Chapter 4

A perishable inventory-queueing model with delayed vacation, negative and impatient customers

4.1 Introduction

In numerous realistic situations the queueing-inventory systems are used to offer central framework for capable design and analysis together with different technical systems also predictions the performance of systems such as mean reorder rate, mean perishable rate, mean number of the customers in the waiting hall, fraction of time server is on vacations and so forth. Queueing-inventory systems with vacations have also found broad applicability in distribution centre, manufacturing system and several other engineering systems. Such queueing-inventory situations may happen in numerous valid time systems such as manufacturing system, etc. Vacation models are give details by their arrangement regulations, according to which when a service stops, a vacation starts. These forecasts facilitate us to look forward to positions of the system and to take correct measures to curtail the queue. However, in queueing models with rest periods (vacations) have been investigated extensively. For a related bibliography see [41, 42, 140, 141, 142].

Daniel and Ramanathan first investigated the concept of rest periods in inventory system with two servers [36]. In [37], they have investigated an inventory system in which the server takes vacation when the stock level is zero. Also, they assumed that the demands that occurred during stock-out period or server vacation period were lost. Sivakumar [129] formulated an inventory model with multiple vacations and infinite orbite. He assumed exponential distribution for inter-demand times, lead times, inter-retrial times and server vacation times. Padmavathi et.al [102] analyzed a retrial inventory system with single and modified multiple vacation for server. They considered two models which different in the way that server go for vacation. Jeganathan et.al [67] studied a perishable inventory system with server interruptions, multiple server vacations and N-Policy. The service starts only when the customer level reaches a prefixed level N, starting from the epoch at which no customer is left behind in the system. Very recently, Yadavalli and Jeganathan [157] analyzed a perishable inventory system with two heterogeneous servers including one with multiple vacations and retrial customers.

In this chapter, we consider a perishable inventory-queueing system with delayed vacation. The customers arriving to the service station are classified as ordinary (positive or regular) and negative customers. The arrival of ordinary customers to the service station increase the queue length by one and the arrival of negative customer to the service station causes one ordinary customer to be removed if any. If the server finds waiting room is empty at a service completion epoch, he will wait for some more time in the system, which is called changeover (delay) time. If the server has been idle for a period of time, the server begins an exponentially distributed vacation which is repeated as long as at least one customer in the system. The various measures of system performance are computed in the steady state case and the long-run total expected cost rate is calculated.

The rest of the chapter is organized as follows. In the next section, we describe the mathematical model and the notations used in this paper are defined. Analysis of the model and the steady state solution of the model are proposed in section 4.3. Some key system performance measures are obtained in section 4.4. Section 4.5 is dedicated to cost analysis and sensitivity investigation.

4.2 Mathematical Model

Consider the following queueing-inventory model consisting of a single server and two types of customers: positive and negative.

The maximum capacity of the inventory is S and the replenishment of the inventory is instantaneous that is $(0, S)$ ordering policy with the reorder rate is infinite. Natural life span of each commodity has negative exponential distribution with parameter $\gamma > 0$. There is a finite waiting room for the customers and the capacity of the waiting room is N including the one at the service point. The customers arrive according to a Poisson process with parameter $\lambda > 0$. The customers are of two kinds: positive and negative. An arriving customer turns out to be a positive customer with probability p and a negative customer with probability $q = (1 - p)$. An arrival of a negative customer eliminates one of the positive customers from the waiting room including the one at the service point. Note that the negative customers affect the model only when waiting room is non-empty. A customer is lost when he finds the waiting hall is full at the arriving instant. Each customer requires a single homogeneous item, having random duration of service time which follows exponential distribution with parameter $\mu > 0$. However, it is not essential that inventory is provided to the customer at the end of his service. More specifically, the item is served with probability r_1 at the end of a service and with probability $r_2 = 1 - r_1$, $(0 \leq r_1 \leq 1)$ the item is not delivered to the customer. If the server finds waiting room is empty at a service completion epoch, he will wait for some more time in the system, which is called changeover (delay) time. The delay time is exponentially distributed with parameter $\beta > 0$. If the delay time is completed before the arrival of a customer, the server begins a vacation whose length is exponentially distributed with parameter $\eta > 0$. At the end of a vacation period, service commences if a customer is present in the waiting hall. Otherwise, the server begins another vacation (multiple vacations). After joining the customers in the waiting room, in this case that the server is busy or on vacation, each customer will wait a random period for service to begin before he/she impatient and leave the waiting room without getting service and this time is distributed as negative exponential with parameter $\theta > 0$. All aforesaid random variables are independent of each other.

Let $X_1(t)$ and $X_3(t)$ respectively, indicate the on-hand inventory level and the number of customers in the waiting room at time t. Further, let the status of the server $X_2(t)$ be defined as follows:

$$X_2(t) = \begin{cases} 0, & \text{if the servers is in delay period (changeover time) at time t,} \\ 1, & \text{if the servers is on vacation at time t,} \\ 2, & \text{if the server is busy at time t,} \end{cases}$$

Then, the behavior of the model can be expressed by a three-dimensional stochastic process $Y(t) = \{(X_1(t), X_2(t), X_3(t)), t \geq 0\}$ with the state space H is defined by

$$\begin{aligned} H &= \{(X_1(t), 0, 0) \cup (X_1(t), 1, 0) \cup (X_1(t), 1, X_3(t)) \cup (X_1(t), 2, X_3(t)) \\ &\quad \mid 1 \leq X_1(t) \leq S; 1 \leq X_3(t) \leq N\} \end{aligned}$$

According to the statements of Poisson process and exponential service time, $\{Y(t), t \geq 0\}$ is a homogeneous CTMC (continuous time Markov chain). We denote its steady-state distribution by $\phi^{(i_1,i_2,i_3)}$:

$$\phi^{(i_1,i_2,i_3)} = \lim_{t\to\infty} Pr[X_1(t) = i_1, X_2(t) = i_2, X_3(t) = i_3 | X_1(0), X_2(0), X_3(0)]$$

In the sequel, $\mathbf{0}$ refers to the zero matrix, $k \in V_i^j$ denotes $k = i, i+1, \ldots j$ and δ is the delta function defined by $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$.

4.3 Analysis of the Model

In this section, we create the system of linear equations for the steady state distribution, and derive a recursive algorithm to compute its solution.

4.3.1 Linear equations for the steady state distribution

The steady-state distribution of the CTMP $Y(t) = \{(X_1(t), X_2(t), X_3(t)), t \geq 0\}$ satisfies the following Chapman-Kolmogorov equations:

For $1 \leq i_1 \leq S, \quad i_2 = 0, \quad i_3 = 0$

$$(p\lambda + \beta + i_1\gamma)\phi^{(i_1,i_2,i_3)} = \bar{\delta}_{i_1S}(i_1+1)\gamma\phi^{(i_1+1,i_2,i_3)} + \delta_{i_1S}\gamma\phi^{(i_1-(i_1-1),i_2,i_3)} + \tag{4.1}$$
$$(q\lambda + (i_3+1)\theta + r_2\mu)\phi^{(i_1,2,i_3+1)} + \bar{\delta}_{i_1S}r_1\mu\phi^{(i_1+1,2,i_3+1)} + \delta_{i_1S}r_1\mu\phi^{(i_1-(i_1-1),2,i_3+1)}$$

For $1 \leq i_1 \leq S, \quad i_2 = 1, \quad 0 \leq i_3 \leq N,$

$$\begin{aligned}(\bar{\delta}_{i_3N}p\lambda + \delta_{i_30}(q\lambda + i_3\theta + \eta) + i_1\gamma)\phi^{(i_1,i_2,i_3)} &= \bar{\delta}_{i_1S}(i_1+1)\gamma\phi^{(i_1+1,i_2,i_3)} + \\ \delta_{i_30}\beta\phi^{(i_1,i_2-1,i_3)} + \delta_{i_1S}\gamma\phi^{(i_1-(i_1-1),i_2,i_3)} &+ \bar{\delta}_{i_3N}(q\lambda + (i_3+1)\theta + \\ \bar{\delta}_{i_30}p\lambda\phi^{(i_1,i_2,i_3-1)})\phi^{(i_1,i_2,i_3+1)}\end{aligned} \tag{4.2}$$

For $1 \leq i_1 \leq S, \quad i_2 = 2, \quad 0 \leq i_3 \leq N,$

$$\begin{aligned}(\bar{\delta}_{i_3N}p\lambda + q\lambda + i_3\theta + \mu + i_1\gamma)\phi^{(i_1,i_2,i_3)} &= \bar{\delta}_{i_1S}(i_1+1)\gamma\phi^{(i_1+1,i_2,i_3)} + \\ \delta_{i_1S}\gamma\phi^{(i_1-(i_1-1),i_2,i_3)} + \delta_{i_31}p\lambda\phi^{(i_1,0,i_3-1)} &+ \bar{\delta}_{i_31}p\lambda\phi^{(i_1,i_2,i_3-1)} + \\ \bar{\delta}_{i_3N}(q\lambda + (i_3+1)\theta + r_2\mu)\phi^{(i_1,i_2,i_3+1)} &+ \bar{\delta}_{i_1S}\bar{\delta}_{i_31}r_1\mu\phi^{(i_1+1,i_2,i_3)} + \\ \delta_{i_1S}\bar{\delta}_{i_31}r_1\mu\phi^{(i_1-(i_1-1),i_2,i_3)} &+ \eta\phi^{(i_1,i_2-1,i_3)}\end{aligned} \tag{4.3}$$

In order to write the steady-state equations in term of matrix form, we rearrange states in H in the following lexicographical ordering:

$$\begin{aligned}(\mathbf{i_1}) &= \{(\mathbf{i_1}, \mathbf{0}), (\mathbf{i_1}, \mathbf{1}), (\mathbf{i_1}, \mathbf{2})\}, & i_1 \in V_1^S, \\ (\mathbf{i_1}, \mathbf{0}) &= \{(i_1, 0, 0)\}, & i_1 \in V_1^S, \\ (\mathbf{i_1}, \mathbf{1}) &= \{(i_1, 1, 0), (i_1, 1, 1), \dots (i_1, 1, N)\}, & i_1 \in V_1^S, \\ (\mathbf{i_1}, \mathbf{2}) &= \{(i_1, 2, 1), (i_1, 2, 2), \dots (i_1, 2, N)\}, & i_1 \in V_1^S,\end{aligned}$$

and define the corresponding vectors

$$\begin{aligned}
\mathbf{\Phi}^{(\boldsymbol{i_1})} &= (\mathbf{\Phi}^{(\mathbf{i_1,0})}, \mathbf{\Phi}^{(\mathbf{i_1,1})}, \mathbf{\Phi}^{(\mathbf{i_1,2})}), \quad \mathbf{i_1} \in \mathbf{V_1^S};\\
\mathbf{\Phi}^{(\mathbf{i_1,0})} &= (\phi^{(i_1,0,0)}), \quad i_1 \in V_1^S;\\
\mathbf{\Phi}^{(\mathbf{i_1,1})} &= (\phi^{(i_1,1,0)}, \phi^{(i_1,1,1)}, \dots, \Phi^{(i_1,1,N)}), \quad i_1 \in V_1^S;\\
\mathbf{\Phi}^{(\mathbf{i_1,2})} &= (\phi^{(i_1,2,1)}, \dots, \phi^{(i_1,2,N)}), \quad i_1 \in V_1^S;
\end{aligned}$$

Then, $\phi^{(i_1,0,0)}$ is the probability that there are $i_1(i_1 \in V_1^S)$, items in the inventory, server is idle and $i_3(i_3 \in V_1^N)$ customers in the waiting room, $\phi^{(i_1,1,i_3)}$ is the probability that there are $i_1 \geq 1$ items in the inventory, server is on vacation and $i_3(i_3 \in V_0^N)$ customers in the waiting room, and $\phi^{(i_1,2,i_3)}$ is the probability that there are $i_1 \geq 1$ items in the inventory, server is busy and $i_3(i_3 \in V_1^N)$ customers in the waiting room. Using these, the system of linear equations $(4.1) - (4.3)$ can be expressed as follows:

$$\mathbf{\Phi}^{(\boldsymbol{i_1})}\Delta_{i_1} + \mathbf{\Phi}^{(\boldsymbol{i_1+1})}\Lambda_{i_1+1} = \mathbf{0}, \qquad i_1 = 1, 2, \dots, S-1, \tag{4.4}$$

$$\mathbf{\Phi}^{(\boldsymbol{i_1})}\Delta_S + \mathbf{\Phi}^{(\boldsymbol{i_1-(S-1)})}\Lambda_1 = \mathbf{0}, \qquad i_1 = S, \tag{4.5}$$

Therefore, the steady state distribution $\mathbf{\Phi} = (\mathbf{\Phi}^{(\mathbf{1})}, \dots, \mathbf{\Phi}^{(\mathbf{S})})$ is the unique solution of the linear system

$$\mathbf{\Phi}\Theta = \mathbf{0} \tag{4.6}$$

and the normalization condition

$$\mathbf{\Phi e} = \sum_{(i_1,i_2,i_3)}\sum\sum \phi^{(i_1,i_2,i_3)} = 1, \tag{4.7}$$

where Θ is the infinitesimal generator (transition rate) matrix of the finite state Markov chain Y given by:

$$[\Theta]_{i_1 j_1} = \begin{array}{c} \\ \mathbf{(1)} \\ \mathbf{(2)} \\ \mathbf{(3)} \\ \vdots \\ \mathbf{(S-1)} \\ \mathbf{(S)} \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{(1)} & \mathbf{(2)} & \mathbf{(3)} & \cdots & \mathbf{(S-1)} & \mathbf{(S)} \end{array} \\ \begin{pmatrix} \Delta_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \Lambda_1 \\ \Lambda_2 & \Delta_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Lambda_3 & \Delta_3 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \Delta_{S-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \Lambda_S & \Delta_S \end{pmatrix} \end{array},$$

where all sub matrixes Δ_{i_1} and Λ_{i_1}, $i_1 \in V_1^S$ are square matrixes of order $2(N+1)$ and we can obtain the sub matrixes of Θ using the following arguments: Now, we describe the transitions in the Markov chain $Y(t)$ as follows:

- Transitions due to the arrival of positive customers:
  From $(i_1, 0, 0)$ to $(i_1, 2, 1)$ by $p\lambda$, $i_1 \in V_1^S$.
  From $(i_1, 1, i_3)$ to $(i_1, 1, i_3 + 1)$ by $p\lambda$, $i_1 \in V_1^S$, $i_3 \in V_0^{N-1}$.
  From $(i_1, 2, i_3)$ to $(i_1, 2, i_3 + 1)$ by $p\lambda$, $i_1 \in V_1^S$, $i_3 \in V_1^{N-1}$.

- Transitions due to the arrival of negative customers:
  From $(i_1, 1, i_3)$ to $(i_1, 1, i_3 - 1)$ by $q\lambda$, $i_1 \in V_1^S$, $i_3 \in V_1^N$.
  From $(i_1, 2, 1)$ to $(i_1, 0, 0)$ by $q\lambda$, $i_1 \in V_1^S$.
  From $(i_1, 2, i_3)$ to $(i_1, 2, i_3 - 1)$ by $q\lambda$, $i_1 \in V_1^S$, $i_3 \in V_2^N$.

- Transitions due to service completion consequent to which an inventoried item is served to the outgoing customer:
  From $(1, 2, 1)$ to $(S, 0, 0)$ by $r_1\mu$.
  From $(i_1, 2, 1)$ to $(i_1 - 1, 0, 0)$ by $r_1\mu$, $i_1 \in V_2^S$.

From $(1,2,i_3)$ to $(1,2,i_3-1)$ by $r_1\mu$, $i_3 \in V_2^N$.

From $(i_1,2,i_3)$ to $(i_1-1,2,i_3-1)$ by $r_1\mu$, $i_1 \in V_2^S$, $i_3 \in V_2^N$.

- Transitions due to service completion for which inventory is not served:

  From $(i_1,2,1)$ to $(i_1,0,0)$ by $r_2\mu$, $i_1 \in V_1^S$.

  From $(i_1,2,i_3)$ to $(i_1,2,i_3-1)$ by $r_2\mu$, $i_1 \in V_1^S$, $i_3 \in V_2^N$.

- Transitions due to an item perishing:

  From $(1,0,0)$ to $(S,0,0)$ by γ.

  From $(1,1,i_3)$ to $(S,1,i_3)$ by γ, $i_3 \in V_0^N$.

  From $(1,2,i_3)$ to $(S,2,i_3)$ by γ, $i_3 \in V_1^N$.

  From $(i_1,0,0)$ to $(i_1-1,0,0)$ by $i_1\gamma$, $i_1 \in V_2^S$.

  From $(i_1,1,i_3)$ to $(i_1-1,1,i_3)$ by $i_1\gamma$, $i_1 \in V_2^S$, $i_3 \in V_0^N$.

  From $(i_1,2,i_3)$ to $(i_1-1,2,i_3)$ by $i_1\gamma$, $i_1 \in V_2^S$, $i_3 \in V_1^N$.

- Transitions due to completion of a vacation:

  From $(i_1,1,i_3)$ to $(i_1,2,i_3)$ by η, $i_1 \in V_1^S$, $i_3 \in V_1^N$.

- Transitions due to the impatient of customers:

  From $(i_1,1,i_3)$ to $(i_1,1,i_3-1)$ by $i_3\theta$, $i_1 \in V_1^S$, $i_3 \in V_1^N$.

  From $(i_1,2,1)$ to $(i_1,0,0)$ by θ, $i_1 \in V_1^S$.

  From $(i_1,2,i_3)$ to $(i_1,2,i_3-1)$ by $i_3\theta$, $i_1 \in V_1^S$, $i_3 \in V_2^N$.

- Transitions due to delay period completion:

  From $(i_1,0,0)$ to $(i_1,1,0)$ by β, $i_1 \in V_1^S$.

- For other transition from (i_1,i_2,i_3) to (j_1,j_2,j_3), except $(i_1,i_2,i_3) \neq (j_1,j_2,j_3)$, the rate is zero.

- To obtain the intensity of passage, $a((i_1,i_2,i_3),(j_1,j_2,j_3))$ of state (i_1,i_2,i_3), we note that the entries in any row of this matrix add to zero. Hence the diagonal entry is equal to the negative of the sum of the other entries in that row. More explicitly,

$$a((i_1,i_2,i_3),(i_1,i_2,i_3)) = -\sum_{i_1}\sum_{i_2}\sum_{i_3}_{(i_1,i_2,i_3)\neq(j_1,j_2,j_3)} a((i_1,i_2,i_3),(j_1,j_2,j_3))$$

Using the above arguments, we get a rate matrix transition:

$$\Theta_{i_1j_1} = \begin{cases} \Delta_{i_1} & j_1 = i_1,\ \ i_1 = 1,2,\ldots,S \\ \Lambda_{i_1} & j_1 = i_1 - 1,\ i_1 = 2,3,\ldots,S-1,S \\ \Lambda_1 & j_1 = S,\ i_1 = 1, \\ \mathbf{0} & \text{Otherwise.} \end{cases}$$

Now we derive a recursive algorithm for the solution of the steady-state equations (4.6) and (4.7). The steady state probability distribution $\mathbf{\Phi}^{(i_1)}, i_1 = 1,2,\ldots,S$, can be obtained using the following algorithm:

Step 1:

$$\mathbf{\Phi}^{(1)}\left[\Lambda_1 + \left\{(-1)^{S-1}\mathop{\Omega}_{r=1}^{S-1}\Delta_r\Lambda_{r+1}^{-1}\right\}\right] = \mathbf{0}$$

and

$$\mathbf{\Phi}^{(1)}\left[\sum_{i_1=2}^{S}\left((-1)^{i_1-1}\mathop{\Omega}_{r=1}^{i_1-1}\Delta_r\Lambda_{r+1}^{-1}\right) + I\right]\mathbf{e} \ = \ 1.$$

Step 2: Calculate the values of

$$\begin{aligned} \mathbf{\Pi_{i_1}} &= (-1)^{i_1-1}\mathop{\Omega}_{k=1}^{r=i_1-1}\Delta_k\Lambda_{k+1}^{-1}, \quad i_1 = 1,2,\ldots,S, \text{ and } \Omega \text{ is defined by} \\ \mathop{\Omega}_{i=r}^{k} c_i &= \begin{cases} c_r c_{r-1}\cdots c_k & \text{if } r \geq k \\ 1 & \text{if } r < k \end{cases} \end{aligned}$$

Step 3: Using step 1 and step 2, calculate the value of $\mathbf{\Phi}^{(\mathbf{i_1})}, i_1 = 1,\ldots,S$. That is,

$$\mathbf{\Phi}^{(\mathbf{i_1})} = \mathbf{\Phi}^{(1)}\mathbf{\Pi_{i_1}},\ i_1 = 1,\ldots,S,$$

Further we consider an inventory model with non perishable items and other assumptions are the same as given in the description of the model. Its infinitesimal generator matrix Θ_1 is given by,

$$
\Theta_1 = \begin{array}{c} \\ (\mathbf{1}) \\ (\mathbf{2}) \\ (\mathbf{3}) \\ \vdots \\ (\mathbf{S-1}) \\ (\mathbf{S}) \end{array}
\begin{array}{c} \begin{array}{cccccc} (\mathbf{1}) & (\mathbf{2}) & (\mathbf{3}) & \cdots & (\mathbf{S-1}) & (\mathbf{S}) \end{array} \\
\left(\begin{array}{cccccc}
\widehat{\Delta} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\Lambda} \\
\widehat{\Lambda} & \widehat{\Delta} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\
\mathbf{0} & \widehat{\Lambda} & \widehat{\Delta} & \ddots & \ddots & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \widehat{\Delta} & \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \widehat{\Lambda} & \widehat{\Delta}
\end{array}\right) \end{array}
$$

All the sub matrices of Θ_1 are square matrices of dimension $2(N+1)$ and the sub matrices of Θ_1 is given by

$$
\widehat{\Lambda} = \begin{array}{c} \\ 0,0 \\ 1,0 \\ 1,1 \\ 1,2 \\ \vdots \\ 1,N \\ 2,1 \\ 2,2 \\ 2,3 \\ \vdots \\ 2,N \end{array}
\begin{array}{c} \begin{array}{ccccccccccc} 0,0 & 1,0 & 1,1 & 1,2 & \cdots & 1,N & 2,1 & 2,2 & 2,3 & \cdots & 2,N \end{array} \\
\left(\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
r_1\mu & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & r_1\mu & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & r_1\mu & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right) \end{array}
$$

$$
\widehat{\Delta} = \begin{array}{c} \\ 0,0 \\ 1,0 \\ 1,1 \\ 1,2 \\ \vdots \\ 1,N-1 \\ 1,N \\ 2,1 \\ 2,2 \\ 2,3 \\ \vdots \\ 2,N-1 \\ 2,N \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc}
0,0 & 1,0 & 1,1 & 1,2 & \cdots & 1,N & 2,1 & 2,2 & 2,3 & \cdots & 2,N
\end{array} \\
\left(\begin{array}{ccccccccccc}
a_0 & \beta & 0 & 0 & \cdots & 0 & p\lambda & 0 & 0 & \cdots & 0 \\
0 & b & p\lambda & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & q\lambda+\theta & c & p\lambda & \cdots & 0 & \eta & 0 & 0 & \cdots & 0 \\
0 & 0 & q\lambda+2\theta & c & \cdots & 0 & 0 & \eta & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & p\lambda & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & d & 0 & 0 & 0 & \cdots & \eta \\
q\lambda+r_2\mu+\theta & 0 & 0 & 0 & \cdots & 0 & f & p\lambda & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & q\lambda+r_2\mu+2\theta & f & p\lambda & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & q\lambda+r_2\mu+3\theta & f & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & i_1\gamma & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & g
\end{array}\right)
\end{array}
$$

where $a_0 = -(p\lambda + \beta)$; $b = -(p\lambda)$; $c = -(\lambda + \eta + i_3\theta), i_3 \in V_1^{N-1}$; $d = -(q\lambda + N\theta + \eta)$; $f = -(\lambda + \mu + i_3\theta), i_3 \in V_1^{N-1}$; $g = q\lambda + \mu + N\theta$.

Let $\mathbf{\Xi} = (\xi_{(\mathbf{1})}, \xi_{(\mathbf{2})}, \ldots, \xi_{(\mathbf{S})})$ be the steady state probability vector of $Y(t)$ process. Then Θ_1 satisfies the equations $\mathbf{\Xi}\Theta_1 = 0$ and $\mathbf{\Xi e} = \mathbf{1}$. We shall find the steady-state probability vectors $\xi_{(\mathbf{1})}, \xi_{(\mathbf{2})}, \ldots, \xi_{(\mathbf{S})}$ of $Y(t)$ by using the same technique as mentioned above.

4.3.2 An Alternative Markov Chain Representation

Let us now reorganize the state space as follows. Consider the state space

$$
\begin{aligned}
\widehat{H} &= \{(0, X_1(t), 0) \cup (0, X_1(t), 1) \cup (X_3(t), X_1(t), 1) \cup (X_3(t), X_1(t), 2) \\
&\quad \mid 1 \leq X_1(t) \leq S; 1 \leq X_3(t) \leq N\}
\end{aligned}
$$

The stochastic process described by this state space $\widehat{H}$ is still a CTMC, wherein the first variable is the number of customers in the waiting room $[X_3(t)]$, the second variable is the inventory level $[X_1(t)]$ and the third variable is the status of the server $[X_2(t)]$. The simply difference between H and $\widehat{H}$ is that we have replaced the order of all the variables $X_1(t), X_2(t)$ and $X_3(t)$ by $X_3(t), X_1(t)$

and $X_2(t)$ respectively. The infinitesimal generator matrix Θ_2 is given by:

$$\Theta_2 = \begin{array}{c} \\ (\mathbf{0}) \\ (\mathbf{1}) \\ (\mathbf{2}) \\ (\mathbf{3}) \\ \vdots \\ (\mathbf{N-1}) \\ (\mathbf{N}) \end{array} \begin{array}{c} \begin{array}{ccccccc} (\mathbf{0}) & (\mathbf{1}) & (\mathbf{2}) & (\mathbf{3}) & \cdots & (\mathbf{N-1}) & (\mathbf{N}) \end{array} \\ \left(\begin{array}{ccccccc} \widehat{F}_0 & \widehat{V}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \widehat{D}_1 & \widehat{F}_1 & \widehat{V}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widehat{D}_2 & \widehat{F}_2 & \widehat{V}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widehat{D}_3 & \widehat{F}_3 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & & \widehat{F}_{N-1} & \widehat{V}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & & \widehat{D}_N & \widehat{F}_N \end{array} \right) \end{array}$$

where all the matrices are square matrices of order $2S$ and the sub matrices of Θ_2 are given by

$$\widehat{V}_0 = \begin{array}{c} \\ 1,0 \\ 2,0 \\ 3,0 \\ 4,0 \\ \vdots \\ S,0 \\ 1,1 \\ 2,1 \\ 3,1 \\ \vdots \\ S,1 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1,1 & 2,1 & 3,1 & 4,1 & \cdots & S,1 & 1,2 & 2,2 & 3,2 & \cdots & S,2 \end{array} \\ \left(\begin{array}{ccccccccccc} 0 & 0 & 0 & 0 & \cdots & 0 & p\lambda & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p\lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & p\lambda & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & p\lambda \\ p\lambda & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & p\lambda & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & p\lambda & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & p\lambda & 0 & 0 & 0 & \cdots & 0 \end{array} \right) \end{array}$$

$$\widehat{V}_1 = \begin{array}{c} \\ 1,1 \\ 2,1 \\ 3,1 \\ 4,1 \\ \vdots \\ S,1 \\ 1,2 \\ 2,2 \\ 3,2 \\ \vdots \\ S,2 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1,1 & 2,1 & 3,1 & 4,1 & \cdots & S,1 & 1,2 & 2,2 & 3,2 & \cdots & S,2 \end{array} \\ \left(\begin{array}{ccccccccccc} p\lambda & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & p\lambda & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & p\lambda & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & p\lambda & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & p\lambda & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & p\lambda & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p\lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & p\lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & p\lambda \end{array} \right) \end{array}$$

$$
\widehat{D}_1 = \begin{array}{c} \\ 1,1 \\ 2,1 \\ 3,1 \\ 4,1 \\ \vdots \\ S,1 \\ 1,2 \\ 2,2 \\ 3,2 \\ \vdots \\ S,2 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 1,0 & 2,0 & 3,0 & 4,0 & \cdots & S,0 & 1,1 & 2,1 & 3,1 & \cdots & S,1 \end{array} \\
\left(\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & \cdots & 0 & q\lambda+\theta & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & q\lambda+\theta & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & q\lambda+\theta & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & q\lambda+\theta \\
q\lambda+\theta+r_2\mu & 0 & 0 & 0 & \cdots & r_1\mu & 0 & 0 & 0 & \cdots & 0 \\
r_1\mu & q\lambda+\theta+r_2\mu & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & r_1\mu & q\lambda+\theta+r_2\mu & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & q\lambda+\theta+r_2\mu & \cdots & 0 & 0 & 0 & 0 & \cdots & p\lambda
\end{array}\right)
\end{array}
$$

For $k \in V_2^N$,

$$
\widehat{D}_k = \begin{array}{c} \\ 1,1 \\ 2,1 \\ 3,1 \\ 4,1 \\ \vdots \\ S,1 \\ 1,2 \\ 2,2 \\ \vdots \\ S,2 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 1,1 & 2,1 & 3,1 & 4,1 & \cdots & S,1 & 1,2 & 2,2 & 3,2 & \cdots & S,2 \end{array} \\
\left(\begin{array}{ccccccccccc}
q\lambda+k\theta & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & q\lambda+k\theta & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & q\lambda+k\theta & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & q\lambda+k\theta & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & q\lambda+k\theta & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & q\lambda+\theta+r_2\mu & 0 & 0 & \cdots & r_1\mu \\
0 & 0 & 0 & 0 & \cdots & 0 & r_1\mu & q\lambda+\theta+r_2\mu & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & q\lambda+\theta+r_2\mu
\end{array}\right)
\end{array}
$$

$$
\widehat{F}_0 = \begin{array}{c} \\ 1,0 \\ 2,0 \\ 3,0 \\ 4,0 \\ \vdots \\ S,0 \\ 1,1 \\ 2,1 \\ 3,1 \\ \vdots \\ S,1 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 1,0 & 2,0 & 3,0 & 4,0 & \cdots & S,0 & 1,1 & 2,1 & 3,1 & \cdots & S,1 \end{array} \\
\left(\begin{array}{ccccccccccc}
h_1 & 0 & 0 & 0 & \cdots & \gamma & \beta & 0 & 0 & \cdots & 0 \\
2\gamma & h_2 & 0 & 0 & \cdots & 0 & 0 & \beta & 0 & \cdots & 0 \\
0 & 3\gamma & h_3 & 0 & \cdots & 0 & 0 & 0 & \beta & \cdots & 0 \\
0 & 0 & 4\gamma & h_4 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & h_S & 0 & 0 & 0 & \cdots & \beta \\
0 & 0 & 0 & 0 & \cdots & 0 & -(p\lambda+\gamma) & 0 & 0 & \cdots & \gamma \\
0 & 0 & 0 & 0 & \cdots & 0 & 2\gamma & -(p\lambda+2\gamma) & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 3\gamma & -(p\lambda+3\gamma) & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -(p\lambda+S\gamma)
\end{array}\right)
\end{array}
$$

For $k \in V_1^N$,

$$
\widehat{F}_k = \begin{array}{c} \\ 1,1 \\ 2,1 \\ 3,1 \\ 4,1 \\ \vdots \\ S,1 \\ 1,2 \\ 2,2 \\ 3,2 \\ \vdots \\ S,2 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 1,1 & 2,1 & 3,1 & 4,1 & \cdots & S,1 & 1,2 & 2,2 & 3,2 & \cdots & S,2 \end{array} \\
\left(\begin{array}{ccccccccccc}
l_1 & 0 & 0 & 0 & \cdots & \gamma & \eta & 0 & 0 & \cdots & 0 \\
2\gamma & l_2 & 0 & 0 & \cdots & 0 & 0 & \eta & 0 & \cdots & 0 \\
0 & 3\gamma & l_3 & 0 & \cdots & 0 & 0 & 0 & \eta & \cdots & 0 \\
0 & 0 & 4\gamma & l_4 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & l_S & 0 & 0 & 0 & \cdots & \eta \\
0 & 0 & 0 & 0 & \cdots & 0 & m_1 & 0 & 0 & \cdots & \gamma \\
0 & 0 & 0 & 0 & \cdots & 0 & 2\gamma & m_2 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 3\gamma & m_3 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & m_S
\end{array}\right)
\end{array}
$$

where $h_i = -(p\lambda + i\gamma + \beta)$, $l_i = -(\lambda + i\gamma + k\theta + \eta)$, $m_i = -(\lambda + i\gamma + k\theta + \mu)$, $1 \leq i \leq S$.

Let $\widehat{\Pi} = (\widehat{\pi}_{(0)}, \widehat{\pi}_{(1)}, \ldots, \widehat{\pi}_{(N)})$ denote the steady-state probability vector of the infinitesimal generator Θ_2. Then the steady-state probability vector must satisfy the relations, $\widehat{\mathbf{\Pi}}\Theta_2 = 0$ and $\widehat{\mathbf{\Pi}}\mathbf{e} = \mathbf{1}$. From the structure of Θ_2, it is noticed that the Markov process under study falls into the class of BDP(birth-and-death process) in the Markovian environment as analyzed by Gaver et.al [52]. Hence we can calculate the limiting probability vectors.

Calculation of the stationary probabilities:

1. Determine recursively the matrices

$$
\begin{aligned}
\widehat{R}_0 &= \widehat{F}_0 \\
\widehat{R}_1 &= \widehat{F}_1 + \widehat{D}_1(-\widehat{R}_0^{-1})\widehat{V}_0, \\
\widehat{R}_i &= \widehat{F}_i + \widehat{D}_i(-\widehat{R}_{i-1}^{-1})\widehat{V}_1, \quad i = 2, 3, \ldots, N.
\end{aligned}
$$

2. Compute recursively the vectors $\widehat{\pi}_{(i)}$ using

$$
\widehat{\pi}_{(i)} = \widehat{\pi}_{(i+1)}\widehat{D}_{i+1}(-\widehat{R}_i^{-1}), \quad i = 0, 1, 2, \ldots, N-1, \tag{4.8}
$$

3. Solve the system of equations

$$\widehat{\pi}_{(N)}\widehat{R}_N = \mathbf{0} \tag{4.9}$$

and

$$\sum_{i=0}^{N}\widehat{\pi}_{(i)}\mathbf{e} = 1. \tag{4.10}$$

From the equations (4.9), vector $\widehat{\pi}_{(N)}$ could be concluded uniquely, up to a multiplicative constant. This constant is decided by using (4.8) and (4.10).

4.4 System characteristics

The system characteristics, such as the mean inventory level η_I, mean reorder rate η_R, mean perishable rate η_P, mean number of the customers in the waiting hall η_W, the effective arrival rate η_A, mean waiting time of a customer Γ, mean reneging rate η_{RR}, mean rate of arrivals of negative customers η_{NA}, and mean balking rate η_L can be obtained from the steady-state probabilities $(\phi^{(i_1,0,0)}, \phi^{(i_1,1,0)}, \ldots, \phi^{(i_1,1,N)}, \phi^{(i_1,2,1)},$
$\ldots, \phi^{(i_1,2,N)})$ where $i_1 = 1, 2, \ldots, S$ and are defined by

$$\eta_I = \sum_{i_1=1}^{S} i_1\left(\phi^{(i_1,0,0)} + \sum_{i_3=1}^{N}\phi^{(i_1,2,i_3)} + \sum_{i_3=0}^{N}\phi^{(i_1,1,i_3)}\right) \tag{4.11}$$

$$\eta_R = \gamma\phi^{(1,0,0)} + \sum_{i_3=1}^{N}(\gamma + r_1\mu)\phi^{(1,2,i_3)} + \sum_{i_3=0}^{N}\gamma\phi^{(i_1,1,i_3)} \tag{4.12}$$

$$\eta_P = \sum_{i_1=1}^{S} i_1\gamma\left(\phi^{(i_1,0,0)} + \sum_{i_3=1}^{N}\phi^{(i_1,2,i_3)} + \sum_{i_3=0}^{N}\phi^{(i_1,1,i_3)}\right) \tag{4.13}$$

$$\eta_W = \sum_{i_1=1}^{S}\sum_{i_2=1}^{2}\sum_{i_3=1}^{N} i_3\phi^{(i_1,i_2,i_3)} \tag{4.14}$$

$$\eta_A = \sum_{i_1=1}^{S} p\lambda\phi^{(i_1,0,0)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N-1} p\lambda\phi^{(i_1,2,i_3)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-1} p\lambda\phi^{(i_1,1,i_3)} \tag{4.15}$$

$$\Gamma = \frac{\eta_W}{\eta_A} \tag{4.16}$$

$$\eta_{RR} = \sum_{i_1=1}^{S}\sum_{i_2=1}^{2}\sum_{i_3=1}^{N} i_3\theta\phi^{(i_1,i_2,i_3)} \tag{4.17}$$

$$\eta_{NA} = \sum_{i_1=1}^{S}\sum_{i_2=1}^{2}\sum_{i_3=1}^{N} q\lambda\phi^{(i_1,i_2,i_3)} \tag{4.18}$$

$$\eta_L = \sum_{i_1=1}^{S}\sum_{i_2=1}^{2} p\lambda\phi^{(i_1,i_2,N)} \tag{4.19}$$

4.5 Cost Analysis and Sensitivity Investigation

Now we define different costs as

c_h : The inventory carrying cost per unit item per unit time.

c_s : Setup cost per order per unit time.

c_p : Failure cost per unit item per unit time.

c_w : Waiting time cost of a customer in the waiting room per unit time.

c_r : Reneging cost per customer per unit time.

c_l : Cost due to loss of customers (due to waiting hall full) per unit per unit time.

c_n : Loss per unit time due to arrival of a negative customer.

We introduce a cost function, defined as the expected total cost (ETC) of the system, is given by

$$ETC(S,N) = c_h\eta_I + c_s\eta_R + c_p\eta_P + c_w(\Gamma = \frac{\eta_W}{\eta_A}) + c_r\eta_{RR} + c_l\eta_L + c_n\eta_N \quad (4.20)$$

where $\eta's$ are given in the system characteristics.

4.5.1 Sensitivity Investigation

In this subdivision, we present a few numerical illustrations to study the result of different parameters and cost values on the expected total cost.
First we study the effect of each cost value on the expected total cost (ETC) of this model subject to fixing remaining cost values.

For expediency, we initial let the cost values: $c_h = 6$, $c_s = 15$, $c_p = 0.1$, $c_w = 5$, $c_r = 5$, $c_l = 5$ and $c_n = 0.001$; and
parameter values: $\lambda = 8$, $\gamma = 0.08$, $\theta = 0.05$, $\mu = 45$, $\eta = 0.005$, $\beta = 0.9$, $S = 8$, $N = 6$, $p = 0.5$, and $r_1 = 0.5$.

The result of different values of costs c_h, c_s, c_p, c_w, c_r, c_l and c_n on the expected total cost (ETC) is depicted in Tables 4.1 and 4.2. Tables 4.1 and 4.2 divulge that ETC increases as c_h, c_s, c_p, c_w, c_r, c_l and c_n increase. We can notice that ETC is more sensitive to c_h than to c_s, c_p, c_w, c_r, c_l and c_n.

Next we deals with the parameters on the ETC. By fixing all the parameters except λ, it is observe from Figure 4.1 that ETC is convex in λ. For certain parameter values, this ETC attains the minimum value 26.690703 at $\lambda = 6.5$. As perishable rate γ increases (keeping remaining parameters fixed), one can clear that ETC attains the minimum value 28.164326 at $\gamma = 0.11$. As service rate μ increases (keeping remaining parameters fixed), one can clear that ETC attains the minimum value 26.690547 at $\mu = 40$. One can study the minimum value of ETC by varying the remaining parameters θ, β, η, S and N (see Figures $4.2 - 4.8$). In all illustrations considered here, ETC has the minimum value.

Table 4.1: Effect of c_h, c_s, c_p, c_w on the expected total cost

c_h	ETC	c_s	ETC	c_p	ETC	c_w	ETC
4	19.6085416	5	25.2752541	0.01	25.5563689	3	23.9695896
5	22.5932000	6	25.3055145	0.02	25.5587566	4	24.7737240
6	25.5778584	7	25.3357750	0.03	25.5611443	5	25.5778584
7	28.5625168	8	25.3660354	0.04	25.5635321	6	26.3819928
8	31.5471753	9	25.3962958	0.05	25.5659198	7	27.1861273
9	34.5318337	10	25.4265563	0.06	25.5683075	8	27.9902617
10	37.5164921	11	25.4568167	0.07	25.5706952	9	28.7943961
11	40.5011505	12	25.4870771	0.08	25.5730830	10	29.5985305

Table 4.2: Effect of c_r, c_l, c_n on the expected total cost

c_r	ETC	c_l	ETC	c_n	ETC
2	25.1487403	1	23.6155806	0.001	25.5778584
3	25.2917797	2	24.1061500	0.002	25.5812664
4	25.4348191	3	24.5967195	0.003	25.5846743
5	25.5778584	4	25.0872890	0.004	25.5880822
6	25.7208978	5	25.5778584	0.005	25.5914902
7	25.8639372	6	26.0684279	0.006	25.5948981
8	26.0069765	7	26.5589973	0.007	25.5983060
9	26.1500159	8	27.0495668	0.008	25.6017140

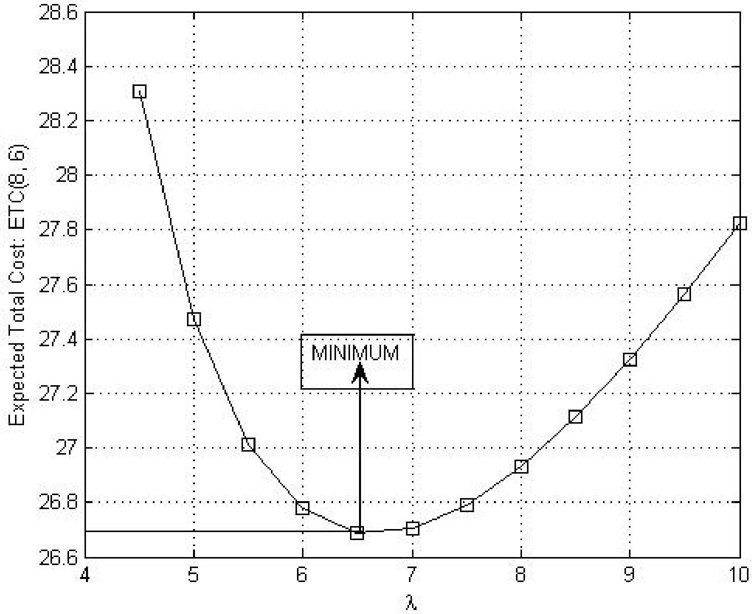

Figure 4.1: ETC versus λ

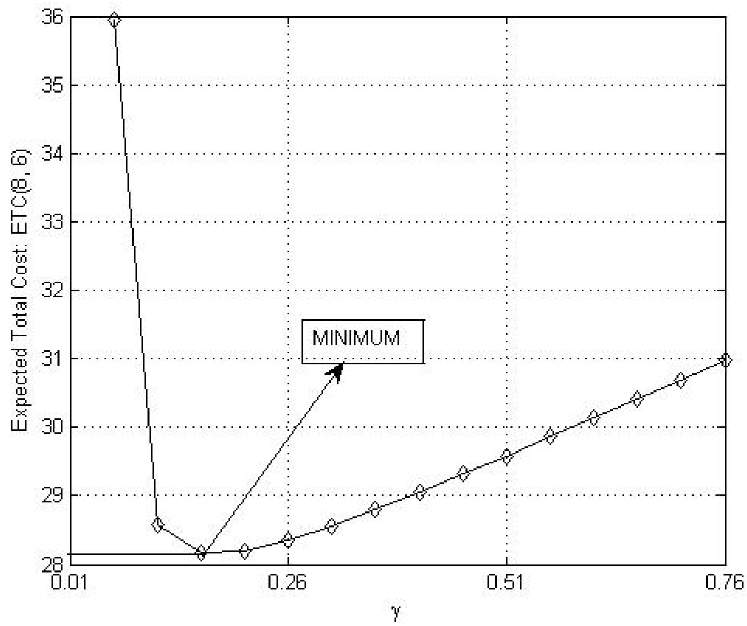

Figure 4.2: ETC versus γ

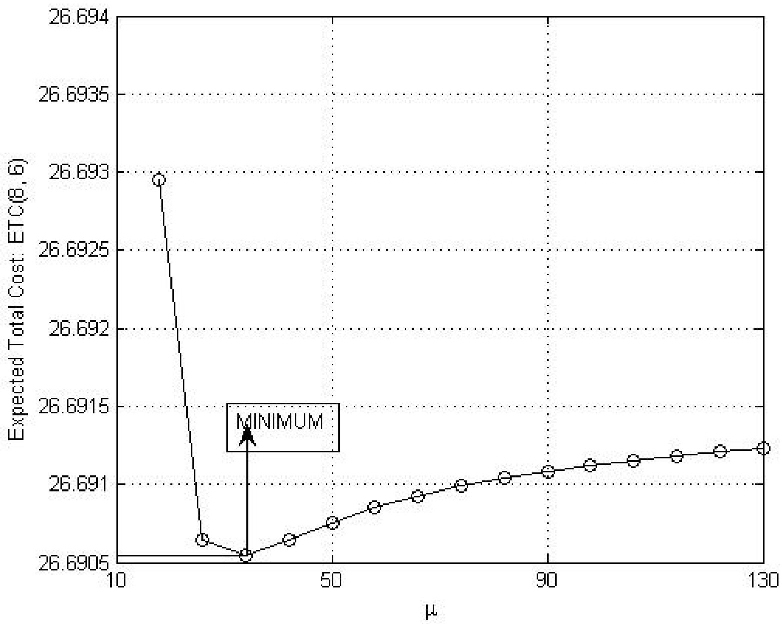

Figure 4.3: ETC versus μ

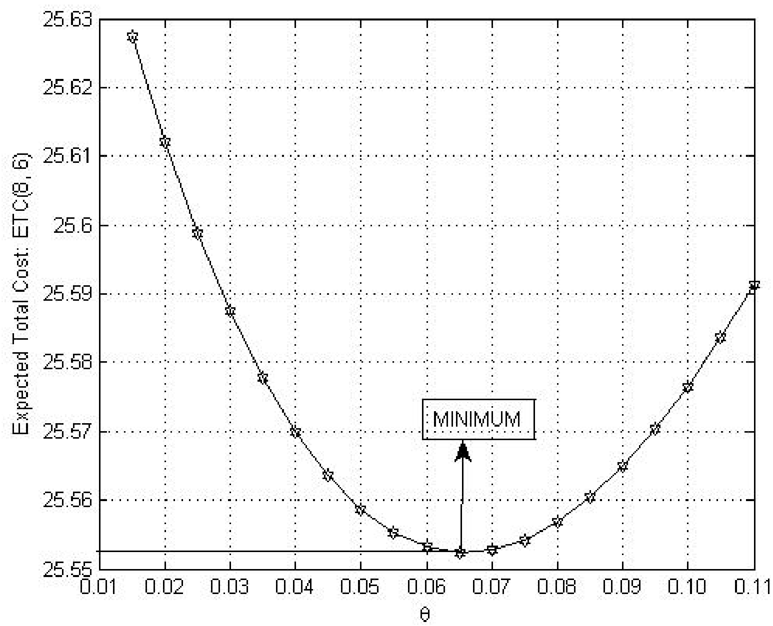

Figure 4.4: ETC versus θ

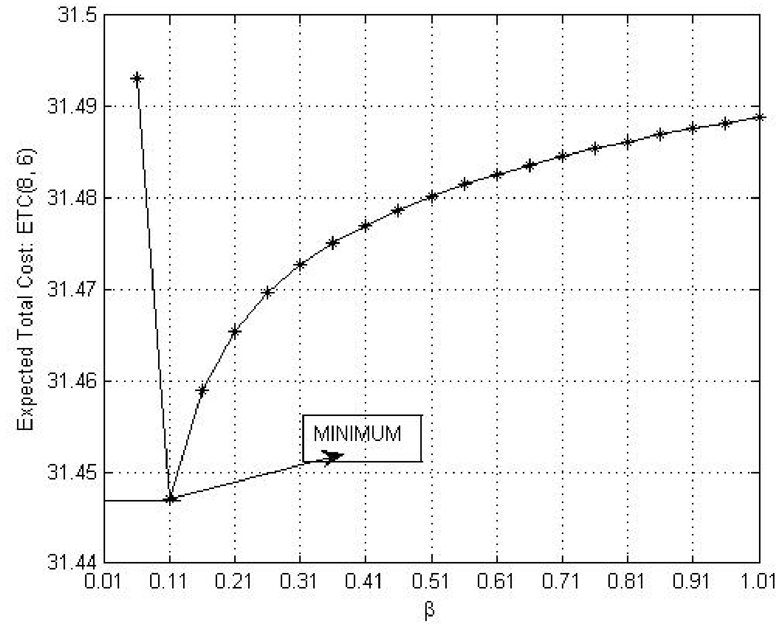

Figure 4.5: ETC versus β

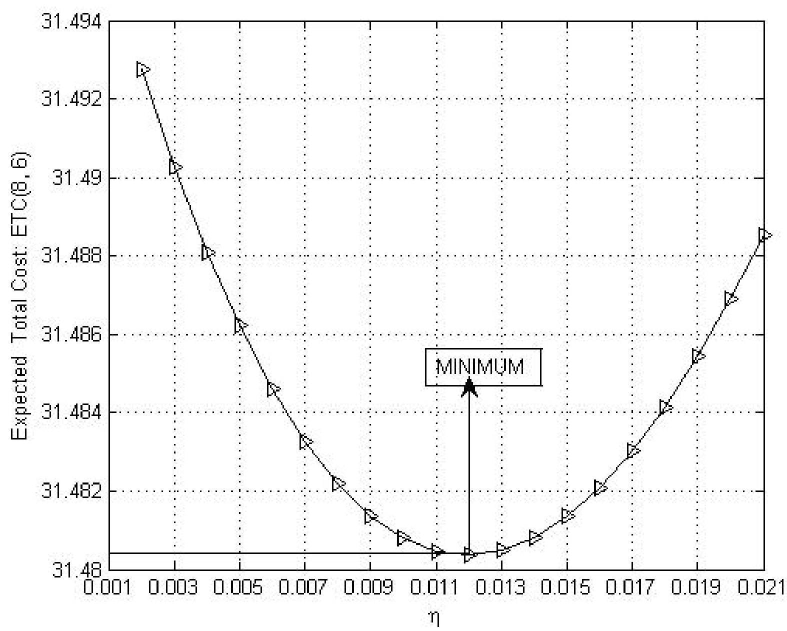

Figure 4.6: ETC versus η

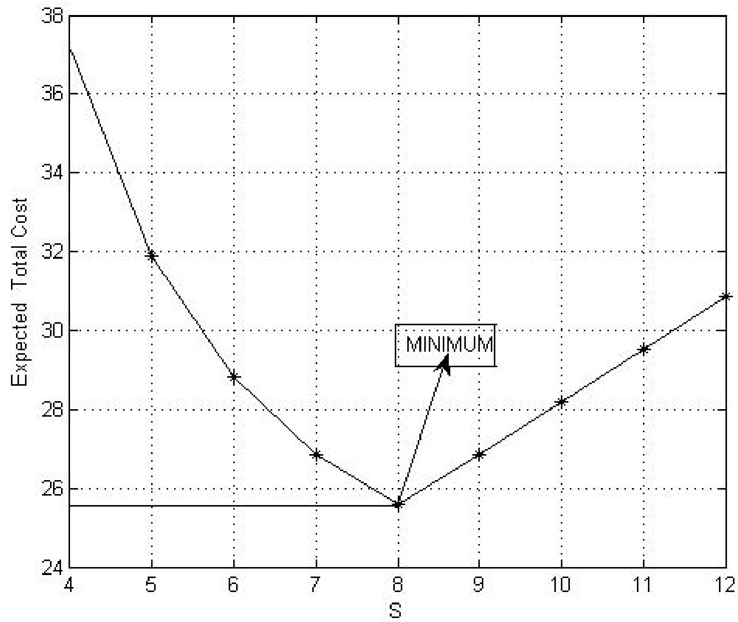

Figure 4.7: ETC versus S

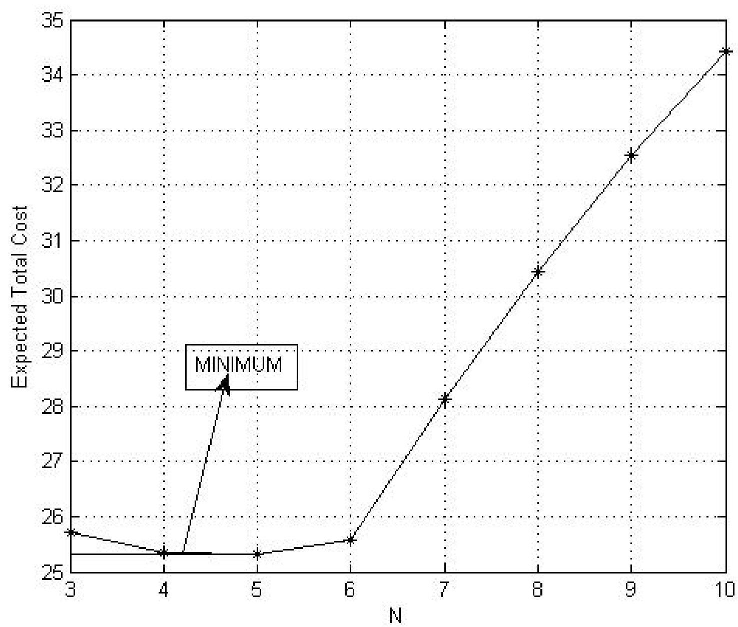

Figure 4.8: ETC versus N

4.6 Concluding Remarks

In this chapter, we studied an $M/M/1$ queueing-inventory system with delayed vacation, negative and impatiemt customers. Based on the ordering of the state space, we discussed two different type of Markov chains for determining the distribution of the number of customers in the waiting hall and server status and inventory level. The major contribution of the paper is the fortitude of the joint distribution of the number of customers in the waiting hall, server status and inventory level and scheming them easily. Various system performance measures are derived and the long-run total expected cost rate is calculated. By assuming a suitable cost structure on the queueing-inventory system, we have presented extensive numerical illustrations to show the effect of change of values for constants on the total expected cost rate.