CHAPTER 5

DESIGN OF OUTPUT FEEDBACK CONTROLLERS FOR DISCRETE TWO-TIME SCALE INTERVAL SYSTEM

5.1 INTRODUCTION

Control of two-time scale systems have been intensively studied for the past three decades and a popular approach adopted to handle these systems is based on the so-called reduced technique by Kokotovic et al (1976). The composite design based on separate designs for slow and fast subsystems has been systematically reviewed in Saksena et al (1984). The research on two-time scale systems in the $H_\infty$ sense is of great practical importance. The problem of $H_\infty$ control of linear two-time scale systems and focused on modeling and control of linear two-time scale systems: Applied to single-link flexible manipulator. The control design technique is extended to robust control of two-time scale discrete interval system in Patre et al(2006). In this chapter a periodic and fast output sampling feedback controllers are designed for the interval system using two-time scale approach for the vibration control of smart cantilever beam.

The chapter is organized as follows: In section 5.2, preliminaries of two time scale system and method for designing state feedback controller are given. A periodic and fast output sampling feedback controller designed for discrete two time scale interval system is presented in section 5.3. In section 5.4, simulation results are given based on given procedure. Finally conclusion is given in section 5.5.
5.2 BACKGROUND

A discrete time interval system possessing two-time-scale nature for the entire range of parameter variation can be described by the following equations.

\[
x_1(k+1) = A_{11}^I x_1(k) + A_{12}^I x_2(k) + B_1^I u(k), \quad (5.1)
\]

\[
x_2(k+1) = A_{21}^I x_1(k) + A_{22}^I x_2(k) + B_2^I u(k), \quad (5.2)
\]

where the state \( x(k) \in \mathbb{R}^n \) is formed by the \( n_1 \) and \( n_2 \) dimensional vector \( x_1(k) \) and \( x_2(k) \) at the discrete instant \( k \) and the control \( u(k) \) is an \( m \) dimensional vector. The matrices \( A_{ij}^I \) (\( i, j = 1, 2 \)) are interval matrices. It is assumed that the system (5.1), (5.2) possesses two-time-scale property for the entire range of parameter variation, that is, it has a cluster of \( n_1 \) eigenvalues distributed near the unit circle and a cluster of \( n_2 \) eigenvalues centered around the origin in the complex plane. Clearly, the \( n_1 \) eigenvalues have large magnitudes compared with small magnitudes of the \( n_2 \) eigenvalues over the period \([0, T]\). The system behavior, therefore, can be approximately decomposed into a slow subsystem with \( n_1 \) eigenvalues and a fast subsystem with \( n_2 \) eigenvalues. In an asymptotically stable system the fast modes corresponding to the eigenvalues of small magnitudes are important only during a short initial period \([0, T_f]\). After that period they are negligible and the behavior of the system can be described by its slow modes. Neglecting the effects of the fast modes is equivalent to letting \( x_2(k+1) = x_2(k) \) in (5.2). Without the fast modes system (5.1), (5.2) reduces to,

\[
\bar{x}_1(k+1) = A_{11}^I \bar{x}_1(k) + A_{12}^I \bar{x}_2(k) + B_1^I \bar{u}(k) \quad \bar{x}(0) = x_{10}, \quad (5.3)
\]

\[
\bar{x}_2(k) = A_{21}^I \bar{x}_1(k) + A_{22}^I \bar{x}_2(k) + B_2^I \bar{u}(k), \quad (5.4)
\]
where a bar indicates a discrete quasi-steady state by Mahmoud, M.S (1982).

Assuming that \([I_2 - A_{22}^I]^{-1}\) exists, where \(I_2\) is the identity matrix with degenerate interval of dimension \(n_2\times n_2\), we can express \(\bar{x}_2(k)\) as

\[
\bar{x}_2(k) = [I_2 - A_{22}^I]^{-1} \left\{ A_{21}^I \bar{x}_1(k) + B_{21}^I \bar{u}(k) \right\}, \tag{5.5}
\]

and, substituting it into (5.3), the slow subsystem of (5.1),(5.2) is given by

\[
x_s(k + 1) = A_s^I x_s(k) + B_s^I u(k), \tag{5.6}
\]

where

\[
A_s^I = A_{11}^I + A_{12}^I[I_2 - A_{22}^I]^{-1} A_{21}^I,
\]

\[
B_s^I = B_1^I + A_{12}^I[I_2 - A_{22}^I]^{-1} B_2^I. \tag{5.7}
\]

Hence \(\bar{x}_i(k) = x_s(k), \bar{x}_2(k)\) and \(\bar{u}(k)\) are the slow components of the corresponding variables in system (5.1),(5.2). The fast subsystem is derived by making the assumptions that \(\bar{x}_1(k) = x_s(k)\) = constant and \(\bar{x}_2(k + 1) = \bar{x}_2(k)\). From (5.2) and (5.5) we get,

\[
x_2(k + 1) - \bar{x}_2(k + 1) = A_{22}^I \left\{ x_2(k) - \bar{x}_2(k) \right\} + B_2^I \left\{ u(k) - u_s(k) \right\}. \tag{5.8}
\]

Defining \(x_f(k) = x_2(k) - \bar{x}_2(k)\) and \(u_f(k) = u(k) - u_s(k)\), the fast subsystem of (5.1), (5.2) can be expressed as,

\[
x_f(k + 1) = A_{22}^I x_f(k) + B_2^I u_f(k); \quad x_f(0) = x_{20} - \bar{x}_2(0). \tag{5.9}
\]

The assumptions used in deriving the fast subsystem are justified by noting that the slow modes of system (5.1), (5.2) have magnitudes which are close to unity and during transients, they are changing very slowly with respect to the fast modes.
5.2.1 State Feedback Control

Low Order Control

Assume that the fast system is stable and the pair \((A_i', B_i')\) is controllable Kolev et al.(1996), Smagina et al.(1997). Neglecting the fast subsystem, consider the lower-order feedback control

\[ u(k) = K_0 x_1(k). \]  \hspace{1cm} (5.10)

If this \(u(k)\) is applied to (5.6), closed loop slow model becomes,

\[ x_s(k+1) = \begin{bmatrix} A_i' + B_i' K_0 \end{bmatrix} x_s(k), \]
\[ = H_0 x_s(k). \]  \hspace{1cm} (5.11)

If the pair \((A_i', B_i')\) is controllable then the closed loop system matrix can be stabilized by the appropriate selection of the gain matrix \(K_0\).

We now apply control (5.10) to the discrete system (5.1),(5.2) to obtain

\[ x_1(k+1) = [A_{i1}'+ B_{i1} K_0] x_1(k) + A_{i2} x_2(k) \]  \hspace{1cm} (5.12)
\[ x_2(k+1) = [A_{i2}'+ B_{i2} K_0] x_1(k) + A_{i22} x_2(k). \]  \hspace{1cm} (5.13)

If the pair \((A_i', B_i')\) is controllable, the matrix \(A_{i22}'\) is stable, and \([I_2-A_{i22}]^{-1}\) exists, then the closed loop discrete system (5.12),(5.13) is asymptotically stable. The significance is that, it reduces the design of stabilizing feedback controllers for discrete interval systems of dimensions \(n_1 + n_2\) to the reduced system of order \(n_1\) when the pair \((A_i', B_i')\) is controllable.

5.2.2 Composite Control

Consider a discrete time uncertain system possessing two-time-scale nature for the entire range of parameter variation described by
\[ x_1(k+1) = A_{11}^I x_1(k) + A_{12}^I x_2(k) + B_1^I u(k), \quad (5.17) \]
\[ x_2(k+1) = A_{21}^I x_1(k) + A_{22}^I x_2(k) + B_2^I u(k), \quad (5.18) \]

It is assumed that \( A_{21}, A_{22}, \) and \( B_2 \) are constant matrices. The problem is to find a linear state feedback which stabilize the uncertain system (5.17), (5.18).

The slow uncertain subsystem can be obtained as in (5.6) with
\[
A_1^I = A_{11}^I + A_{12}^I [I_2 - A_{22}^I]^{-1} A_{21}^I
\]
\[
B_1^I = B_1^I + A_{12}^I [I_2 - A_{22}^I]^{-1} B_2^I
\]
(5.19)

The fast subsystem can be obtained as,
\[ x_f(k+1) = A_{22} x_f(k) + B_2 u_f(k). \quad (5.20) \]

Assume that the pair \((A_{22}, B_2)\) and the pair \((A_s, B_s)\) is controllable. Let \( u_s(k) = k_0 x_s(k) \) and \( u_f(k) = K_f x_f(k) \) be designed to stabilize the slow and fast subsystem in (5.19) and (5.20) respectively.

By virtue of equation (5.5),
\[
\bar{x}_2(k) = [I_2 - A_{22}]^{-1} \{ A_{21} \bar{x}_1(k) + B_2 \bar{u}(k) \},
\]
The composite control
\[ u_c(k) = u_s(k) + u_f(k) = K_0 x_s(k) + K_f x_f(k), \]
Can be written as,
\[
u_c(k) = \{(I_m - K_f (I_2 - A_{22})^{-1} B_2) K_0 - K_f (I_2 - A_{22})^{-1} A_{21}\} x_s(k) + K_f \{(I_2 - A_{22})^{-1} (A_{21} + B_2 K_0) x_s(k) + x_f(k)\}
\quad (A_{21} + B_2 K_0) x_s(k) + x_f(k)\}
(5.21)
If we replace $x_s(k)$ by $x_1(k)$ and $\bar{x}_s(k) + x_j(k)$ by $x_2(k)$, we get the composite controller in terms of the states of the higher order system as

$$u_c(k) = [(I_m - K_f(I_2 - A_{22})^{-1}B_2)K_0 - K_f(I_2 - A_{22})^{-1}A_{21}]x_1(k) + K_fx_2(k).$$

(5.22)

If $[I_2 - A_{22}]^{-1}$ exists and the pairs $(A_s, B_s)$ and $(A_{22}, B_2)$ are stabilizable, then the linear state feedback control

$$u_c(k) = [(I_m - K_f(I_2 - A_{22})^{-1}B_2)K_0 - K_f(I_2 - A_{22})^{-1}A_{21}]x_1(k) + K_fx_2(k).$$

(5.23)

Stabilizes $5.8, 5.9$ where $K_0$ and $K_f$ are designed to make $(A_s + B_sK_0)$ and $(A_{22} + B_2K_f)$ stable matrices respectively.

### 5.3 CONTROLLER DESIGN

In this section, a fast output sampling feedback controller and periodic output feedback controller for discrete two-time scale interval system is presented. The design procedure is given as follows.

#### 5.3.1 FOS for Two-Time Scale Interval Model

Let $(\Phi_s, \Gamma_s, C^T)$ be the discrete time system obtained by sampling $(A, B, C^T)$ at a rate $\frac{1}{\tau}$ ($\tau = 0.001$ sec).
\[
\Phi_x = \begin{bmatrix}
0.9901 & 1.0000 & -0.0000 & -0.0000 \\
-0.2781 & -0.0000 & 1.0000 & -0.0000 \\
0.9873 & -0.0000 & -0.0000 & 1.0000 \\
-0.9904 & 0.0000 & 0.0000 & -0.0000 \\
\end{bmatrix}, \quad \Gamma_x = \begin{bmatrix}
0.0007 \\
0.0002 \\
0.0000 \\
0.0001 \\
\end{bmatrix}
\] (5.24)

and its interval form is

\[
\Phi'_x = \begin{bmatrix}
[0.9286, 1.0463] & [0.9164, 1.0691] & [-0.0630, 0.0776] & [-0.0460, 0.0458] \\
[-0.3629, -0.1751] & [-0.1297, 0.1400] & [0.9794, 0.9922] & [-0.1087, 0.1231] \\
[0.9673, 0.9787] & [-0.1080, 0.1222] & [-0.1082, 0.1225] & [0.9794, 0.9922] \\
[-1.0587, -0.9077] & [-0.0769, 0.0626] & [-0.0456, 0.0455] & [-0.0628, 0.0772] \\
\end{bmatrix}
\]

\[
\Gamma'_x = \begin{bmatrix}
[0.0007, 0.0007] \\
[0.0002, 0.0002] \\
[-0.0010, -0.0010] \\
[0.0001, 0.0001] \\
\end{bmatrix}
\] (5.25)

A slow and fast subsystems for the system in equation (5.25) is obtained by using equation (5.1) and (5.2) where

\[
A'_{11} = \begin{bmatrix}
[0.9286, 1.0463] & [0.9164, 1.0691] \\
[0.3629, -0.1751] & [-0.1297, 0.1400] \\
\end{bmatrix}
\]

\[
A'_{12} = \begin{bmatrix}
[-0.0630, 0.0776] & [-0.0460, 0.0458] \\
[0.9794, 0.9922] & [-0.1087, 0.1231] \\
\end{bmatrix}
\]

\[
A'_{21} = \begin{bmatrix}
[0.9673, 0.9787] & [-0.1080, 0.1222] \\
[-1.0587, -0.9077] & [-0.0769, 0.0626] \\
\end{bmatrix}
\]

\[
A'_{22} = \begin{bmatrix}
[-0.1082, 0.1225] & [0.9794, 0.9922] \\
[-0.0456, -0.0455] & [-0.0628, 0.0772] \\
\end{bmatrix}
\]
The slow subsystem is

\[ A_s' = \begin{bmatrix}
1.0209 & 1.0300 \\
0.2493 & 0.3202 \\
-0.1147 & -0.0255 \\
0.7484 & 0.7923
\end{bmatrix} \]

(5.26)

\[ B_s' = \begin{bmatrix}
0.00092 & 0.00092 \\
-0.00034 & 0.00024
\end{bmatrix} \]

And fast subsystem is

\[ A_f' = \begin{bmatrix}
-0.1082 & 0.1225 \\
0.9794 & 0.9922 \\
-0.0456 & -0.0455 \\
-0.0628 & 0.0772
\end{bmatrix} \]

(5.27)

\[ B_f' = \begin{bmatrix}
-0.0010 & -0.0010 \\
0.0001 & 0.0001
\end{bmatrix} \]

Stabilizing state feedback controller is designed using LQR technique for slow and fast interval subsystem in equation (5.26) and (5.27). The state feedback gains are

\[ K_o = \begin{bmatrix}
-129.24 & 182.300
\end{bmatrix} \]

\[ K_f = \begin{bmatrix}
123 & 65
\end{bmatrix} \]

The composite state feedback controller is designed for the system in equation (5.25) using equation (5.21). The composite controller gain obtained is

\[ k_i' = \begin{bmatrix}
55.9733 & 58.5322
\end{bmatrix} \]
The fast output sampling feedback controller for slow (lower and upper bound), fast (lower and upper bound) and nominal (lower and upper bound) are designed by following the procedure given in section (3.3). The controller gains are

**For the slow subsystem**

**Lower bound**

\[
L_s = \begin{bmatrix} -33.2862 & -20.7804 & 76.1581 & -23.9287 \\
\end{bmatrix}
\]

**Upper bound**

\[
L_s = \begin{bmatrix} -33.4073 & -20.7920 & 76.2527 & -23.8104 \\
\end{bmatrix}
\]

**For the fast subsystem**

**Lower bound**

\[
L_f = \begin{bmatrix} 7.7333 & 12.3518 & 16.1506 & 18.7590 \\
\end{bmatrix}
\]

**Upper bound**

\[
L_f = \begin{bmatrix} 7.7525 & 12.3457 & 16.1279 & 18.7242 \\
\end{bmatrix}
\]

**For the nominal system**

**Lower bound**

\[
K_c = \begin{bmatrix} 4.3243 & -18.3222 & 38.4332 & 55.9733 \\
\end{bmatrix}
\]

**Upper bound**

\[
K_c = \begin{bmatrix} 5.2322 & -21.2773 & 41.7643 & 58.5322 \\
\end{bmatrix}
\]
The open loop, close loop response with FOS and control signal obtained in simulation for slow (lower and upper bound), fast (lower and upper bound) and nominal (lower and upper bound) are shown in Figures 5.1, 5.2, 5.3, 5.4, and 5.5 respectively.

Figure 5.1  Response with Fast Output Sampling Feedback for Two-Time Scale Slow System-Lower Bound

Figure 5.2  Response with Fast Output Sampling Feedback for Two-Time Scale slow system- upper bound
Figure 5.3  Response with Fast Output Sampling Feedback for Two-Time Scale Fast System-Lower Bound

Figure 5.4  Response with Fast Output Sampling Feedback for Two-Time Scale Fast System-Upper Bound
5.3.2 POF for Two-Time Scale Interval Model

Stabilizing output injection gain is designed for slow and fast interval subsystem in equation (5.26) and (5.27). The output injection gains are

\[ G_0 = [86.7624 \ 25.7514]; \]

\[ G_f = [0.142 \ -0.322] \]

The composite output injection gain is designed for the system in equation (5.25) using equation (5.21). The composite controller gain obtained is

\[
G'_c = \begin{bmatrix}
-30.2227 & -26.1627 \\
-18.5434 & -15.7653 \\
-8.8532 & -4.8906 \\
0.2477 & 0.3065
\end{bmatrix}
\]

Figure 5.5  Response with Fast Output Sampling Feedback for Two-Time Scale Nominal System – Composite Controller
The periodic output feedback controller for slow (lower and upper bound), fast (lower and upper bound) and nominal (lower and upper bound) are designed by following the procedure given in section (4.3). The controller gains are

For the slow subsystem
Lower bound
\[ K_s = \begin{bmatrix} -33.4073 & -20.7920 & 76.2527 & -23.8104 \end{bmatrix} \]

Upper bound
\[ K_s = \begin{bmatrix} -33.5073 & -20.8820 & 76.3527 & -23.9124 \end{bmatrix} \]

For the fast subsystem
Lower bound
\[ K_f = \begin{bmatrix} 8.8126 & 14.2791 & 18.1235 & 22.1105 \end{bmatrix} \]

Upper bound
\[ K_f = \begin{bmatrix} 8.9802 & 14.3819 & 18.3213 & 22.2110 \end{bmatrix} \]

For the nominal system
Lower bound
\[ K_c = \begin{bmatrix} -29.5423 & -17.4322 & -7.8643 & 0.3855 \end{bmatrix} \]

Upper bound
\[ K_c = \begin{bmatrix} -28.2527 & -16.2734 & -6.7130 & 0.2877 \end{bmatrix} \]

The open loop, close loop response with POF and control signal obtained in simulation for slow (lower and upper bound), fast (lower and upper bound) and nominal (lower and upper bound) are shown in Figures 5.6, 5.7, 5.8, 5.9, and 5.102 respectively
Figure 5.6  Response with Periodic Output Feedback for Two-Time Scale Slow System-Lower Bound

Figure 5.7  Response with Periodic Output Feedback for Two-Time Scale Slow System-Upper Bound
Figure 5.8  Response with Periodic Output Feedback for Two-Time Scale Fast System-Lower Bound

Figure 5.9  Response with Periodic Output Feedback for Two-Time Scale Fast System-Upper Bound
A fast and periodic output feedback controller to suppress the first vibration mode of a smart cantilever beam are designed and evaluated through simulation. The discrete two-time scale interval system is decomposed into slow and fast discrete interval subsystems with maintaining good accuracy. Stabilizing state feedback controller and output injection controller are designed using LQR technique for slow and fast interval subsystem. The composite state feedback controller and composite output injection gain are designed for the fourth order original discrete two-time scale interval system. It is observed that the state feedback gain and output injection gain designed to stabilize the slow model stabilizes the actual full order system (provided the fast modes are stable). It is also observed from simulation results that the composite controller formed from the subsystem controller controls the original discrete two-time scale interval system.