CHAPTER 5

A SINGLE SERVER RETRIAL QUEUE WITH BERNOULLI SCHEDULE AND TWO PHASES OF HETEROGENEOUS SERVICE AND RECURRENT DEMAND OPTION

5.1 INTRODUCTION

This chapter addresses the steady state behavior of an M/G/1 retrial queue with infinite waiting space in which arriving customers who find the server busy join (i) the retrial group with probability $p$ in order to seek service again after a random amount of time, or (ii) the infinite waiting space (priority group) with probability $q (=1-p)$ where they wait to be served.

The single server retrial queue with priority calls have been studied by Choi et al (1990, 1995, 1999) for many applications in telecommunication and mobile communication. The retrial queues with feedback or recurrent behavior are having considerable attention of literature. Krishnakumar et al (2002b) have analysed an M/G/1 retrial queue with feedback and starting failures using supplementary variable technique. Farahmand et al (1996, 2001) have analysed some retrial queueing models with recurrent demand option in which the leaving customer either join the orbit or leave the system permanently.

Priority mechanisms are an invaluable scheduling method that allows customers to receive different quality of service. Service priority is clearly today a main feature of the operation of almost any manufacturing system. The role of quality and service performance is crucial aspects in
customer perceptions, and firms must dedicate special attention to them when designing and implementing their operations. For this reason, the priority queue has received considerable attention in the literature. Atencia and Morena (2005) surveyed the retrial queueing systems with priority customers comprehensively and general retrial time. Choi and Park (1990) surveyed the retrial queueing systems with priority customers and exponential retrial time.

The main motivation of this paper is from the mobile cellular communication systems. **Mobile cellular communication** systems are governed by numerous base stations, each of them with a specific influence zone (called cell). These cells are overlapped and all together cover the service area. The base station in each cell can serve calls which are being served in contiguous cells (because they are in an overlap region). The base station in a cell manages two sorts of calls: **originating and handoff calls**. An originating call is a call initiated in its cell. Otherwise, a handoff call is initiated in a contiguous cell and enters the overlap zone of both cells. A handoff call undergoes a failure during the communication when it escapes from the overlap region and the base station does not provide it a free channel. A blocked call repeats its request later after a random time. It is reasonable that the policy of the mobile companies tends to avoid failure in a current communication of a handoff call before than the inconvenience caused by a block of an originating call (the loss of a handoff call is a very important aspect for the quality of the service). Consequently, the base station gives priority to a handoff call over an originating call. Our retrial queue can model this system considering a **handoff call** as a **Priority call** and an **originating call** as a **Retrial call**. At the end of call connection, if the caller is not satisfied with the call, he will repeat the call again. Thus, this kind of system can be modeled as the single server retrial queue with Bernoulli schedule and two phases of heterogeneous service and recurrent demand on option.
5.2 MATHEMATICAL MODEL

If the primary call, on arrival finds the server busy with infinite waiting space the customers join (i) the retrial group with probability $p$ in order to seek service again after a random amount of time, or (ii) the infinite waiting space with probability $q (=1-p)$ where they wait to be served. The server provides preliminary first essential service (FES) and second essential service (SES) to the calls from priority group or calls from the retrial group. On completion of SES, the recipient of SES has an option of leaving the system completely with probability $1-\beta$, or returning to the retrial group with probability $\beta$ if there are no customers in the priority group.

The time between two successive repeated attempts of each call in orbit is assumed to be exponentially distributed with rate ‘$\nu$’. Let $S_1(x)$ $(s_1(x)) \{S_1^0(x)\}$ be the cumulative distribution function (probability density function) {Laplace – Stieltjes transform} [remaining service time] of FES. $S_2(x)$ $(s_2(x)) \{S_2^0(x)\}$ be cumulative distribution function (probability density function) {Laplace-Stieltjes transform} [remaining service time] of SES. $N_1(t)$ denotes the number of customers in the priority group at time $t$. $N_2(t)$ denotes the number of customers in retrial group.

The server state is denoted as,

$$C(t) = \begin{cases} 
0 & \text{if the server is idle} \\
1 & \text{if the server is doing FES} \\
2 & \text{if the server is doing SES} 
\end{cases}$$

Now the system state probabilities are defined as follows,

$$P_{n,j}(t)dt = \Pr\{N_2(t) = j, C(t) = 0\} \quad j \geq 0 \quad \text{and}$$

$$P_{n,i,j}(x,t)dt = \Pr\{N_1(t) = i, N_2(t) = j, C(t) = n, x \leq S_n^0(t) \leq x + dt\}; \quad i \geq 0; \quad j \geq 0; \quad n = 1, 2;$$
5.3 **SYSTEM SIZE DISTRIBUTION**

The following equations are obtained for the queueing system:

\[
P_{0,i}(t + \Delta t) = P_{0,i}(t)(1 - \lambda\Delta t - jv\Delta t) + (1 - \beta)P_{2,0,i}(0,t)\Delta t + \beta P_{2,0,i-1}(0,t)\Delta t
\]

\[
P_{1,0,i}(x - \Delta t, t + \Delta t) = P_{1,0,i}(x,t)(1 - \lambda\Delta t) + p\lambda\Delta t P_{1,0,i-1}(x,t)
\]

\[
+ (j + 1)vP_{0,j+1}(t)s_{1}(x)\Delta t + P_{2,1,i}(0,t)s_{1}(x)\Delta t + \lambda s_{1}(x)P_{0,i}(t)\Delta t
\]

\[
P_{1,i,j}(x - \Delta t,t + \Delta t) = P_{1,i,j}(x,t)(1 - \lambda\Delta t) + \lambda p\Delta t P_{1,i,j-1}(x,t) + \lambda q\Delta t P_{1,i,j-1}(x,t)
\]

\[
+ P_{1,i,j-1}(0,t)s_{1}(x)\Delta t
\]

\[
P_{2,0,i}(x - \Delta t, t + \Delta t) = P_{2,0,i}(x,t)(1 - \lambda\Delta t) + \lambda p\Delta t P_{2,0,i-1}(x,t) + P_{1,0,i}(0,t)s_{2}(x)\Delta t
\]

\[
P_{2,i,j}(x - \Delta t, t + \Delta t) = P_{2,i,j}(x,t)(1 - \lambda\Delta t) + \lambda p\Delta t P_{2,i,j-1}(x,t) + \lambda q\Delta t P_{2,i,j-1}(x,t)
\]

\[
+ P_{1,i,j}(0,t)s_{2}(x)\Delta t
\]

The Steady State equations of the above equations are,

\[
(\lambda + jv) P_{0,i} = (1 - \beta)P_{2,0,i}(0) + \beta P_{2,0,i-1}(0) \tag{5.1}
\]

\[
- \frac{d}{dx}P_{1,0,i}(x) = -\lambda P_{1,0,i}(x) + \lambda P_{0,i}(x) s_{1}(x) + (j + 1)vP_{0,j+1}(x)s_{1}(x)
\]

\[
+ \lambda p P_{1,0,i-1}(x) + P_{1,i,j}(0)s_{1}(x) \tag{5.2}
\]

\[
- \frac{d}{dx}P_{1,i,j}(x) = -\lambda P_{1,i,j}(x) + \lambda pP_{1,i,j-1}(x) + \lambda qP_{1,i,j-1}(x) + P_{2,j+1}(0)s_{2}(x)
\]

\[
+ P_{1,i,j-1}(0)s_{2}(x) \tag{5.3}
\]

\[
- \frac{d}{dx}P_{2,0,i}(x) = -\lambda P_{2,0,i}(x) + \lambda pP_{2,0,i-1}(x) + P_{1,0,i}(0)s_{2}(x) \tag{5.4}
\]

\[
- \frac{d}{dx}P_{2,i,j}(x) = -\lambda P_{2,i,j}(x) + \lambda pP_{2,i,j-1}(x) + \lambda qP_{2,i,j-1}(x) + P_{1,i,j-1}(0)s_{2}(x) \tag{5.5}
\]

Let \(LST(P_{1,i,j}(x)) = \tilde{P}_{1,i,j}(\theta)\); \(LST(P_{2,i,j}(x)) = \tilde{P}_{2,i,j}(\theta)\); \(i \geq 0; j \geq 0\)
Taking LST of steady state equations (5.2), (5.3) (5.4) and (5.5), we have,

\[(\theta - \lambda) \tilde{P}_{i,0,j}(\theta) = P_{i,0,j}(0) - \lambda p \tilde{P}_{i,1,j-1}(\theta) - \lambda P_{0,j} \tilde{S}_1(\theta) - (j + 1)v P_{0,j+1} \tilde{S}_1(\theta) - P_{2,1,j}(0) \tilde{S}_1(\theta) \] (5.6)

\[(\theta - \lambda) \tilde{P}_{1,i,j}(\theta) = P_{2,j}(0) - \lambda p \tilde{P}_{1,i+1,j-1}(\theta) - \lambda q \tilde{P}_{1,i-1,j}(\theta) - P_{2,j+1}(0) \tilde{S}_1(\theta) \] (5.7)

\[(\theta - \lambda) \tilde{P}_{2,0,j}(\theta) = P_{2,0,j}(0) - \lambda p \tilde{P}_{2,0,j-1}(\theta) - \lambda P_{1,0,j} \tilde{S}_1(\theta) \] (5.8)

\[(\theta - \lambda) \tilde{P}_{2,i,j}(\theta) = P_{2,i,j}(0) - \lambda p \tilde{P}_{2,i-1,j-1}(\theta) - \lambda q \tilde{P}_{2,i+1,j}(\theta) - P_{2,i+1,j}(0) \tilde{S}_1(\theta) \] (5.9)

Now, to obtain the steady state probability generating function (PGF) of the number of customers in orbit at an arbitrary time epoch, the following probability generating functions of complex $z_2$ are defined:

\[P_0(z_2) = \sum_{j=0}^{\infty} P_{0,j} z_2^j; \]

\[\tilde{P}_{n,i}(z_2,\theta) = \sum_{j=0}^{\infty} \tilde{P}_{n,i,j}(\theta) z_2^j; \quad P_{n,i}(z_2,0) = \sum_{j=0}^{\infty} P_{n,i,j}(0) z_2^j; \quad n=1,2; \quad i \geq 0\]

Using equations (5.1), (5.6) - (5.9) and above PGFs, the following equations are obtained as,

\[\lambda P_0(z_2) + vz_2 P_0'(z_2) = (1 - \beta + \beta z_2) P_{2,0}(z_2,0) \] (5.10)

\[\theta - \lambda + \lambda pz_2 \tilde{P}_{1,0}(z_2,\theta) = P_{1,0}(z_2,0) - \lambda P_0(z) \tilde{S}_1(\theta) - v P_0'(z) \tilde{S}_1(\theta) - P_{2,1}(z_2,0) \tilde{S}_1(\theta) \] (5.11)

\[\theta - \lambda + \lambda pz_2 \tilde{P}_{1,i}(z_2,\theta) = P_{1,i}(z_2,0) - q \lambda \tilde{P}_{1,i+1}(z_2,\theta) \tilde{S}_1(\theta) - P_{2,i}(z_2,0) \tilde{S}_1(\theta) \] (5.12)

\[\theta - \lambda + \lambda pz_2 \tilde{P}_{2,0}(z_2,\theta) = P_{2,0}(z_2,0) - P_{1,0}(z_2,0) \tilde{S}_2(\theta) \] (5.13)

\[\theta - \lambda + \lambda pz_2 \tilde{P}_{2,i}(z_2,\theta) = P_{2,i}(z_2,0) - P_{1,i}(z_2,0) \tilde{S}_2(\theta) - q \lambda \tilde{P}_{2,i+1}(z_2,\theta) \] (5.14)
Next, to obtain the steady state probability generating function (PGF) of the number of customers in priority group at an arbitrary time epoch, the following probability generating functions of complex $z_1$ are defined,

$$\tilde{P}_n(z_1, \theta) = \sum_{i=0}^{\infty} \tilde{P}_n(z_1, \theta)z_1^i ; \quad P_n(z_1, \theta) = \sum_{i=0}^{\infty} P_n(z_1, \theta)z_1^i ; \quad n = 1, 2$$

Using the equations (5.11) – (5.14) and above PGFs the following equations are obtained as,

$$(\theta - \lambda + \lambda p z_2 + \lambda q z_1)\tilde{P}_1(z_1, z_2, \theta) = P_1(z_1, z_2, 0) - \lambda P_0(z)\bar{S}_1(\theta) - v P_0'(z)\bar{S}_1(\theta)$$
$$- \frac{\bar{S}_1(\theta)}{z_1} P_2(z_1, z_2, 0) + \frac{\bar{S}_1(\theta)}{z_1} P_{2,0}(z_2, 0) \quad (5.15)$$

$$(\theta - \lambda + \lambda p z_2 + \lambda q z_1)\tilde{P}_2(z_1, z_2, \theta) = P_2(z_1, z_2, 0) - P_1(z_1, z_2, 0)\bar{S}_2(\theta) \quad (5.16)$$

By choosing, $\theta = \lambda - \lambda p z_2 - \lambda q z_1$, the equations (5.15) and (5.16) become as follows,

$$P_1(z_1, z_2, 0) = \frac{\bar{S}_1(\lambda - \lambda p z_2 - \lambda q z_1)}{z_1} P_2(z_1, z_2, 0) - \frac{\bar{S}_1(\lambda - \lambda p z_2 - \lambda q z_1)}{z_1} P_{2,0}(z_2, 0)$$
$$+ \frac{\bar{S}_1(\lambda - \lambda p z_2 - \lambda q z_1)}{z_1} \left[v P_0'(z_2) + \lambda P_0(z_2)\right] \quad (5.17)$$

$$P_2(z_1, z_2, 0) = \frac{\bar{S}_2(\lambda - \lambda p z_2 - \lambda q z_1)}{z_1} P_1(z_1, z_2, 0) \quad (5.18)$$

From the equations (5.17) and (5.18), we have,

$$\left(z_1 - \bar{S}_1(\lambda - \lambda p z_2 - \lambda q z_1)\bar{S}_2(\lambda - \lambda p z_2 - \lambda q z_1)\right)P_1(z_1, z_2, 0)$$
$$= -\bar{S}_1(\lambda - \lambda p z_2 - \lambda q z_1) P_{2,0}(z_2, 0) + z_1\bar{S}_1(\lambda - \lambda p z_2 - \lambda q z_1) \left[v P_0'(z_2) + \lambda P_0(z_2)\right] \quad (5.19)$$

Now consider the function

$$f(z_1, z_2) = z_1 - \frac{\bar{S}_1(\lambda - \lambda p z_2 - \lambda q z_1)}{z_1} \bar{S}_2(\lambda - \lambda p z_2 - \lambda q z_1)$$
For each fixed $z_2$ with $|z_2| < 1$, regard $f(z_1, z_2)$ as a function $z_1$. On the unit circle $|z_1| = 1$, we see that $\text{Rd}(\lambda - p\bar{z}_2 - q\bar{z}_1) = \lambda(1 - p\text{Rd}(z_2) - q\text{Rd}(z_1)) > 0$. It is easy to see that $|\tilde{S}_1(\theta)\tilde{S}_2(\theta)| < 1$ for $\text{Re}(\theta) > 0$. Thus we must have $|\tilde{S}_1(\lambda - \lambda p z_2 - \lambda q z_1)\tilde{S}_2(\lambda - \lambda p z_2 - \lambda q z_1)| < |z_1|_{|z_1| = 1}$. By Roche's theorem it follows that for each $z_2$ with $|z_2| < 1$; there is a unique solution $z_1 = \phi(z_2)$ of the equation $f(z_1, z_2) = 0$ in the unit circle, i.e.,

$$f(\phi(z_2), z_2) = \phi(z_2) - \tilde{S}_1(\lambda - \lambda p z_2 - \lambda q \phi(z_2))\tilde{S}_2(\lambda - \lambda p z_2 - \lambda q \phi(z_2)) = 0$$

On the other hand, since

$$\left. \frac{\partial f(z_1, z_2)}{\partial z_1} \right|_{z_1 = 0, z_2 = 0} = 1 - q\rho > 0 \text{ where } \rho = \lambda[E(S_1) + E(S_2)]$$

We conclude by the implicit function theorem that $z_1 = \phi(z_2)$ is analytic on $|z_2| < 1$ and is continuous $z_2 = 1$ and $\phi(1) = 1$. We need to know the first and second derivative of $\phi(z_1)$ at $z_2 = 1$ for later use. It is easily seen that,

$$\phi'(1) = \frac{p\rho}{1 - q\rho}; \phi''(1) = \frac{p^2\lambda^2}{(1 - q\rho)^3}$$

(5.20)

Now, substituting $z_1 = \phi(z_2)$ in the equation (5.19), we have,

$$P_{z_0}(0, z_2) = v\phi(z_2)P_{0}(z_2) + \lambda\phi(z_2)P_{\phi}(z_2)$$

(5.21)

By equating the equation (5.10) and (5.21), we get,

$$\frac{P_{0}(z_2)'}{P_{0}(z_2)} = \frac{\lambda}{v} \left\{ \frac{1 - \phi(z_2)(1 - \beta + \beta z_2)}{\phi(z_2)(1 - \beta + \beta z_2) - z_2} \right\}$$

(5.22)
whose solution is,

\[ P_0(z_2) = P_0(1) \exp \left\{ -\lambda \left( \frac{1 - \phi(u)(1 - \beta + \beta u)}{v} \right) \right\} \tag{5.23} \]

Substituting (5.23) and (5.24) into the equation (5.21), yields,

\[ P_{2,0}(0, z_2) = \lambda \phi(z_2) P_0(z_2) \left( \frac{1 - z_2}{\phi(z_2)(1 - \beta + \beta z_2) - z_2} \right) \tag{5.24} \]

From the equations (5.23), (5.24) and considering \( \theta = 0 \), the equations (5.15) and (5.16) become,

\[ (-\lambda + p\lambda z_2 + q\lambda z_2) \bar{P}_0(0, z_1, z_2) \]
\[ = \frac{\lambda P_0(z_2)\left( S_i(\lambda - \lambda p z_2 - \lambda q z_2) - 1 \right)}{z_2 - S_i(\lambda - \lambda p z_2 - \lambda q z_2) S_i(\lambda - \lambda p z_2 - \lambda q z_2)} \left( \frac{1 - z_2}{\phi(z_2)(1 - \beta + \beta z_2) - z_2} \right) \tag{5.25} \]

\[ (-\lambda + p\lambda z_2 + q\lambda z_2) \bar{P}_2(0, z_1, z_2) \]
\[ = \frac{\lambda P_2(z_2)\left( S_i(\lambda - \lambda p z_2 - \lambda q z_2) - 1 \right) S_i(\lambda - \lambda p z_2 - \lambda q z_2)}{z_2 - S_i(\lambda - \lambda p z_2 - \lambda q z_2) S_i(\lambda - \lambda p z_2 - \lambda q z_2)} \left( \frac{1 - z_2}{\phi(z_2)(1 - \beta + \beta z_2) - z_2} \right) \tag{5.26} \]

From the equations (5.25) and (5.26), we obtain the

\[ \sum_{j=1}^{3} \bar{P}_j(0, z_1, z_2) \]
\[ = \frac{P_j(z_2)\left( S_i(\lambda - \lambda p z_2 - \lambda q z_2) - 1 \right) S_i(\lambda - \lambda p z_2 - \lambda q z_2)}{z_2 - S_i(\lambda - \lambda p z_2 - \lambda q z_2) S_i(\lambda - \lambda p z_2 - \lambda q z_2)} \left( \frac{1 - z_2}{\phi(z_2)(1 - \beta + \beta z_2) - z_2} \right) \tag{5.27} \]

From the normalizing condition, \( P_0(1) + \sum_{j=1}^{3} P_j(0,1,1) = 1 \), we immediately obtain the steady state condition as,

\[ \rho < \frac{1 - \beta}{p + (1 - \beta)q} \tag{5.28} \]
Moreover, we obtain \( \sum_{i=1}^{2} \bar{P}_i(0,1,1) = \frac{\rho P_0(1)}{1 - \rho - \beta(1 - q \rho)} \) as the probability that the server is busy and \( P_0(1) = 1 - \frac{\rho}{1 - \beta(1 - q \rho)} \) as the probability that server is idle.

The stationary distribution of this model has the following generating functions:

\[
P_0(z_2) = \left(1 - \frac{\rho}{1 - \beta(1 - q \rho)}\right) \exp \left(-\frac{\lambda}{v} \int_{z_2}^{1} \frac{1 - \phi(u)(1 - \beta + \beta u)}{\phi(u)(1 - \beta + \beta u) - u} \, du\right)
\]

\[
\sum_{i=1}^{2} \bar{P}_i(0, z_1, z_2) = \frac{(\bar{S}_1(\lambda - \lambda p z_2 - \lambda q z_2) \bar{S}_2(\lambda - \lambda p z_2 - \lambda q z_2) - 1)}{(z_1 - \bar{S}_1(\lambda - \lambda p z_2 - \lambda q z_2) \bar{S}_2(\lambda - \lambda p z_2 - \lambda q z_2) + \rho z_2 + q z_2 - 1)} \left(1 - \frac{\rho}{1 - \beta(1 - q \rho)}\right) \exp \left(-\frac{\lambda}{v} \int_{z_2}^{1} \frac{1 - \phi(u)(1 - \beta + \beta u)}{\phi(u)(1 - \beta + \beta u) - u} \, du\right)
\]

Equation (5.30) represents the joint generating function of the priority group and retrial group.

5.3.1 Remarks

(1) The probability generating function of the number of customers in the priority queue is given by

\[
R(z) = P_0(1) + \sum_{i=1}^{2} \bar{P}_i(0, z, 1) = \left(1 - \frac{\rho}{1 - \beta(1 - q \rho)}\right) \left[1 + \frac{\rho q}{q(1 - \rho - \beta(1 - q \rho))} \frac{\bar{S}_1(q \lambda - q \lambda z) \bar{S}_2(q \lambda - q \lambda z) - 1}{z - \bar{S}_1(q \lambda - q \lambda z) \bar{S}_2(q \lambda - q \lambda z)}\right]
\]

(2) The probability generating function of the number of customers in the retrial group is given by
Using the PGFs (5.31) and (5.32), some performance measures for the system at the stationary regime are obtained as follows

### 5.4 PERFORMANCE MEASURES

Some useful results of our model are listed below:

a) **The mean number of customers in the orbit (retrial group)**

\[
E[N_2] = L_t \frac{d}{dz} S(z) = \frac{q^{\lambda^2 \beta_2} p}{2(1 - \beta - \beta(1-q\rho))(1-\beta(1-q\rho))} + \frac{\lambda \rho (p \rho + \beta(1-q\rho))}{\nu(1 - \rho - \beta(1-q\rho))(1 - \beta(1-q\rho))}
\]

(5.33)

where \( \beta_2 = E[S_1^2] + E[S_2^2] + 2E[S_1]E[S_2] \)

b) **The mean number of customers in priority group**

\[
E[N_1] = L_t \frac{d}{dz} R(z) = \frac{q^{\lambda^2 \beta_2}}{2(1 - \beta - \beta(1-q\rho))(1-\beta(1-q\rho))}
\]

(5.34)

where \( \beta_2 = E[S_1^2] + E[S_2^2] + 2E[S_1]E[S_2] \)

c) **The probability that the server is idle is**

\[
P_0(1) = 1 - \frac{\rho}{1 - \beta(1-q\rho)}
\]

(5.35)
d) The probability that server is busy is

\[
\sum_{i=1}^{2} \bar{P}_i(0,1,1) = \frac{\rho}{1 - \beta(1 - q\rho)} \tag{5.36}
\]

e) Mean waiting time in priority group \((W_1)\) and Mean waiting time in retrial group \((W_2)\)

If an arriving customer who obtains service immediately is regarded as a member of the priority queue and the orbit with probability \(q\) and \(p\) respectively, then the mean waiting time for each group is given by

\[
W_1 = \frac{E[N_1]}{\lambda q} \tag{5.37}
\]

\[
W_2 = \frac{E[N_2]}{\lambda p} \tag{5.38}
\]

5.5 PARTICULAR CASES

In this section, some interesting particular cases are discussed.

Case I: If there is no second phase of service and no recurrent customers, then \(S(\lambda - \lambda p z - \lambda q z) = 1\) and \(\beta = 0\). So, the equations (5.29) and (5.30) are written as,

\[
P_0(z_2) = (1 - \rho) \exp\left\{ -\frac{\lambda}{v} \left( 1 - \frac{1 - \phi(u)}{\phi(u) - u} \right) \int_{z_2}^{1} \phi(u) - u \, du \right\} \tag{5.39}
\]

\[
\sum_{i=1}^{2} \bar{P}_i(0, z_1, z_2) = \frac{\left( \bar{S}(\lambda - \lambda p z_2 - \lambda q z_1) - 1 \right)}{(z_2 - S(\lambda - \lambda p z_2 - \lambda q z_1))(p z_2 + q z_1 - 1)} \left( \frac{(1 - z_2)(z_2 - \phi(z_2))}{\phi(z_2) - z_2} \right) \tag{5.40}
\]

\[
\times (1 - \rho) \exp\left\{ -\frac{\lambda}{v} \left( 1 - \frac{1 - \phi(u)}{\phi(u) - u} \right) \int_{z_2}^{1} \phi(u) - u \, du \right\}
\]

where \(\phi(z_2) = z_2\)
Equations (5.39) and (5.40) agree the generating functions of an M/G/1 retrial queue with Bernoulli schedule in the stationary regime obtained by Choi and Park (1990).

Case II: If there is no second phase of service and there is no priority group of customers then \( \tilde{S}_i(\lambda - \lambda p z_i - q z_i) = 1 \) and \( p = 1 \). So, the equation (5.32) is written as,

\[
S(z) = P_0(z) + \sum_{j=1}^{\infty} \bar{P}_j(0,1,z) \\
= \left( \frac{(\tilde{S}_i(\lambda - \lambda z) \beta - 1)(z - 1)}{\tilde{S}_i(\lambda - \lambda z)(1 - \beta + \beta z) - z} \right) P_0(z) 
\]

(5.41)

where

\[
P_0(z) = \left( 1 - \frac{\rho}{1 - \beta} \right) \exp \left\{ -\frac{\lambda}{\nu} \left[ \frac{1 - \tilde{S}_i(\lambda - \lambda u)(1 - \beta + \beta u)}{\tilde{S}_i(\lambda - \lambda u)(1 - \beta + \beta u) - u} du \right] \right\} 
\]

(5.42)

Equations (5.41) and (5.42) agree the generating functions of retrial queue with recurrent demand options obtained by Farahmand and Smith (1996).

5.6 SPECIAL CASES

Further, by specifying service time random variables as Erlang and Exponential distribution, some special cases of this model are discussed below:

Case I: Single server retrial queue with Bernoulli schedule, two phases of essential Erlangian service \( (k_1\text{-Erlang for FES, } k_2\text{-Erlang for SES}) \) and recurrent demand option
It is assumed that the service time is an k-Erlang with probability density function,
\[ s(x) = \frac{(k \theta)^x e^{-k \theta x}}{(k-1)!}, \quad k>0 \]; where u is the parameter.

Hence the PGF of the retrial queue size distribution and priority group are given as follows when

\[
\tilde{S}_i(\lambda - \lambda z_2 p - \lambda z_1 q) = \left( \frac{u_i k_i}{u_i k_i + \lambda (1 - z_2 p - z_1 q)} \right)^{k_i}; \quad i = 1, 2.
\]

\[
P_0(z_2) = \left(1 - \frac{\rho}{1 - \beta (1 - q \rho)}\right) \exp \left\{ -\frac{\lambda}{v} \int_{z_2}^{1 - \phi(u)(1 - \beta + \beta u) - u} 1 - \phi(u)(1 - \beta + \beta u) - u \right\} \tag{5.43}
\]

\[
\sum_{i=1}^{2} \tilde{p}(0, z_1, z_2)
\]

\[
= \left( \frac{u_1 k_1}{u_1 k_1 + \lambda (1 - z_2 p - z_1 q)} \right)^{k_1} \left( \frac{u_2 k_2}{u_2 k_2 + \lambda (1 - z_2 p - z_1 q)} \right)^{k_2} - 1
\]

\[
\times 1 - \frac{\rho}{1 - \beta (1 - q \rho)} \exp \left\{ -\frac{\lambda}{v} \int_{z_2}^{\phi(u)(1 - \beta + \beta u) - u} 1 - \phi(u)(1 - \beta + \beta u) - u \right\} \tag{5.44}
\]

The steady state condition is obtained as \( \rho = \lambda \left( \frac{1}{u_1} + \frac{1}{u_2} \right) < \frac{1 - \beta}{1 - \beta q} \)

**Case II:** Single server retrial queue with Bernoulli schedule and two phases of Hyper Exponential service time and recurrent demand option

Considering the case of Hyper Exponential service time random variable, the pdf of Hyper Exponential service time is given as follows,
\[ s(x) = cu_1 e^{-u_1 x} + (1 - c)u_2 e^{-u_2 x}. \]
Then \( \bar{S}_i(\lambda - \lambda z_p - \lambda z_q) = \left( \frac{u_{1,i} c}{u_{1,i} + \lambda (1 - p z_2 - q z_1)} + \frac{u_{2,i} (1 - c)}{u_{2,i} + \lambda (1 - p z_2 - q z_1)} \right); i=1,2 \)

Hence the Probability Generating Functions of the model are given by

\[
P_0(z_2) = \left( 1 - \frac{\rho}{1 - \beta (1 - q \rho)} \right) \exp \left\{ -\frac{\lambda}{v} \left( 1 - \phi(u)(1 - \beta + \beta u) - u \right) \right\}
\]

\[
\sum_{i=1}^{2} \frac{1}{\bar{S}_i(\lambda - \lambda p z_2 - \lambda q z_1) \bar{S}_i(\lambda - \lambda p z_2 - \lambda q z_1)} \left( \frac{(1 - z_2)(z_2 - \phi(z_2))}{(z_1 - \lambda p z_2 + \lambda q z_1) (z_1 - \lambda p z_2 + \lambda q z_1)(p z_2 + q z_1 - 1) (\phi(z_2)(1 - \beta + \beta z_2) - z_2)} \right)
\]

The steady state condition is given as \( \rho = \lambda \left( \sum_{i=1}^{2} \frac{c + 1 - c}{u_{1,i} u_{2,i}} \right) < \frac{1 - \beta}{1 - \beta q} \)

### 5.7 NUMERICAL RESULTS

In this section, some numerical results are obtained to study the effect of Bernoulli schedule probability 'p' and the effect of recurrent probability \( \beta \) on mean orbit size are obtained. Mobile cellular communication systems are governed by numerous base stations, each of them with a specific influence zone (called cell). These cells are overlapped and all together cover the service area. The base station in each cell can serve call which are being served in contiguous cells (because are in an overlap region). The base station in a cell manages two sorts of calls: originating and handoff calls. Consequently, the base station gives priority to a handoff call over an originating call. Call arrivals follow Poisson distribution with mean arrival rate \( \lambda \). This service process is done in two phases: (i) connection establishment (FES) and (ii) transferring information (SES). After the
completion of SES, the unsatisfied customer either joins the retrial group with probability $\beta$ only if there are no customers in priority group or leaves the system with probability $1 - \beta$. So, this system can be modeled as the single server retrial queue with Bernoulli schedule and two phases of heterogeneous service and recurrent demand option.

Hence, it is reasonable that the effect of Bernoulli schedule probability and effect of recurrent probability are compared with average number of calls in retrial group and average number of calls in priority group. Moreover, the effect of retrial rate is compared with average number of handoff calls.

In Table 5.1(a), 5.1(b) and 5.1(c) and Figure 5.1(a)–5.1(c), the mean orbit size is compared with varying values of the Bernoulli schedule probability ‘p’ and with varying retrial rate when FES and SES distribution follow Exponential, Erlangian of order two and Hyper – Exponential, respectively. It is observed that

- the average handoff call size is decreased if the retrial rate ‘v’ increases.

- the average handoff call size is increased if the Bernoulli vacation probability ‘p’ is increased.

In Table 5.2 and Figure 5.2, the mean number of priority call size is compared with varying values of Bernoulli schedule ‘p’ when FES and SES distribution follow Exponential, Erlangian of order two and Hyper – Exponential, respectively. It is observed that the increase in Bernoulli Schedule ‘p’ if mean number of priority calls is decreased.

Thus the theoretical development of the model is justified with the numerical results, which are consistent with the fact that
- the average handoff call size is decreased when the retrial rate \( v \) increases.

- the average handoff call size is increased when the Bernoulli vacation probability \( p \) is increased.

- the increase in Bernoulli Schedule \( p \) when mean number of priority calls is decreased.

**Table 5.1(a) Mean Orbit size (L) when FES and SES follow Exponential distribution**

(retrial rate \( v \), Bernoulli Schedule probability \( p \))

<table>
<thead>
<tr>
<th>( p )</th>
<th>( v )</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.17142857</td>
<td>0.117857</td>
<td>0.1</td>
<td>0.091071</td>
<td>0.085714</td>
</tr>
<tr>
<td>0.1</td>
<td>0.24595456</td>
<td>0.171348</td>
<td>0.14648</td>
<td>0.134045</td>
<td>0.126585</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.32066651</td>
<td>0.224017</td>
<td>0.191801</td>
<td>0.175692</td>
<td>0.166027</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.39604666</td>
<td>0.276288</td>
<td>0.236368</td>
<td>0.216409</td>
<td>0.204433</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.47251993</td>
<td>0.328525</td>
<td>0.280527</td>
<td>0.256528</td>
<td>0.242128</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.5504701</td>
<td>0.381047</td>
<td>0.324573</td>
<td>0.296335</td>
<td>0.279393</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.6302521</td>
<td>0.434138</td>
<td>0.368767</td>
<td>0.336081</td>
<td>0.316469</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.71220141</td>
<td>0.488059</td>
<td>0.413344</td>
<td>0.375987</td>
<td>0.353573</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.79664159</td>
<td>0.543051</td>
<td>0.458521</td>
<td>0.416256</td>
<td>0.390897</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.88389035</td>
<td>0.599347</td>
<td>0.504499</td>
<td>0.457075</td>
<td>0.428621</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.97426471</td>
<td>0.657169</td>
<td>0.551471</td>
<td>0.498621</td>
<td>0.466912</td>
<td></td>
</tr>
</tbody>
</table>

(FES and SES follow Exponential distribution with service rate of FES \( s_1 = 8 \) and service rate of SES \( s_2 = 16 \) )
Table 5.1(b) Mean Orbit size (L) when FES and SES follow Erlangian of order two

(retrial rate v, Bernoulli schedule probability ‘p’)

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>v</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>0</td>
<td>0.78947368</td>
<td>0.690789</td>
<td>0.657895</td>
<td>0.641447</td>
<td>0.631579</td>
</tr>
<tr>
<td>0.1</td>
<td>1.76881942</td>
<td>1.465307</td>
<td>1.364137</td>
<td>1.313551</td>
<td>1.2832</td>
</tr>
<tr>
<td>0.2</td>
<td>2.74136829</td>
<td>2.189898</td>
<td>2.006074</td>
<td>1.914162</td>
<td>1.859015</td>
</tr>
<tr>
<td>0.3</td>
<td>3.80737086</td>
<td>2.951908</td>
<td>2.666754</td>
<td>2.524177</td>
<td>2.438631</td>
</tr>
<tr>
<td>0.4</td>
<td>5.0488837</td>
<td>3.814935</td>
<td>3.403619</td>
<td>3.197961</td>
<td>3.074566</td>
</tr>
<tr>
<td>0.5</td>
<td>6.56</td>
<td>4.845714</td>
<td>4.274286</td>
<td>3.988571</td>
<td>3.817143</td>
</tr>
<tr>
<td>0.6</td>
<td>8.4717608</td>
<td>6.132966</td>
<td>5.353368</td>
<td>4.963568</td>
<td>4.729689</td>
</tr>
<tr>
<td>0.7</td>
<td>10.9894415</td>
<td>7.812861</td>
<td>6.754001</td>
<td>6.224571</td>
<td>5.906913</td>
</tr>
<tr>
<td>0.8</td>
<td>14.4666903</td>
<td>10.11805</td>
<td>8.668498</td>
<td>7.943723</td>
<td>7.508859</td>
</tr>
<tr>
<td>0.9</td>
<td>19.5796079</td>
<td>13.49183</td>
<td>11.46257</td>
<td>10.44794</td>
<td>9.839164</td>
</tr>
<tr>
<td>1</td>
<td>27.8125</td>
<td>18.90625</td>
<td>15.9375</td>
<td>14.45312</td>
<td>13.5625</td>
</tr>
</tbody>
</table>

(FES and SES follow Erlangian of order two with service rate of FES $s_1$=8 and service rate of SES $s_2$ =16)
Table 5.1(c) Mean Orbit size (L) when FES and SES follow Hyper Exponential

(retrial rate v, Bernoulli schedule probability 'p')

<table>
<thead>
<tr>
<th>p</th>
<th>v</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.0628</td>
<td>0.0368</td>
<td>0.0282</td>
<td>0.0239</td>
<td>0.0213</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0766</td>
<td>0.0472</td>
<td>0.0373</td>
<td>0.0324</td>
<td>0.0294</td>
<td>0.0263</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0905</td>
<td>0.0574</td>
<td>0.0464</td>
<td>0.0408</td>
<td>0.0372</td>
<td>0.0334</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1044</td>
<td>0.0676</td>
<td>0.0553</td>
<td>0.0492</td>
<td>0.0455</td>
<td>0.0419</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1182</td>
<td>0.0777</td>
<td>0.0645</td>
<td>0.0574</td>
<td>0.0534</td>
<td>0.0497</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1321</td>
<td>0.0878</td>
<td>0.0731</td>
<td>0.0655</td>
<td>0.0611</td>
<td>0.0571</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1461</td>
<td>0.0979</td>
<td>0.0817</td>
<td>0.0736</td>
<td>0.0689</td>
<td>0.0649</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1600</td>
<td>0.1080</td>
<td>0.0904</td>
<td>0.0820</td>
<td>0.0767</td>
<td>0.0730</td>
</tr>
<tr>
<td>0.8</td>
<td>0.1741</td>
<td>0.1182</td>
<td>0.0992</td>
<td>0.0897</td>
<td>0.0840</td>
<td>0.0803</td>
</tr>
<tr>
<td>0.9</td>
<td>0.1882</td>
<td>0.1281</td>
<td>0.1076</td>
<td>0.0976</td>
<td>0.0915</td>
<td>0.0878</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.2044</td>
<td>0.1378</td>
<td>0.1163</td>
<td>0.1055</td>
<td>0.0990</td>
</tr>
</tbody>
</table>

(FES and SES follow Hyper Exponential distribution \( \lambda = \alpha e^{\beta x} + \beta e^{\alpha (1-c)} \) with service rate of FES \( s_1 = 8 \) and service rate of SES \( s_2 = 16 \))

Table 5.2 Mean Size of Priority Group

<table>
<thead>
<tr>
<th>p</th>
<th>Exponential dist.</th>
<th>Erlangian of order 2</th>
<th>Hyper Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
<td>1.513158</td>
<td>0.05561</td>
</tr>
<tr>
<td>0.1</td>
<td>0.171279</td>
<td>1.064377</td>
<td>0.049225</td>
</tr>
<tr>
<td>0.2</td>
<td>0.145349</td>
<td>0.78125</td>
<td>0.043054</td>
</tr>
<tr>
<td>0.3</td>
<td>0.121775</td>
<td>0.585199</td>
<td>0.037084</td>
</tr>
<tr>
<td>0.4</td>
<td>0.10021</td>
<td>0.440501</td>
<td>0.031303</td>
</tr>
<tr>
<td>0.5</td>
<td>0.080367</td>
<td>0.328571</td>
<td>0.025699</td>
</tr>
<tr>
<td>0.6</td>
<td>0.062013</td>
<td>0.238787</td>
<td>0.020262</td>
</tr>
<tr>
<td>0.7</td>
<td>0.044951</td>
<td>0.164631</td>
<td>0.014982</td>
</tr>
<tr>
<td>0.8</td>
<td>0.029017</td>
<td>0.101878</td>
<td>0.009851</td>
</tr>
<tr>
<td>0.9</td>
<td>0.014073</td>
<td>0.04767</td>
<td>0.004859</td>
</tr>
<tr>
<td>1</td>
<td>1.52E-17</td>
<td>4.99E-17</td>
<td>5.32E-18</td>
</tr>
</tbody>
</table>
Figure 5.1 (Continued)
Figure 5.1 Mean orbit size vs Bernoulli schedule probability when
(a) FES and SES follow Exponential Distribution (b) FES and SES follow Erlang 2 Distribution (c) FES and SES follow Hyper Exponential

Figure 5.2 Mean Priority group Size Vs Bernoulli schedule probability