ for each $x \in X$. Then $\varphi$ is upper (lower) cl-supercontinuous but $\varphi$ is not upper (lower) perfectly continuous. Moreover, $\varphi$ is not upper (lower) almost perfectly continuous.

Example 2.1.6. Let $X = \{a, b, c\}$ be endowed with the topology $\mathcal{S}_X = \{\emptyset, X, \{a\}\}$ and let $Y = \{x, y, z\}$ be given the topology $\mathcal{S}_Y = \{\emptyset, Y, \{x\}\}$. Define $\varphi : X \to Y$ by $\varphi(a) = \{x, y\}$, $\varphi(b) = \{x\}$, $\varphi(c) = \{y, z\}$. Clearly $\varphi$ is upper almost cl-supercontinuous but $\varphi$ is not upper cl-supercontinuous since $\{x\}$ is an open set in $Y$ but $\varphi^{-1}(\{x\}) = \{b\}$ is not cl-open in $X$. Moreover, $\varphi$ is not lower cl-supercontinuous. Further observe that $\varphi$ is upper (lower) almost perfectly continuous but not upper (lower) perfectly continuous.

Example 2.1.7. Let $X$ be a completely regular space which is not zero dimensional. Then every upper (lower) semicontinuous multifunction $\varphi : X \to Y$ is upper (lower) $z$-supercontinuous but not necessarily upper (lower) cl-supercontinuous.

Example 2.1.8. Let $X = \{a, b, c, d\}$ be equipped with the topology $\mathcal{S}_X = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and let $Y = \{p, q, r, s\}$ be endowed with topology $\mathcal{S}_Y = \{\emptyset, Y, \{p\}, \{r\}, \{p, q\}, \{p, r\}, \{p, q, r\}, \{p, r, s\}\}$. Define $\varphi : X \to Y$ by $\varphi(a) = \{q\}$, $\varphi(b) = \{p, q\}$, $\varphi(c) = \{p\}$, $\varphi(d) = \{p\}$. The sets $\{r\}$ and $\{p, q\}$ are regular open sets. Clearly $\varphi$ is upper (lower) almost $z$-supercontinuous but not upper (lower) $z$-supercontinuous.
2.2 Strongly continuous multifunctions

**Theorem 2.2.1.** If \( \varphi : X \to Y \) is a strongly continuous multifunction and \( \psi : Y \to Z \) is any multifunction, then \( \psi \circ \varphi \) is strongly continuous.

**Proof.** Let \( B \) be a subset of \( Z \). Then \( \psi^{-1}(B) \) is a subset of \( Y \). Since \( \varphi \) is strongly continuous, \( \varphi^{-1}(\psi^{-1}(B)) = (\psi \circ \varphi)^{-1}(B) \) is clopen in \( X \), and so \( \psi \circ \varphi \) is strongly continuous \( \square \)

**Theorem 2.2.2.** Let \( \varphi : X \to Y \) be strongly continuous and let \( A \subset X \). Then \( \varphi|_A : A \to Y \) is strongly continuous.

**Proof.** Let \( B \) be a subset of \( Y \). Since \( \varphi \) is strongly continuous, \( \varphi^{-1}(B) \) is clopen in \( X \). Again since \( (\varphi|_A)^{-1}(B) = A \cap \varphi^{-1}(B) \) is clopen in \( A \), the multifunction \( \varphi|_A : A \to Y \) is strongly continuous \( \square \)

**Theorem 2.2.3.** Let \( \varphi : X \to Y \) and \( \psi : X \to Y \) be two strongly continuous multifunctions. Then \( \varphi \cup \psi : X \to Y \) defined by \( (\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x) \) for each \( x \in X \), is strongly continuous.

**Proof.** Let \( B \) be a subset of \( Y \). Since \( \varphi \) and \( \psi \) are strongly continuous, \( \varphi^{-1}(B) \) and \( \psi^{-1}(B) \) are clopen sets in \( X \). Since \( (\varphi \cup \psi)^{-1}(B) = \varphi^{-1}(B) \cap \psi^{-1}(B) \) and since finite intersection of clopen sets is clopen, \( (\varphi \cup \psi)^{-1}(B) \) is clopen in \( X \). Thus \( \varphi \cup \psi \) is strongly continuous. \( \square \)

2.3 Upper perfectly continuous multifunctions

**Theorem 2.3.1.** A multifunction \( \varphi : X \to Y \) is upper perfectly continuous if and only if for every closed set \( A \subset Y \) the set \( \varphi^{-1}_+(A) \) is a clopen subset of \( X \).
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Proof. Suppose \( \varphi : X \to Y \) is upper perfectly continuous. Let \( A \) be a closed subset of \( Y \). Then \( Y - A \) is an open set in \( Y \). Since \( \varphi \) is upper perfectly continuous, \( \varphi^{-1}(Y - A) = X - \varphi^{-1}(A) \) is clopen in \( X \) and so \( \varphi^{-1}(A) \) is clopen in \( X \).

Conversely suppose that \( \varphi^{-1}(A) \) is clopen in \( X \) for every closed \( A \subset Y \). Let \( U \) be an open subset of \( Y \). Then \( Y - U \) is a closed subset of \( Y \). By hypothesis, \( \varphi^{-1}(Y - U) = X - \varphi^{-1}(U) \) is clopen in \( X \) and so \( \varphi^{-1}(U) \) is clopen in \( X \). Thus \( \varphi \) is upper perfectly continuous. \( \square \)

Theorem 2.3.2. Let \( \varphi : X \to Y \) and \( \psi : Y \to Z \) be multifunctions. The following statements are true.

(a) If \( \varphi \) is upper perfectly continuous and \( \psi \) is upper semicontinuous, then the multifunction \( \psi \circ \varphi \) is upper perfectly continuous.

(b) If \( \varphi \) is upper almost perfectly continuous and \( \psi \) is upper completely continuous, then \( \psi \circ \varphi \) is upper perfectly continuous.

Proof. To prove (a) let \( B \) be an open subset of \( Z \). Since \( \psi \) is upper semicontinuous, \( \psi^{-1}(B) \) is open in \( Y \). Again since \( \varphi \) is upper perfectly continuous, \( \varphi^{-1}(\psi^{-1}(B)) = (\psi \circ \varphi)^{-1}(B) \) is clopen in \( X \) and so \( \psi \circ \varphi \) is upper perfectly continuous.

(b) Let \( B \) be an open subset of \( Z \). Since \( \psi \) is upper completely continuous, \( \psi^{-1}(B) \) is regular open in \( Y \). Again since \( \varphi \) is upper almost perfectly continuous, \( \varphi^{-1}(\psi^{-1}(B)) = (\psi \circ \varphi)^{-1}(B) \) is clopen in \( X \) and so \( \psi \circ \varphi \) is upper perfectly continuous. \( \square \)

Corollary 2.3.3. If \( \varphi : X \to Y \) is upper perfectly continuous and if \( Z \) is a superspace of \( Y \), then \( \psi : X \to Z \) defined by \( \psi(x) = \varphi(x) \) for each \( x \in X \) is upper perfectly continuous.

Proof. This is immediate in view of Theorem 2.3.1 and the fact that \( \psi = \iota \circ \varphi \), where \( \iota \) denotes the inclusion mapping is upper semicontinuous. \( \square \)
Theorem 2.3.4. If \( \varphi : X \rightarrow Y \) is upper perfectly continuous and \( \varphi(X) \) is endowed with the subspace topology, then the multifunction \( \varphi : X \rightarrow \varphi(X) \) is upper perfectly continuous.

Proof. This is immediate in view of the fact that for every open set \( V \subset Y \), \( \varphi^{-1}(V) = \varphi^{-1}(V \cap \varphi(X)) \).

Theorem 2.3.5. Let \( \varphi : X \rightarrow Y \) be an upper perfectly continuous multifunction and let \( A \subset X \). Then the multifunction \( \varphi_A = \varphi|_A : A \rightarrow Y \) is upper perfectly continuous.

Proof. Let \( U \) be an open set in \( Y \). Since \( \varphi \) is upper perfectly continuous, \( \varphi^{-1}(U) \) is clopen in \( X \). Since \( (\varphi_A)^{-1}(U) = A \cap \varphi^{-1}(U) \), which is clopen in \( A \) and so \( \varphi_A \) is upper perfectly continuous.

Theorem 2.3.6. Let \( \Omega = \{X_\alpha : \alpha \in \Lambda\} \) be a locally finite clopen cover of \( X \) and let \( \varphi : X \rightarrow Y \) be a multifunction. For each \( \alpha \in \Lambda \), let \( \varphi_\alpha = \varphi|_{X_\alpha} : X_\alpha \rightarrow Y \) be the restriction of \( \varphi \) to \( X_\alpha \).

Then \( \varphi \) is upper perfectly continuous if and only if each \( \varphi_\alpha \) is upper perfectly continuous.

Proof. Necessity is easy to see. To prove sufficiency, let \( V \) be an open set in \( Y \). Then \( \varphi^{-1}(V) = \bigcup_{\alpha \in \Lambda}(\varphi_\alpha)^{-1}(V) = \bigcup_{\alpha \in \Lambda}(\varphi^{-1}(V) \cap X_\alpha) \). Each \( \varphi^{-1}(V) \cap X_\alpha \) is clopen in \( X_\alpha \) and hence in \( X \). Thus \( \varphi^{-1}(V) \) is open being the union of clopen sets. In view of local finiteness of \( \Omega \), the collection \( \{\varphi^{-1}(V) \cap X_\alpha : \alpha \in \Lambda\} \) is a locally finite collection of clopen sets. Therefore \( \varphi^{-1}(V) \) is also closed being the union of a locally finite collection of clopen sets and hence clopen. Consequently, \( \varphi \) is upper perfectly continuous.

Theorem 2.3.7. Let \( \varphi : X \rightarrow Y \) and \( \psi : X \rightarrow Y \) be two upper perfectly continuous multifunctions. Then \( \varphi \cup \psi : X \rightarrow Y \) defined by \( (\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x) \) for each \( x \in X \), is upper perfectly continuous.
Proof. Let \( U \) be an open set in \( Y \). Since \( \phi \) and \( \psi \) are upper perfectly continuous, \( \phi^{-1}(U) \) and \( \psi^{-1}(U) \) are clopen sets in \( X \). Again since \( (\phi \cup \psi)^{-1}(U) = \phi^{-1}(U) \cap \psi^{-1}(U) \) and since the finite intersection of clopen sets is clopen, \( (\phi \cup \psi)^{-1}(U) \) is clopen in \( X \). Thus \( \phi \cup \psi \) is upper perfectly continuous. \( \square \)

**Theorem 2.3.8.** Let \( \phi : X \rightarrow Y \) be upper perfectly continuous. Then \( [\phi^{-1}(B)]_{cl} \subset \phi^{-1}(\overline{B}) \) for every subset \( B \) of \( Y \).

**Proof.** Let \( B \) be any subset of \( Y \). Then \( \overline{B} \) is a closed subset of \( Y \). Since \( \phi \) is upper perfectly continuous, \( \phi^{-1}(\overline{B}) \) is clopen in \( X \). Again since \( \phi^{-1}(B) \subset \phi^{-1}(\overline{B}) \), \( [\phi^{-1}(B)]_{cl} \subset [\phi^{-1}(\overline{B})]_{cl} = \phi^{-1}(\overline{B}) \). \( \square \)

**Remark 2.3.9.** Converse of Theorem 2.3.8 is not true. For example, let \( X = Y = \mathbb{R} \), the set of real numbers and let \( \mathcal{Z} \) be the upper limit topology on \( X \) and let \( U \) be the usual topology on \( Y \). Define \( \phi : X \rightarrow Y \) by \( \phi(x) = \{ x \} \) for each \( x \in X \). Then it is easily verified that for every subset \( B \) of \( Y \), \( \phi^{-1}(B) \subset \phi^{-1}(\overline{B}) \) but \( \phi \) is not upper perfectly continuous.

**Theorem 2.3.10.** Let \( \phi : X \rightarrow Y \) be a multifunction and let \( g : X \rightarrow X \times Y \) defined by \( g(x) = \{ x \} \times \phi(x) \) for each \( x \in X \) be the graph multifunction. If \( g \) is upper perfectly continuous, then \( \phi \) is upper perfectly continuous and the space \( X \) is equipped with a partition topology.

**Proof.** Suppose that \( g \) is upper perfectly continuous. By Theorem 2.3.2(a) the multifunction \( \phi = p_y \circ g \) is upper perfectly continuous, where \( p_y : X \times Y \rightarrow Y \) denotes the projection mapping which is upper semicontinuous as well as upper almost completely continuous. Now to show that \( X \) is endowed with a partition topology, let \( U \) be an open set in \( X \). Then \( U \times Y \) is an open set in \( X \times Y \). Since \( g \) is upper perfectly continuous, \( g^{-1}(U \times Y) \) is clopen in \( X \). It is easily verified that \( g^{-1}(U \times Y) = U \), and so \( U \) is clopen in \( X \). Thus \( X \) is endowed with a partition topology. \( \square \)
Theorem 2.3.11. If \( \varphi : X \to Y \) is an upper perfectly continuous multifunction where \( Y \) is a regular space and \( \varphi(x) \) is closed for each \( x \in X \), then the graph \( \Gamma_\varphi \) of \( \varphi \) is a cl-closed subset of \( X \times Y \) with respect to \( X \).

Proof. Let \((x, y) \notin \Gamma_\varphi \). Then \( y \notin \varphi(x) \). Since \( Y \) is a regular space, there exist disjoint open sets \( V_y \) and \( V_{\varphi(x)} \) containing \( y \) and \( \varphi(x) \), respectively. Since \( \varphi \) is upper perfectly continuous, \( U_x = \varphi^{-1}(V_{\varphi(x)}) \) is a clopen set containing \( x \). We assert that \((U_x \times V_y) \cap \Gamma_\varphi = \emptyset \). For, if \((h, k) \in (U_x \times V_y) \cap \Gamma_\varphi \), then \( h \in \varphi^{-1}(V_x) \), \( k \in V_y \) and \( k \in \varphi(h) \). Hence \( \varphi(h) \subset V_{\varphi(x)} \) and \( k \in \varphi(h) \cap V_y \) which contradicts the fact that \( V_y \) and \( V_{\varphi(x)} \) are disjoint. Thus the graph \( \Gamma_\varphi \) is cl-closed in \( X \times Y \) with respect to \( X \).

\( \square \)

Theorem 2.3.12. Let \( \varphi : X \to Y \) be an upper perfectly continuous multifunction such that \( \varphi(x) \) is compact for each \( x \in X \). If \( A \) is a mildly compact set in \( X \), then \( \varphi(A) \) is compact.

Proof. Let \( \Omega \) be an open cover of \( \varphi(A) \). Then \( \Omega \) is also an open cover of \( \varphi(a) \) for each \( a \in A \). Since each \( \varphi(a) \) is compact, there exists a finite subset \( \beta_a \subset \Omega \) such that \( \varphi(a) \subset \bigcup_{B \in \beta_a} B = V_a \) (say). Again since \( \varphi \) is upper perfectly continuous, \( U_a = \varphi^{-1}(V_a) \) a clopen set containing \( a \). Let \( Q = \{U_a \mid a \in A\} \). Then \( Q \) is a clopen covering of \( A \). Since \( A \) is mildly compact, there exists a finite subset \( \{a_1, \ldots, a_n\} \) of \( A \) such that \( A \subset \bigcup_{i=1}^n U_{a_i} \subset \bigcup_{i=1}^n \varphi^{-1}(V_{a_i}) \). Therefore \( \varphi(A) \subset \varphi(\bigcup_{i=1}^n \varphi^{-1}(V_{a_i})) = \bigcup_{i=1}^n \varphi(\varphi^{-1}(V_{a_i})) \subset \bigcup_{i=1}^n V_{a_i} \), where \( V_{a_i} = \bigcup_{B \in \beta_{a_i}} B \), \( i = 1, \ldots, n \) and each \( \beta_{a_i} \) is finite. Thus \( \varphi(A) \) is compact.

\( \square \)

We may recall that a multifunction \( \varphi : X \to Y \) is called nonmingled [79] if for \( x, y \in X \), \( x \neq y \) the image sets \( \varphi(x) \) and \( \varphi(y) \) are either disjoint or identical.

Theorem 2.3.13. Let \( \varphi : X \to Y \) be an open and upper perfectly continuous nonmingled multifunction such that \( \varphi(x) \) is paracompact for each \( x \in X \). If \( A \) is cl-paracompact, then \( \varphi(A) \) is paracompact.
Let \( V \) be a neighborhood of \( y \) of \( \Omega \). Then \( \Psi \) which covers \( \Psi \) is an open refinement of \( y \). Since \( \varphi \) is upper perfectly continuous, \( U_x = \varphi^{-1}(V_x) \) is a clopen set containing \( x \).

Now \( \{U_x \mid x \in A\} \) is a clopen cover of \( A \). Since \( A \) is cl-paracompact, it has a locally finite open refinement \( \Omega = \{W_\alpha \mid \alpha \in \Lambda\} \) such that \( A \subseteq \bigcup_{\alpha \in \Lambda} W_\alpha \). So for each \( \alpha \in \Lambda \) there exists \( x_\alpha \in A \) such that \( W_\alpha \subset U_{x_\alpha} \) and hence \( \varphi(W_\alpha) \subset \varphi(U_{x_\alpha}) \subset \bigcup \Psi_{x_\alpha} \). Let \( \mathcal{R}_\alpha = \{ \varphi(W_\alpha) \cap \mathcal{V} \mid \mathcal{V} \in \Psi_{x_\alpha} \} \), and let \( \mathcal{R} = \{ R \mid R \in \mathcal{R}_\alpha, \alpha \in \Lambda \} \). We shall show that \( \mathcal{R} \) is a locally finite open refinement of \( \Psi \) which covers \( \varphi(A) \). Since \( \varphi \) is open, each \( \varphi(W_\alpha) \) is open and so each \( R \in \mathcal{R} \) is open. Let \( R \in \mathcal{R} \). Then \( R \in \mathcal{R}_\alpha \) for some \( \alpha \in \Lambda \), i.e. \( R = \varphi(W_\alpha) \cap \mathcal{V} \subset \mathcal{V} \subset U \) for some \( U \in \Psi \). This shows that \( \mathcal{R} \) is an open refinement of \( \Psi \). Now to show that \( \mathcal{R} \) is locally finite, let \( y \in \varphi(A) \).

Then \( y \in \varphi(x) \) for some \( x \in A \). Since \( \Omega \) is locally finite, for this \( x \in A \) we can choose an open neighborhood \( G_x \) of \( x \) which intersects only finitely many members \( W_{\alpha_1}, W_{\alpha_2}, \ldots, W_{\alpha_n} \) of \( \Omega \). Since \( \varphi \) is a nonmingled open multifunction, it follows that \( H_0 = \varphi(G_x) \) is an open neighborhood of \( y \) which intersects only finitely many members \( \varphi(W_{\alpha_1}), \varphi(W_{\alpha_2}), \ldots, \varphi(W_{\alpha_n}) \) of the family \( \{ \varphi(W_\alpha) \mid \alpha \in \Lambda \} \). Furthermore, each \( \mathcal{R}_\alpha \) \((k = 1, \ldots, n)\) is locally finite, hence there exists an open neighborhood \( H_k(k = 1, \ldots, n) \) of \( y \) which intersects only finitely many members of \( \mathcal{R}_\alpha \) \((k = 1, \ldots, n)\). Finally let \( H = \bigcap_{k=1}^n H_k \). Then \( H \) is an open neighborhood of \( y \) which intersects at most finitely many members of \( \mathcal{R} \). Hence \( \mathcal{R} \) is locally finite. Thus \( \varphi(A) \subset \varphi(\bigcup_{\alpha \in \Lambda} W_\alpha) \subset \bigcup_{\alpha \in \Lambda} \varphi(W_\alpha) \subset \bigcup_{\alpha \in \Lambda} (\bigcup \mathcal{R}_\alpha) = \bigcup \{ R : R \in \mathcal{R} \} \). So \( \mathcal{R} \) is a locally finite open refinement of \( \Psi \) which covers \( \varphi(A) \) and thus \( \varphi(A) \) is paracompact. \( \square \)

**Corollary 2.3.14.** Let \( \varphi : X \to Y \) be an open, upper perfectly continuous, nonmingled multifunction from a space \( X \) onto \( Y \) such that \( \varphi(x) \) is a paracompact set in \( Y \) for each \( x \in X \). If \( X \) is a cl-paracompact space, then \( Y \) is paracompact.
Theorem 2.3.15. Let \( \varphi : X \to Y \) be an open, and upper perfectly continuous, nonmingled multifunction from a space \( X \) into a \( P \)-space \( Y \) such that \( \varphi(x) \) is para-Lindelöf. If \( A \) is a cl-para-Lindelöf, then so is \( \varphi(A) \).

Proof. Similar to that of Theorem 2.3.13 and hence omitted.

Theorem 2.3.16. Let \( \varphi : X \to Y \) be an upper perfectly continuous multifunction such that \( \varphi(x) \cap \varphi(y) = \emptyset \) for each \( x \neq y \) in \( X \) and \( \varphi(x) \) is closed for each \( x \in X \). If \( Y \) is a normal space, then \( X \) is an ultra Hausdorff space.

Proof. Let \( x, y \in X, x \neq y \). Then \( \varphi(x) \cap \varphi(y) = \emptyset \). Since \( Y \) is normal, there exist disjoint open sets \( U \) and \( V \) containing \( \varphi(x) \) and \( \varphi(y) \) respectively. Since \( \varphi \) is upper perfectly continuous, \( \varphi^{-1}(U) \) and \( \varphi^{-1}(V) \) are disjoint clopen sets containing \( x \) and \( y \) respectively and so \( X \) is ultra Hausdorff.

Theorem 2.3.17. Let \( \varphi, \psi : X \to Y \) be upper perfectly continuous multifunctions from a topological space \( X \) into a normal space \( Y \) such that \( \varphi(x) \) and \( \psi(x) \) are closed for each \( x \in X \). Then the set \( E = \{ x \in X : \varphi(x) \cap \psi(x) \neq \emptyset \} \) is a cl-closed subset of \( X \).

Proof. To prove that \( E \) is a cl-closed, we shall show that \( X \setminus E \) is cl-open. To this end, let \( x \in X \setminus E \). Then \( \varphi(x) \cap \psi(x) = \emptyset \). Since \( Y \) is normal, there exist disjoint open sets \( U \) and \( V \) containing \( \varphi(x) \) and \( \psi(x) \) respectively. Since \( \varphi \) and \( \psi \) are upper perfectly continuous, \( \varphi^{-1}(U) \) and \( \psi^{-1}(V) \) are clopen sets containing \( x \). Let \( G_1 = \varphi^{-1}(U) \) and \( G_2 = \psi^{-1}(V) \). Then \( G = G_1 \cap G_2 \) is a clopen set containing \( x \). Since \( U \) and \( V \) are disjoint, \( G \subset X \setminus E \) and hence \( X \setminus E \) is cl-open.

Theorem 2.3.18. Let \( \varphi : X \to Y \) be an upper perfectly continuous multifunction from a topological space \( X \) into a normal space \( Y \) such that \( \varphi(x) \) is closed for each \( x \in X \). Then the set \( A = \{ (x, y) \in X \times X : \varphi(x) \cap \varphi(y) \neq \emptyset \} \) is a cl-closed subset of \( X \times X \).
Proof. Let \((x,y) \notin A\). Then \(\varphi(x) \cap \varphi(y) = \emptyset\). Since \(Y\) is normal, there exist disjoint open sets \(U\) and \(V\) containing \(\varphi(x)\) and \(\varphi(y)\) respectively. Since \(\varphi\) is upper perfectly continuous, \(\varphi^{-1}(U)\) and \(\varphi^{-1}(V)\) are disjoint clopen sets containing \(x\) and \(y\) respectively. Let \(G_1 = \varphi^{-1}(U)\) and \(G_2 = \varphi^{-1}(V)\). Then \(G_1 \times G_2\) is a clopen set in \(X \times X\) containing \((x,y)\). We claim that \((G_1 \times G_2) \cap A = \emptyset\). For if \((G_1 \times G_2) \cap A \neq \emptyset\), then \((x_1,y_1) \in G_1 \times G_2\) and \((x_1,y_1) \in A\). This in turn implies that \(U\) and \(V\) are not disjoint which is a contradiction. Thus \((G_1 \times G_2) \cap A = \emptyset\) and so \(G_1 \times G_2 \subset X \times X \setminus A\). Hence \(X \times X \setminus A\) being the union of clopen sets is cl-open and so \(A\) is a cl-closed subset of \(X \times X\).

2.4 Lower perfectly continuous multifunctions

**Theorem 2.4.1.** A multifunction \(\varphi : X \rightarrow Y\) is lower perfectly continuous if and only if for every closed set \(A \subset Y\) the set \(\varphi^{-1}(A)\) is a clopen subset of \(X\).

**Proof.** Suppose \(\varphi : X \rightarrow Y\) is lower perfectly continuous. Let \(A\) be a closed subset of \(Y\). Then \(Y \setminus A\) is an open set in \(Y\). Since \(\varphi\) is lower perfectly continuous, \(\varphi^{-1}(Y \setminus A) = X \setminus \varphi^{-1}(A)\) is clopen in \(X\) and so \(\varphi^{-1}(A)\) is clopen in \(X\).

Conversely suppose that \(\varphi^{-1}(A)\) is clopen in \(X\) for every closed \(A \subset Y\). Let \(U\) be an open subset of \(Y\). Then \(Y \setminus U\) is a closed subset of \(Y\). By hypothesis, \(\varphi^{-1}(Y \setminus U) = X \setminus \varphi^{-1}(U)\) is clopen in \(X\) and so \(\varphi^{-1}(U)\) is clopen in \(X\). Thus \(\varphi\) is lower perfectly continuous. \(\square\)

**Theorem 2.4.2.** Let \(\varphi : X \rightarrow Y\) and \(\psi : Y \rightarrow Z\) be multifunctions. The following statements are true.

(a) If \(\varphi\) lower perfectly continuous and \(\psi\) is lower semicontinuous, then the multifunction \(\psi \circ \varphi\) is lower perfectly continuous.
(b) If $\phi$ is lower almost perfectly continuous and $\psi$ is lower completely continuous, then $\psi \circ \phi$ is lower perfectly continuous.

Proof. To prove $(a)$ let $B$ be an open subset of $Z$. Since $\psi$ is lower semicontinuous, $\psi^{-1}(B)$ is open in $Y$. Again since $\phi$ is lower perfectly continuous, $\phi^{-1}(\psi^{-1}(B)) = (\psi \circ \phi)^{-1}(B)$ is clopen in $X$ and so $\psi \circ \phi$ is lower perfectly continuous.

(b) Let $B$ be an open subset of $Z$. Since $\psi$ is lower completely continuous, $\psi^{-1}(B)$ is regular open in $Y$. Again since $\phi$ is lower almost perfectly continuous, $\phi^{-1}(\psi^{-1}(B)) = (\psi \circ \phi)^{-1}(B)$ is clopen in $X$ and so $\psi \circ \phi$ is lower perfectly continuous.

\[\square\]

**Theorem 2.4.3.** Let $\phi : X \rightarrow Y$ be a lower perfectly continuous multifunction and let $A \subset X$.

Then the multifunction $\phi_{|A} : A \rightarrow Y$ is lower perfectly continuous.

Proof. Let $U$ be an open set in $Y$. Since $\phi$ is lower perfectly continuous, $\phi^{-1}(U)$ is clopen in $X$. Since $(\phi_{|A})^{-1}(U) = A \cap \phi^{-1}(U)$, which is clopen in $A$ and so $\phi_{|A}$ is lower perfectly continuous.

\[\square\]

**Theorem 2.4.4.** If $\phi : X \rightarrow Y$ and $\psi : X \rightarrow Y$ are lower perfectly continuous multifunctions, then the multifunction $\phi \cup \psi : X \rightarrow Y$ defined by $(\phi \cup \psi)(x) = \phi(x) \cup \psi(x)$ for each $x \in X$, is lower perfectly continuous.

Proof. Let $U$ be an open set in $Y$. Since $\phi$ and $\psi$ are lower perfectly continuous, $\phi^{-1}(U)$ and $\psi^{-1}(U)$ are clopen sets in $X$. Again since $(\phi \cup \psi)^{-1}(U) = \phi^{-1}(U) \cup \psi^{-1}(U)$ and since the finite union of clopen sets is clopen, $(\phi \cup \psi)^{-1}(U)$ is clopen in $X$. Thus $\phi \cup \psi$ is lower perfectly continuous.

\[\square\]
2.5 Upper almost perfectly continuous multifunctions

Theorem 2.5.1. A multifunction $\varphi : X \rightarrow Y$ is upper almost perfectly continuous if and only if for every regular closed set $A \subset Y$ the set $\varphi_+^{-1}(A)$ is a clopen subset of $X$.

Proof. Suppose $\varphi : X \rightarrow Y$ is upper almost perfectly continuous. Let $A$ be a regular closed subset of $Y$. Then $Y - A$ is a regular open set in $Y$. Since $\varphi$ is upper almost perfectly continuous, $\varphi_+^{-1}(Y - A) = X - \varphi_+^{-1}(A)$ is clopen in $X$ and so $\varphi_+^{-1}(A)$ is clopen in $X$.

Conversely, suppose that $\varphi_+^{-1}(A)$ is clopen in $X$ for every regular closed $A \subset Y$. Let $U$ be a regular open subset of $Y$. Then $Y - U$ is a regular closed subset of $Y$. By hypothesis, $\varphi_+^{-1}(Y - U) = X - \varphi_+^{-1}(U)$ is clopen in $X$ and so $\varphi_+^{-1}(U)$ is clopen in $X$. Thus $\varphi$ is upper almost perfectly continuous.

Theorem 2.5.2. Let $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ be multifunctions. The following statements are true.

(a) If $\varphi$ is upper almost perfectly continuous and $\psi$ is upper almost completely continuous, then their composition $\psi \circ \varphi$ is upper almost perfectly continuous.

(b) If $\varphi$ is upper perfectly continuous and $\psi$ is upper almost continuous, then $\psi \circ \varphi$ is upper almost perfectly continuous.

Proof. To prove (a) let $B$ be a regular open subset of $Z$. Since $\psi$ is upper almost completely continuous, $\psi_+^{-1}(B)$ is regular open in $Y$. Again since $\varphi$ is upper almost perfectly continuous, $\varphi_+^{-1}(\psi_+^{-1}(B)) = (\psi \circ \varphi)_+^{-1}(B)$ is clopen in $X$ and so $\psi \circ \varphi$ is upper almost perfectly continuous.

(b) Let $B$ be a regular open subset of $Z$. Since $\psi$ is upper almost continuous, $\psi_+^{-1}(B)$ is open in $Y$. Again since $\varphi$ is upper perfectly continuous, $\varphi_+^{-1}(\psi_+^{-1}(B)) = (\psi \circ \varphi)_+^{-1}(B)$ is clopen in $X$ and so $\psi \circ \varphi$ is upper almost perfectly continuous.
The following theorem embodies a sufficient condition for the preservation of upper almost perfectly continuity under the expansion of range.

**Theorem 2.5.3.** Let \( \varphi : X \to Y \) be an upper almost perfectly continuous multifunction and let \( Z \) be a superspace of \( Y \) such that the intersection of every regular open set in \( Z \) with \( Y \) is a regular open set in \( Y \). Then the multifunction \( \psi : X \to Z \) defined by \( \psi(x) = \varphi(x) \) for each \( x \in X \) is upper almost perfectly continuous.

**Proof.** Let \( W \) be a regular open set in \( Z \). By hypothesis, \( W \cap Y \) is regular open set in \( Y \). Again since \( \varphi : X \to Y \) is upper almost perfectly continuous, \( \varphi^{-1}(W \cap Y) \) is a clopen set in \( X \). Now it is clear that \( \psi^{-1}(W) = \varphi^{-1}(W \cap Y) \) and so \( \psi : X \to Z \) is upper almost perfectly continuous.

We may recall that a subspace \( S \) of a space \( X \) is said to be \( \delta \)-embedded in \( X \) if every regular open set in \( S \) is the intersection of a regular open set in \( X \) with \( S \). (see Section 1.2, p.10)

**Theorem 2.5.4.** If a multifunction \( \varphi : X \to Y \) is upper almost perfectly continuous and \( \varphi(X) \) is \( \delta \)-embedded in \( Y \), then the multifunction \( \varphi : X \to \varphi(X) \) is also upper almost perfectly continuous.

**Proof.** Let \( V \) be a regular open set in \( \varphi(X) \). Since \( \varphi(X) \) is \( \delta \)-embedded in \( Y \), there exists a regular open set \( W \) in \( Y \) such that \( V = W \cap \varphi(X) \). Since \( \varphi \) is upper almost perfectly continuous, \( \varphi^{-1}(W) \) is clopen in \( X \). Now \( \varphi^{-1}(V) = \varphi^{-1}(W \cap \varphi(X)) = \varphi^{-1}(W) \) and hence \( \varphi \) is upper almost perfectly continuous.

**Theorem 2.5.5.** Let \( \varphi : X \to Y \) be an upper almost perfectly continuous multifunction and let \( A \subset X \). Then the multifunction \( \varphi_A = \varphi|_A : A \to Y \) is upper almost perfectly continuous.

**Proof.** Let \( U \) be a regular open set in \( Y \). Since \( \varphi \) is upper almost perfectly continuous, \( \varphi^{-1}(U) \) is clopen in \( X \). Now \( (\varphi_A)^{-1}(U) = A \cap \varphi^{-1}(U) \), which is clopen in \( A \) and so \( \varphi_A \) is upper almost perfectly continuous.
Theorem 2.5.6. Let $\Omega = \{X_\alpha : \alpha \in \Lambda\}$ be a locally finite clopen cover of $X$ and let $\varphi : X \rightarrow Y$ be a multifunction. For each $\alpha \in \Lambda$, let $\varphi_\alpha = \varphi_{|X_\alpha} : X_\alpha \rightarrow Y$ be the restriction of $\varphi$ to $X_\alpha$. Then $\varphi$ is upper almost perfectly continuous if and only if each $\varphi_\alpha$ is upper almost perfectly continuous.

Proof. Necessity is easy to see. To prove sufficiency, let $V$ be a regular open set in $Y$. Then $\varphi^{-1}(V) = \bigcup_{\alpha \in \Lambda} (\varphi_\alpha)^{-1}(V) = \bigcup_{\alpha \in \Lambda} (\varphi^{-1}(V) \cap X_\alpha)$. Each $\varphi^{-1}(V) \cap X_\alpha$ is clopen in $X_\alpha$ and hence in $X$. Thus $\varphi^{-1}(V)$ is open being the union of clopen sets. In view of local finiteness of $\Omega$, the collection $\{\varphi^{-1}(V) \cap X_\alpha : \alpha \in \Lambda\}$ is a locally finite collection of clopen sets. Therefore $\varphi^{-1}(V)$ is also closed being the union of a locally finite collection of clopen sets and hence clopen. Consequently, $\varphi$ is upper almost perfectly continuous. \hfill $\Box$

Theorem 2.5.7. Let $\varphi : X \rightarrow Y$ and $\psi : X \rightarrow Y$ be two upper almost perfectly continuous multifunctions. Then $\varphi \cup \psi : X \rightarrow Y$ defined by $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$ for each $x \in X$, is upper almost perfectly continuous.

Proof. Let $U$ be a regular open set in $Y$. Since $\varphi$ and $\psi$ are upper almost perfectly continuous, $\varphi^{-1}(U)$ and $\psi^{-1}(U)$ are clopen sets in $X$. Again since $(\varphi \cup \psi)^{-1}(U) = \varphi^{-1}(U) \cap \psi^{-1}(U)$ and since the finite intersection of clopen sets is clopen, $(\varphi \cup \psi)^{-1}(U)$ is clopen in $X$. Thus $\varphi \cup \psi$ is upper almost perfectly continuous. \hfill $\Box$

Theorem 2.5.8. Let $\varphi : X \rightarrow Y$ be a multifunction and let $g : X \rightarrow X \times Y$ defined by $g(x) = \{x\} \times \varphi(x)$ for each $x \in X$ be the graph multifunction. If $g$ is upper almost perfectly continuous, then $\varphi$ is upper almost perfectly continuous and the space $X$ is equipped with an almost partition topology.

Proof. Suppose that $g$ is upper almost perfectly continuous. By Theorem 2.5.2(a) the multifunction $\varphi = p_Y \circ g$ is upper almost perfectly continuous, where $p_Y : X \times Y \rightarrow Y$ denotes the
projection mapping which is upper semicontinuous as well as upper almost completely continuous. Now to show that $X$ is endowed with an almost partition topology, let $U$ be a regular open set in $X$. Then $U \times Y$ is a regular open set in $X \times Y$. Since $g$ is upper almost perfectly continuous, $g^{-1}(U \times Y)$ is clopen in $X$. It is easily verified that $g^{-1}(U \times Y) = U$, and so $U$ is clopen in $X$. Thus $X$ is endowed with an almost partition topology.

**Theorem 2.5.9.** If $\phi : X \to Y$ is an upper almost perfectly continuous multifunction where $Y$ is a regular space and $\phi(x)$ is closed for each $x \in X$, then the graph $\Gamma_\phi$ of $\phi$ is a cl-closed subset of $X \times Y$ with respect to $X$.

**Proof.** Let $(x, y) \notin \Gamma_\phi$. Then $y \notin \phi(x)$. Since $Y$ is a regular space, there exist disjoint open sets $V_y$ and $V_{\phi(x)}$ containing $y$ and $\phi(x)$, respectively. It is easily verified that the sets $V_y$ and $V_{\phi(x)}$ may be chosen to be regular open. Since $\phi$ is upper almost perfectly continuous, $U_x = \phi^{-1}(V_{\phi(x)})$ is a clopen set containing $x$. We assert that $(U_x \times V_y) \cap \Gamma_\phi = \emptyset$. For, if $(h, k) \in (U_x \times V_y) \cap \Gamma_\phi$, then $h \in \phi^{-1}(V_x)$, $k \in V_y$ and $k \in \phi(h)$. Hence $\phi(h) \subset V_{\phi(x)}$ and $k \in \phi(h) \cap V_y$ which contradicts the fact that $V_y$ and $V_{\phi(x)}$ are disjoint. Thus the graph $\Gamma_\phi$ is cl-closed in $X \times Y$ with respect to $X$. □

**Theorem 2.5.10.** Let $\phi : X \to Y$ be an upper almost perfectly continuous multifunction such that $\phi(x) \cap \phi(y) = \emptyset$ for each $x \neq y$ in $X$ and $\phi(x)$ is closed for each $x \in X$. If $Y$ is a normal space, then $X$ is an ultra Hausdorff space.

**Proof.** Let $x, y \in X, x \neq y$. Then $\phi(x) \cap \phi(y) = \emptyset$. Since $Y$ is normal, there exist disjoint open sets $U_1$ and $V_1$ containing $\phi(x)$ and $\phi(y)$ respectively. Then $U = \overline{U_1}$ and $V = \overline{V_1}$ are disjoint regular open sets containing $\phi(x)$ and $\phi(y)$, respectively. Since $\phi$ is upper almost perfectly continuous, $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are disjoint clopen sets containing $x$ and $y$ respectively and so $X$ is ultra Hausdorff. □
Theorem 2.5.11. Let \( \varphi, \psi : X \rightarrow Y \) be upper almost perfectly continuous multifunctions from a topological space \( X \) into a normal space \( Y \) such that \( \varphi(x) \) and \( \psi(x) \) are closed for each \( x \in X \). Then the set \( E = \{ x \in X : \varphi(x) \cap \psi(x) \neq \emptyset \} \) is a cl-closed subset of \( X \).

**Proof.** To prove that \( E \) is a cl-closed, we shall show that \( X \setminus E \) is cl-open. To this end, let \( x \in X \setminus E \). Then \( \varphi(x) \cap \psi(x) = \emptyset \). Since \( Y \) is normal, there exist disjoint open sets \( U_1 \) and \( V_1 \) containing \( \varphi(x) \) and \( \psi(x) \) respectively. Then \( U = U_1^\circ \) and \( V = V_1^\circ \) are disjoint regular open sets containing \( \varphi(x) \) and \( \psi(x) \) respectively. Since \( \varphi \) and \( \psi \) are upper almost perfectly continuous, \( \varphi^{-1}(U) \) and \( \psi^{-1}(V) \) are clopen sets containing \( x \). Let \( G_1 = \varphi^{-1}(U) \) and \( G_2 = \psi^{-1}(V) \). Then \( G = G_1 \cap G_2 \) is a clopen set containing \( x \). Since \( U \) and \( V \) are disjoint, \( G \subset X \setminus E \) and hence \( X \setminus E \) is cl-open.

**Theorem 2.5.12.** Let \( \varphi : X \rightarrow Y \) be an upper almost perfectly continuous multifunction from a topological space \( X \) into a normal space \( Y \) such that \( \varphi(x) \) is closed for each \( x \in X \). Then the set \( A = \{ (x,y) \in X \times X : \varphi(x) \cap \varphi(y) \neq \emptyset \} \) is a cl-closed subset of \( X \times X \).

**Proof.** Let \( (x,y) \notin A \). Then \( \varphi(x) \cap \varphi(y) = \emptyset \). Since \( Y \) is normal, there exist disjoint open sets \( U \) and \( V \) containing \( \varphi(x) \) and \( \varphi(y) \) respectively. As in the proof of Theorem 2.5.11 the sets \( U \) and \( V \) may be taken to be regular open. Since \( \varphi \) is upper almost perfectly continuous, \( \varphi^{-1}(U) \) and \( \varphi^{-1}(V) \) are disjoint clopen sets containing \( x \) and \( y \) respectively. Let \( G_1 = \varphi^{-1}(U) \) and \( G_2 = \varphi^{-1}(V) \). Then \( G_1 \times G_2 \) is a clopen set in \( X \times X \) containing \( (x,y) \). We claim that \( (G_1 \times G_2) \cap A = \emptyset \). For if \( (G_1 \times G_2) \cap A \neq \emptyset \), then \( (x_1,y_1) \in G_1 \times G_2 \) and \( (x_1,y_1) \in A \). This in turn implies that \( U \) and \( V \) are not disjoint which is a contradiction. Thus \( (G_1 \times G_2) \cap A = \emptyset \) and so \( G_1 \times G_2 \subset X \times X \) \( \setminus A \). Hence \( (X \times X) \setminus A \) being the union of clopen sets is cl-open and so \( A \) is a cl-closed subset of \( X \times X \). \( \square \)
2.6 Lower almost perfectly continuous multifunctions

Theorem 2.6.1. A multifunction \( \varphi : X \rightarrow Y \) is lower almost perfectly continuous if and only if for every regular closed set \( A \subset Y \) the set \( \varphi^{-1}(A) \) is a clopen subset of \( X \).

Proof. Similar to that of Theorem 2.5.1. \( \square \)

Theorem 2.6.2. Let \( \varphi : X \rightarrow Y \) and \( \psi : Y \rightarrow Z \) be multifunctions. The following statements are true.

(a) If \( \varphi \) is lower almost perfectly continuous and \( \psi \) is lower almost completely continuous, then their composition \( \psi \circ \varphi \) is lower almost perfectly continuous.

(b) If \( \varphi \) is lower perfectly continuous and \( \psi \) is lower almost continuous, then \( \psi \circ \varphi \) is lower almost perfectly continuous.

Proof. To prove (a) let \( B \) be a regular open subset of \( Z \). Since \( \psi \) is lower almost completely continuous, \( \psi^{-1}_+(B) \) is regular open in \( Y \). Again since \( \varphi \) is lower almost perfectly continuous, \( \varphi^{-1}_+(\psi^{-1}_+(B)) = (\psi \circ \varphi)^{-1}_+(B) \) is clopen in \( X \) and so \( \psi \circ \varphi \) is lower almost perfectly continuous.

(b) Let \( B \) be a regular open subset of \( Z \). Since \( \psi \) is lower almost continuous, \( \psi^{-1}_+(B) \) is open in \( Y \). Again since \( \varphi \) is lower perfectly continuous, \( \varphi^{-1}_+(\psi^{-1}_+(B)) = (\psi \circ \varphi)^{-1}_+(B) \) is clopen in \( X \) and so \( \psi \circ \varphi \) is lower almost perfectly continuous. \( \square \)

Theorem 2.6.3. Let \( \varphi : X \rightarrow Y \) be a almost lower perfectly continuous multifunction and let \( A \subset X \). Then the multifunction \( \varphi_A : A \rightarrow Y \) is lower almost perfectly continuous.

Proof. Let \( U \) be a regular open set in \( Y \). Since \( \varphi \) is lower almost perfectly continuous, \( \varphi^{-1}_+(U) \) is clopen in \( X \). Now \( (\varphi_A)^{-1}_+(U) = A \cap \varphi^{-1}_+(U) \), which is clopen in \( A \) and so \( \varphi_A \) is lower almost perfectly continuous. \( \square \)
Theorem 2.6.4. If $\varphi : X \to Y$ and $\psi : X \to Y$ are lower almost perfectly continuous multifunctions, then the multifunction $\varphi \cup \psi : X \to Y$ defined by $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$ for each $x \in X$, is lower almost perfectly continuous.

Proof. Let $U$ be a regular open set in $Y$. Since $\varphi$ and $\psi$ are lower almost perfectly continuous, $\varphi^{-1}(U)$ and $\psi^{-1}(U)$ are clopen sets in $X$. Again since $(\varphi \cup \psi)^{-1}(U) = \varphi^{-1}(U) \cup \psi^{-1}(U)$ and since the finite union of clopen sets is clopen, $(\varphi \cup \psi)^{-1}(U)$ is clopen in $X$. Thus $\varphi \cup \psi$ is lower almost perfectly continuous. \hfill \Box

Corollary 2.6.5. If a multifunction $\varphi : X \to Y$ is lower almost perfectly continuous and $\varphi(X)$ is $\delta$-embedded in $Y$, then the multifunction $\varphi : X \to \varphi(X)$ is lower almost perfectly continuous.

Proof. Let $V$ be a regular open set in $\varphi(X)$. Since $\varphi(X)$ is $\delta$-embedded, there exist a regular open set $W$ in $Y$ such that $V = W \cap \varphi(X)$. Since $\varphi$ is lower almost perfectly continuous, $\varphi^{-1}(W)$ is clopen in $X$. Now $\varphi^{-1}(V) = \varphi^{-1}(W \cap \varphi(X)) = \varphi^{-1}(W)$ and hence $\varphi$ is lower almost perfectly continuous. \hfill \Box