Chapter 11

Flow and heat transfer of couple stress fluid in a vertical channel in the presence of heat source/sink

11.1 Introduction

The study of convective flow and heat transfer in a vertical channel has gained interest in recent years because of its industrial applications occurring in the design of cooling systems for electronic devices, chemical processing equipments, microelectronic cooling, solar energy collection, and nuclear reactors (Aung, 1987; Barletta, 1999; Cheng, 1978; Lavine, 1988; Rohsenow et al., 1998). Tao (1960) studied the laminar fully developed mixed convection in a vertical channel with a uniform wall temperature. Habchi and Acharya (1986) studied the laminar fully developed mixed convection in a vertical channel with asymmetric heating where one plate is heated and the other is adiabatic. Thereafter, numerous investigations have been made on the mixed convection flow with symmetric/asymmetric heating under different physical situations (see Aung and Worku, 1986a; Cheng et al., 1990; Umavathi and Malashetty,
The effect of viscous dissipation may become very important in several flow configurations occurring in the engineering practice. In fact, viscous dissipation affects strongly the heat transfer process whenever the operating fluid has a low thermal conductivity, a high viscosity, and flows in ducts with a small cross-section and a small wall heat flux. All these features may occur, for instance, in the micro-channel flows. As is well known, the effect of viscous heating increases with the square of the mass flow rate and, as a consequence, becomes especially important under conditions of forced convection.

Quite a number of physical phenomena involve combined free and forced convection driven by heat source/sink. The study of heat source in moving fluids is important in view of several physical problems such as those dealing with chemical reactions and those concerned with dissociating fluids. Possible heat generation/absorption effects may change the temperature distribution and, therefore, the particle deposition rate. This may occur in such applications related to nuclear reactor cores, fire and combustion modeling, electronic chips and semi conductor wafers. In fact, the literature is replete with examples dealing with heat transfer in laminar flow of viscous fluids. Other investigations dealing with internal heat generation or absorption can be seen in the works of Sparrow and Cess (1961) and Chamkha (2002). Mahanthi and Gaur (2009) investigated the effect of the viscosity and thermal conduction on the steady free-convective flow of a viscous incompressible fluid along an isothermal vertical plate in the presence of heat sink. Recently, Umavathi et al. (2012e) studied the effects of viscous dissipation and heat source/sink on fully developed mixed convection for the laminar flow in a parallel-plate vertical channel. Umavathi and Sultana (2012) investigated the fully developed mixed convection for a laminar flow of a micropolar fluid mixture in a vertical channel with a heat source/sink.

In the past, the laminar free and forced convection heat transfer in the hydrodynamic
Flow and heat transfer of couple stress fluid in a vertical channel ···

entrance region of a flat rectangular channel has been investigated either for the
temperature boundary condition of the first kind, characterized by prescribed wall
temperature (Hwang and Fan, 1964; Sparrow, 1955; Stephan, 1959), or for the boundary
condition of the second kind, expressed by the prescribed wall temperature heat flux
(Hwang and Fan, 1964; Siegal and Sparrow, 1959). A more realistic condition in many
applications, however, will be temperature boundary condition of third kind: the local
wall heat flux is a linear function of the local wall temperature. Heat transfer in laminar
region of a flat channel for the temperature boundary condition of third kind was
explored by Javeri (1977). Javeri (1978) investigated the influence of the temperature
boundary condition of the third kind on the laminar heat transfer in the thermal
entrance region of a rectangular channel. Later Zanchini (1998) analyzed the effect of
viscous dissipation on mixed convection in a vertical channel with boundary conditions
of third kind. Recently, Umavathi and Veershetty (2012) analyzed the combined free
and forced convection flow in a parallel plate vertical channel filled with porous matrix
in the fully developed region with boundary conditions of third kind.

During recent years the study of convection heat and mass transfer in non-Newtonian
fluids has received much attention and this is because the traditional Newtonian
fluids cannot precisely describe the characteristics of the real fluids. The study of a
couple stress fluid is very useful in understanding various physical problems, because it
possesses the mechanism to describe rheologically complex fluids such as liquid crystals,
colloidal fluids, liquids containing long-chain molecules as polymeric suspensions, animal
and human blood and lubrication.

The theory of couple stress fluid proposed by Stokes (1966), defines the rotational
field in terms of the velocity field for setting up the constitutive relationship between
the stress and strain rate. Stokes theory is the simplest generalization of the classical
theory of fluids, which allows for polar effects such as the presence of couple stresses,
body couples and a non-symmetric stress tensor. Some theoretical studies of the couple
stress model to biomechanics problems has been proposed in the study of peristaltic transport by Srivastava (1986), Shehawey and Mekheimer (1994) and blood flow in the microcirculation by Pal et al. (1988). In lubrication problems many authors have investigated the couple stress effects on different lubrication problems (Chiang et al., 2004; Jian et al., 2006; Naduvinamani et al., 2005; Rong-Fang Lu, 2007). The couple-stress fluid may be considered as a special case of a non-Newtonian fluid which is intended to take into account of the particle size effects. Moreover, the couple stress fluid model is one of the numerous models that proposed to describe response characteristics of non-Newtonian fluids. The constitutive equations in these fluid models can be very complex and involving a number of parameters, also the out coming flow equations lead to boundary value problems in which the order of differential equations is higher than the Navier–Stokes equations. A review of couple stress (polar) fluid dynamics was reported by Stokes (1984). Some recent investigations regarding such fluids are mentioned in the studies (Patil and Kulkarni, 2008; Srinivasacharya and Kaladhar, 2012a; Srinivasacharya et al., 2009).

The aim of this work is to study the laminar, fully developed flow and heat transfer of a couple stress fluid in a vertical channel for boundary conditions of third kind with a heat source/sink. Thus, keeping in view the applications of couple stress fluid, in this work we extend the analysis of Zanchini (1998) for a non-Newtonian fluid in the presence of a heat source/sink. The semi-exact solutions are derived using DTM, and the effect of the mixed convection parameter, Brinkman number, couple stress parameter, and heat generation or absorption parameter on the flow and heat transfer characteristics is studied.

11.2 Mathematical Formulation

Consider the steady and laminar flow of a non-Newtonian fluid (couple stresses fluid) in the fully developed region of a parallel-plate vertical channel as shown in Figure
11.1. The $X$-axis lies on the axial plane of the channel and its direction is opposite to the gravitational field. The $Y$-axis is orthogonal to the walls. The channel is assumed to occupy the region of space $-b/2 \leq Y \leq b/2$. The flow regime is subjected to the influence of couple stress fluid is as shown in Figure 11.1.

![Figure 11.1: Schematic diagram of the vertical channel.](image)

The physical properties characterizing the fluid, namely, the thermal conductivity, the thermal diffusivity, the dynamic viscosity and the thermal expansion coefficient of the fluid are assumed to be constants. As customary, the Boussinesq approximation and the equation of state

$$\rho = \rho_0 [1 - \beta (T - T_0)],$$  \hspace{1cm} (11.1)

will be adopted. Moreover, it will be assumed that the only non-zero component of the velocity field $U$ is the $X$-component $U$. Thus, since $\nabla U = 0$, one has

$$\frac{\partial U}{\partial X} = 0,$$  \hspace{1cm} (11.2)

so that $U$ depends only on $Y$. The momentum equations governing the motion of an incompressible couple stress fluid in the absence of body couple are given by (Stokes,
Flow and heat transfer of couple stress fluid in a vertical channel ...

\[ \beta g \rho_0(T - T_0) - \frac{\partial P}{\partial X} + \mu \frac{d^2U}{dY^2} - \eta_c \frac{d^4U}{dY^4} = 0, \quad (11.3) \]

\[ \frac{\partial P}{\partial Y} = 0, \quad (11.4) \]

where \( P = p + \rho_0 g X \). Since on account of equation (11.4) \( P \) depends only on \( X \), equation (11.3) can be rewritten as

\[ T - T_0 = \frac{1}{\beta g \rho_0} \frac{dP}{dX} - \frac{\nu}{\beta g} \frac{d^2U}{dY^2} + \frac{\eta_c}{\beta g \rho_0} \frac{d^4U}{dY^4}, \quad (11.5) \]

From equation (11.5), one obtains,

\[ \frac{\partial T}{\partial X} = \frac{1}{\beta g \rho_0} \frac{d^2P}{dX^2}, \quad (11.6) \]

\[ \frac{\partial T}{\partial Y} = - \frac{\nu}{\beta g} \frac{d^3U}{dY^3} + \frac{\eta_c}{\beta g \rho_0} \frac{d^5U}{dY^5}, \quad (11.7) \]

\[ \frac{\partial^2 T}{\partial Y^2} = - \frac{\nu}{\beta g} \frac{d^4U}{dY^4} + \frac{\eta_c}{\beta g \rho_0} \frac{d^6U}{dY^6}, \quad (11.8) \]

Both the walls of the channel will assumed to have a negligible thickness and to exchange heat by convection with an external fluid. In particular, at \( Y = -b/2 \) the external convection coefficient will be considered as uniform with the value \( h_1 \) and the fluid in the region \( Y < -b/2 \) will be assumed to have a uniform reference temperature \( T_1 \). At \( Y = b/2 \) the external convection coefficient will be considered as uniform with the value \( h_2 \) and the fluid in the region \( Y > b/2 \) will be assumed to have a uniform reference temperature \( T_2 \) (\( T_2 \geq T_1 \)). Therefore, the boundary conditions on the temperature field can be expressed as

\[ -K \frac{\partial T}{\partial Y} \bigg|_{Y=-b/2} = h_1[T_1 - T(X, -b/2)], \quad (11.9) \]
Flow and heat transfer of couple stress fluid in a vertical channel ... 

\[-K \frac{\partial T}{\partial Y} \bigg|_{Y=b/2} = h_2[T(X,b/2) - T_2]. \tag{11.10}\]

On account of equation (11.7), equations (11.9) and (11.10) can be rewritten as

\[\left( \frac{d^3U}{dY^3} - \frac{\eta_c}{\rho_0 \nu} \frac{d^5U}{dY^5} \right) \bigg|_{Y=-b/2} = \frac{\beta g h_1}{K \nu} [T_1 - T(X,-b/2)], \tag{11.11}\]

\[\left( \frac{d^3U}{dY^3} - \frac{\eta_c}{\rho_0 \nu} \frac{d^5U}{dY^5} \right) \bigg|_{Y=b/2} = \frac{\beta g h_2}{K \nu} [T(X,b/2) - T_2]. \tag{11.12}\]

It is easily verified that the equations (11.11) and (11.12) imply that \( \frac{\partial T}{\partial X} \) is zero both at \( Y = -L/2 \) and at \( Y = L/2 \). Since equation (11.6) ensures that \( \frac{\partial T}{\partial X} \) does not depend on \( Y \), one is led to the conclusion that \( \frac{\partial T}{\partial X} = 0 \) everywhere. Therefore, the temperature \( T \) depends only on \( Y \) i.e., \( T = T(Y) \). Thus, on account of equation (11.6), there exists a constant \( A \) such that

\[\frac{dP}{dX} = A. \tag{11.13}\]

For the problem under exam, the energy balance equation in the presence of viscous dissipation and heat source/sink can be written as,

\[\frac{\partial^2 T}{\partial Y^2} = -\frac{\nu}{\alpha C_p} \left( \frac{dU}{dY} \right)^2 + \frac{Q_h}{K} \left( \frac{1}{\beta g \rho_0} \frac{dP}{dX} - \frac{\nu}{\beta g} \frac{d^2U}{dY^2} + \frac{\eta_c}{\beta g \rho_0} \frac{d^4U}{dY^4} \right). \tag{11.14}\]

Equations (11.8) and (11.14) yield a differential equation for \( U \), namely

\[\eta_c \frac{d^6U}{dY^6} = \left( \mu + \eta_c \frac{Q_h}{K} \right) \frac{d^4U}{dY^4} \left( \frac{dU}{dY} \right)^2 - \mu \rho_0 g \beta \left( \frac{d^2U}{dY^2} \right)^2 \pm \frac{Q_h}{K} \left( \frac{d^2U}{dY^2} - \frac{A}{\nu \rho_0} \right). \tag{11.15}\]

The boundary conditions on \( U \) are

\[U = 0, \quad \frac{d^2U}{dY^2} = 0 \quad \text{at} \quad Y = \pm \frac{b}{2} \tag{11.16}\]

together with the equations (11.11) and (11.12), which on account of equation (11.5),
can be rewritten as

\[
\left( \frac{d^3 U}{dY^3} - \frac{h_1 d^2 U}{K dY^2} - \frac{\eta_c d^3 U}{\mu dY^5} + \frac{h_1 \eta_c d^4 U}{K \mu dY^4} \right) \bigg|_{Y=-b/2} = -\frac{Ah_1}{\mu K} - \frac{\beta g h_1}{K \nu} (T_0 - T_1)
\]

\[
\left( \frac{d^3 U}{dY^3} + \frac{h_2 d^2 U}{K dY^2} - \frac{\eta_c d^3 U}{\mu dY^5} - \frac{h_2 \eta_c d^4 U}{K \mu dY^4} \right) \bigg|_{Y=b/2} = \frac{Ah_2}{\mu k} - \frac{\beta g h_2}{K \nu} (T_2 - T_0). \quad (11.17)
\]

Equations (11.15)–(11.17) determine the velocity distribution, they can be written in a dimensionless form by means of the following dimensionless parameters

\[
u = \frac{U}{U_0}, \quad \theta = \frac{T - T_0}{\Delta T}, \quad \gamma = \frac{Y}{D}, \quad Gr = \frac{g \beta \Delta T D^3}{\nu^2}, \quad Re = \frac{U_0 D}{\nu}, \quad Br = \frac{\mu U_0^2}{K \Delta T},
\]

\[
Pr = \frac{\nu}{\alpha}, \quad GR = \frac{Gr}{Re}, \quad R_l = \frac{T_2 - T_1}{\Delta T}, \quad a_c^2 = \frac{\eta_c}{\mu D^2}, \quad Bi_1 = \frac{h_1 D}{K}, \quad Bi_2 = \frac{h_2 D}{K},
\]

\[
S = \frac{Bi_1 Bi_2}{Bi_1 Bi_2 + 2Bi_1 + 2Bi_2}, \quad \phi_h = \frac{Q_h D^2}{K}
\]

where \( D = 2b \) is the hydraulic diameter. The reference velocity \( U_0 \) and the reference temperature \( T_0 \) are given by

\[
U_0 = -\frac{AD^2}{48\mu}; \quad T_0 = \frac{T_1 + T_2}{2} + S \left( \frac{1}{Bi_1} - \frac{1}{Bi_2} \right) (T_2 - T_1). \quad (11.19)
\]

The reference temperature difference \( \Delta T \) is given by

\[
\Delta T = T_2 - T \quad \text{if } T_1 < T_2 \quad \text{if } T_1 < T_2 \quad (11.20)
\]

\[
\Delta T = \frac{\nu^2}{C_p D^2} \quad \text{if } T_1 = T_2. \quad (11.21)
\]

Therefore, as in Barletta (1999), the value of the dimensionless parameter \( R_l \) can be either 0 or 1. More precisely, \( R_l \) equals 1 for asymmetric fluid temperatures \( T_1 < T_2 \), and equals 0 for symmetric fluid temperatures \( T_1 = T_2 \).
The dimensionless mean velocity $\bar{u}$ is given by

$$\bar{u} = 2 \int_{-1/4}^{1/4} u \, dy.$$  \hspace{1cm} (11.22)

On account of equation (11.13), for upward flow $A < 0$, so that $Re$ and $GR$ are positive. For downward flow $A > 0$, while $U_0$, $Re$ and $GR$ negative. By employing the dimensionless quantities defined in equation (11.18), equations (11.15)–(11.17) can be rewritten as

$$a_c^2 \frac{d^6 u}{dy^6} = \left(1 \mp a_c^2 \phi_h\right) \frac{d^4 u}{dy^4} - GR Br \left(\frac{d u}{d y}\right)^2 \pm \phi_h \frac{d^2 u}{d y^2} \pm 48 \phi_h,$$ \hspace{1cm} (11.23)

$$u = 0, \quad \frac{d^2 u}{d y^2} = 0 \quad \text{at} \quad y = \pm \frac{1}{4}$$ \hspace{1cm} (11.24)

$$(d^2 u \left(\frac{d^2 u}{d y^2} - \frac{1}{B_i} d^3 u \frac{d^3 u}{d y^3} - a_c^2 \frac{d^4 u}{B_i d^4 y^4} + a_c^2 \frac{d^5 u}{B_i d^5 y^5}\right)\bigg|_{y=-1/4} = -48 + \frac{R_i G R S}{2} \left(1 + \frac{4}{B_i}\right),$$

$$\left(d^2 u \left(\frac{d^2 u}{d y^2} + \frac{1}{B_i} d^3 u \frac{d^3 u}{d y^3} - a_c^2 \frac{d^4 u}{B_i d^4 y^4} + a_c^2 \frac{d^5 u}{B_i d^5 y^5}\right)\bigg|_{y=1/4} = -48 - \frac{R_i G R S}{2} \left(1 + \frac{4}{B_i}\right).$$  \hspace{1cm} (11.25)

When $a_c = 0$, i.e., in the absence of couple stresses, we get the classical results for clear viscous fluid (see e.g., Zanchini, 1998).

Similarly equation (11.14) using equation (11.18) yield

$$\frac{d^2 \theta}{d y^2} + Br \left(\frac{d u}{d y}\right)^2 \pm \frac{\phi_h}{G R} \left(\frac{d^2 u}{d y^2} + 48\right) = 0,$$ \hspace{1cm} (11.26)

where $\phi_h$ is plus sign relates to heat absorption and minus sign relates to heat generation.

Also from equation (11.5) and (11.18) one obtains

$$\theta = -\frac{1}{G R} \left(48 + \frac{d^2 u}{d y^2} - a_c^2 \frac{d^4 u}{d y^4}\right).$$  \hspace{1cm} (11.27)

Equations (11.23)–(11.27) show that the dimensionless velocity profile and the
dimensionless temperature profile depend on six parameters: the mixed convection parameter $GR$, the Brinkman number $Br$, couple stress parameter $a_c$, the temperature difference ratio $R_t$ and the Biot numbers $Bi_1$ and $Bi_2$.

A Nusselt number can be defined at each boundary, as follows:

\[
Nu_1 = \frac{D}{R_t [T (b/2) - T(-b/2)] + (1 - R_t) \Delta T} \left. \frac{dT}{dY} \right|_{Y=-b/2},
\]

\[
Nu_2 = \frac{D}{R_t [T (b/2) - T(-b/2)] + (1 - R_t) \Delta T} \left. \frac{dT}{dY} \right|_{Y=b/2}.
\]

By employing equation (11.23), equation (11.28) can be written as

\[
Nu_1 = \frac{1}{R_t [\theta (1/4) - \theta(-1/4)] + (1 - R_t)} \left. \frac{d\theta}{dy} \right|_{y=-1/4},
\]

\[
Nu_2 = \frac{1}{R_t [\theta (1/4) - \theta(-1/4)] + (1 - R_t)} \left. \frac{d\theta}{dy} \right|_{y=1/4}.
\]

### 11.3 Method of Solutions

#### 11.3.1 Solution using DTM

Now we apply the Differential Transform method into equation (11.23). Taking the differential transform of equation (11.23) with respect to $k$, according to Table 1.1, gives:

\[
\bar{U} (k+6) = \frac{k^6}{a_7^2 (k+6)!} ((1 \mp a_c^2 \phi_h) (k+1) (k+2) (k+3) (k+4) \bar{U} (k+4)
-GR Br \sum_{r=0}^{k} (k-r+1) (r+1) \bar{U} (k-r+1) \bar{U} (r+1)
\pm \phi_h (k+1) (k+2) \bar{U} (k+2) \pm 48 \phi_h \delta (k))
\]

where $\bar{U} (k)$ is the transformed functions of $u (y)$.  

264
The transformed initial conditions are as follows

$$
\tilde{U} (0) = a_1, \quad \tilde{U} (1) = a_2, \quad \tilde{U} (2) = \frac{a_3}{2!}, \quad \tilde{U} (3) = \frac{a_4}{3!}, \quad \tilde{U} (4) = \frac{a_5}{4!}, \quad \tilde{U} (5) = \frac{a_6}{5!}.
$$

Using the conditions as given in equations (11.16)–(11.17), one can evaluate the unknowns $a_1, a_2, a_3, a_4, a_5$ and $a_6$. By using the DTM and the transformed boundary conditions, above equations that finally leads to the solution of a system of algebraic equations.

The temperature distribution in the channel is evaluating by substituting the solution of velocity in equation (11.27).

### 11.3.2 Solution using PM

In this section, both buoyancy forces and viscous dissipation are considered as non-negligible. First, equations (11.24)–(11.25) are solved by regular perturbation series method. Then the dimensionless temperature field is determined by means of equation (11.28). Let us consider the dimensionless parameter

$$
\epsilon = GRBr = Re Pr \frac{\beta g D}{c_p},
$$

which is independent of the reference temperature difference $\Delta T$. For fixed values of $Re, GR, Bi_2$ and $Bi_2$, the solution of equations (11.24) and (11.25) can be expressed by the perturbation expansion

$$
u(y) = u_0(y) + \epsilon u_1(y) + \epsilon^2 u_2(y) + \ldots = \sum_{n=0}^{\infty} \epsilon^n u_n(y).
$$

To obtain the solution to equations (11.23)–(11.25), with the form of equation (11.33), one first substitutes equation (11.33) into equation (11.23)–(11.25) and collects terms having power of $\epsilon$. Then, one equates the coefficient of $\epsilon$ to zero. Thus, one
obtains a sequence of boundary value problems which can be solved in succession and yields the unknown functions \( u_n(y) \). The boundary value problem for \( n = 0 \) is

\[
a_c^2 \frac{d^6 u_0}{dy^6} = (1 \mp a_c^2 \phi_h) \frac{d^4 u_0}{dy^4} \pm \phi_h \frac{d^2 u_0}{dy^2} \pm 48 \phi_h, \tag{11.34}
\]

with the boundary conditions

\[
u_0 = 0, \quad \frac{d^2 u_0}{dy^2} = 0 \quad \text{at} \quad y = \pm \frac{1}{4} \tag{11.35}
\]

\[
\left( \frac{d^2 u_0}{dy^2} - \frac{1}{Bi_1} \frac{d^4 u_0}{dy^4} - a_c^2 \frac{d^4 u_0}{dy^4} + \frac{a_c^2}{Bi_1} \frac{d^6 u_0}{dy^6} \right) \bigg|_{y=1/4} = -48 + \frac{R_t GR S}{2} \left( \frac{1}{Bi_1} + \frac{4}{Bi_2} \right),
\]

\[
\left( \frac{d^2 u_0}{dy^2} + \frac{1}{Bi_2} \frac{d^4 u_0}{dy^4} - a_c^2 \frac{d^4 u_0}{dy^4} + \frac{a_c^2}{Bi_2} \frac{d^6 u_0}{dy^6} \right) \bigg|_{y=-1/4} = -48 - \frac{R_t GR S}{2} \left( \frac{1}{Bi_1} + \frac{4}{Bi_2} \right). \tag{11.36}
\]

The solution of the equations (11.34)–(11.36) is given by

\[
u_0 = C_1 + C_2 y + C_3 \sinh \left( \frac{y}{a_c} \right) + C_4 \cosh \left( \frac{y}{a_c} \right) + C_5 \sinh \left( \sqrt{\phi_h y} \right) + C_6 \cosh \left( \sqrt{\phi_h y} \right) - 24 y^2 \tag{11.37}
\]

\[
\theta_0 = \frac{\phi_h (a_c^2 \phi_h - 1)}{GR} \left( C_5 \sinh \left( \sqrt{\phi_h y} \right) + C_6 \cosh \left( \sqrt{\phi_h y} \right) \right), \tag{11.38}
\]

for heat absorption.

\[
u_0 = D_1 + D_2 y + D_3 \cosh \left( \frac{y}{a_c} \right) + D_4 \sinh \left( \frac{y}{a_c} \right) + D_5 \cos \left( \sqrt{\phi_h y} \right) + D_6 \sin \left( \sqrt{\phi_h y} \right) - 24 y^2 \tag{11.39}
\]

\[
\theta_0 = \frac{\phi_h (a_c^2 \phi_h + 1)}{GR} \left( D_5 \cos \left( \sqrt{\phi_h y} \right) + D_6 \sin \left( \sqrt{\phi_h y} \right) \right), \tag{11.40}
\]

for heat generation.

The boundary value problem for every integer \( n > 0 \) is

\[
a_c^2 \frac{d^6 u_n}{dy^6} = (1 \mp a_c^2 \phi_h) \frac{d^4 u_n}{dy^4} \pm \phi_h \frac{d^2 u_n}{dy^2} - \sum_{j=0}^{n-1} \frac{du_j}{dy} \frac{du_{n-j-1}}{dy}, \tag{11.41}
\]

266
Flow and heat transfer of couple stress fluid in a vertical channel ···

with the boundary conditions

\[
\begin{align*}
    u_n &= 0, \quad \frac{d^2 u_n}{dy^2} = 0 \quad \text{at} \quad y = \pm \frac{1}{4} \\
    \left( \frac{d^2 u_n}{dy^2} - \frac{1}{Bi_1} \frac{d^3 u_n}{dy^3} - a_c^2 \frac{d^4 u_n}{dy^4} + \frac{a_c^2}{Bi_1} \frac{d^5 u_n}{dy^5} \right) \bigg|_{y = -1/4} &= 0, \quad \text{and} \\
    \left( \frac{d^2 u_n}{dy^2} + \frac{1}{Bi_2} \frac{d^3 u_n}{dy^3} - a_c^2 \frac{d^4 u_n}{dy^4} + \frac{a_c^2}{Bi_2} \frac{d^5 u_n}{dy^5} \right) \bigg|_{y = 1/4} &= 0. \quad (11.43)
\end{align*}
\]

Since \( u_0(y) \) is the known function given by equations (11.37) and (11.39) an iterative solution of equations (11.41)–(11.43) is possible for heat absorption and yields the functions \( u_n(y) \), \( n > 0 \). As a consequence of equations (11.27), (11.38), (11.40) and (11.41), the dimensionless temperature \( \theta \) can be written in the form

\[
\begin{align*}
    \theta(y) &= \frac{\phi_h (a_c^2 \phi_h - 1)}{GR} \left( C_5 \sinh \left( \sqrt{\phi_h} y \right) + C_6 \cosh \left( \sqrt{\phi_h} y \right) \right) + \frac{1}{GR} \sum_{n=1}^{\infty} e^n \left[ \frac{d^4 u_n}{dy^4} - a_c^2 \frac{d^2 u_n}{dy^2} \right], \quad (11.44) \\
    \theta(y) &= \frac{\phi_h (a_c^2 \phi_h + 1)}{GR} \left( D_5 \cos \left( \sqrt{\phi_h} y \right) + D_6 \sin \left( \sqrt{\phi_h} y \right) \right) + \frac{1}{GR} \sum_{n=1}^{\infty} e^n \left[ \frac{d^4 u_n}{dy^4} - a_c^2 \frac{d^2 u_n}{dy^2} \right], \quad (11.45)
\end{align*}
\]

for heat absorption,

for heat generation.

The constants appeared in the above solutions are listed in the appendix.
Flow and heat transfer of couple stress fluid in a vertical channel

11.3.2.1 Appendix

\[ B_1 = -48 + \frac{R_l GR_s}{2} \left( 1 + \frac{4}{B_{i1}} \right), \quad l_1 = \phi_h (\phi_h a^2 - 1) \left( \sinh \left( \frac{\sqrt{\phi_h}}{4} \right) + \frac{\sqrt{\phi_h}}{B_{i1}} \cosh \left( \frac{\sqrt{\phi_h}}{4} \right) \right), \]

\[ E_1 = 48 + B_1, \quad l_2 = \phi_h (1 - \phi_h a^2) \left( \cosh \left( \frac{\sqrt{\phi_h}}{4} \right) + \frac{\sqrt{\phi_h}}{B_{i1}} \sinh \left( \frac{\sqrt{\phi_h}}{4} \right) \right), \]

\[ B_2 = -48 - \frac{R_l GR_s}{2} \left( 1 + \frac{4}{B_{i2}} \right), \quad l_3 = \phi_h (1 - \phi_h a^2) \left( \sinh \left( \frac{\sqrt{\phi_h}}{4} \right) + \frac{\sqrt{\phi_h}}{B_{i2}} \cosh \left( \frac{\sqrt{\phi_h}}{4} \right) \right), \]

\[ E_2 = 48 + B_2, \quad l_4 = \phi_h (1 - \phi_h a^2) \left( \cosh \left( \frac{\sqrt{\phi_h}}{4} \right) + \frac{\sqrt{\phi_h}}{B_{i2}} \sinh \left( \frac{\sqrt{\phi_h}}{4} \right) \right), \]

\[ C_6 = \frac{E_{11} l_5 - E_{12} l_6}{l_{12} l_3 - l_{14} l_5}, \quad C_4 = \frac{a^2}{\cosh \left( \frac{1}{4a} \right)} \left( 48 - \phi_h C_6 \cosh \left( \frac{\sqrt{\phi_h}}{4} \right) \right), \quad C_3 = \frac{-a^2 \phi_h C_5 \sinh \left( \frac{\sqrt{\phi_h}}{4} \right)}{\sinh \left( \frac{1}{4a} \right)}, \]

\[ C_2 = 4 (a^2 \phi_h - 1) C_5 \sinh \left( \frac{\sqrt{\phi_h}}{4} \right), \quad C_1 = (a^2 \phi_h - 1) C_6 \cosh \left( \frac{\sqrt{\phi_h}}{4} \right) - 48 a^2 + \frac{3}{2}, \]

\[ l_5 = \phi_h (\phi_h a^2 + 1) \left( \frac{\sqrt{\phi_h}}{B_{i1}} \sin \left( \frac{\sqrt{\phi_h}}{4} \right) - \cos \left( \frac{\sqrt{\phi_h}}{4} \right) \right), \]

\[ l_6 = \phi_h (1 + \phi_h a^2) \left( \sin \left( \frac{\sqrt{\phi_h}}{4} \right) + \frac{\sqrt{\phi_h}}{B_{i1}} \cos \left( \frac{\sqrt{\phi_h}}{4} \right) \right), \]

\[ l_7 = \phi_h (1 + \phi_h a^2) \left( \frac{\sqrt{\phi_h}}{B_{i2}} \sin \left( \frac{\sqrt{\phi_h}}{4} \right) - \cos \left( \frac{\sqrt{\phi_h}}{4} \right) \right), \]

\[ l_8 = \phi_h (1 + \phi_h a^2) \left( \frac{\sqrt{\phi_h}}{B_{i2}} \cos \left( \frac{\sqrt{\phi_h}}{4} \right) - \sin \left( \frac{\sqrt{\phi_h}}{4} \right) \right), \]

\[ D_5 = \frac{E_{11} l_5 - E_{12} l_6}{l_{12} l_5 - l_{14} l_7}, \quad D_6 = \frac{E_{11} l_7 - E_{12} l_5}{l_{12} l_7 - l_{14} l_9}, \quad D_3 = \frac{a^2}{\cosh \left( \frac{1}{4a} \right)} \left( 48 + \phi_h D_5 \cos \left( \frac{\sqrt{\phi_h}}{4} \right) \right), \]

\[ D_4 = \frac{a^2 \phi_h D_6 \sin \left( \frac{\sqrt{\phi_h}}{4} \right)}{\sinh \left( \frac{1}{4a} \right)}, \quad D_2 = -4 D_4 \sin \left( \frac{1}{4a} \right) - 4 D_6 \sin \left( \frac{\sqrt{\phi_h}}{4} \right), \]

\[ D_1 = \frac{3}{2} - D_3 \cosh \left( \frac{1}{4a} \right) - D_5 \cos \left( \frac{\sqrt{\phi_h}}{4} \right). \]

11.4 Results and Discussion

In the previous section, an analytical solution for the problem of the mixed convective flow of couple stress fluid with heat source/sink through a vertical channel was obtained using the Differential Transform method and regular perturbation method. The product of the mixed convection parameter \( GR = Gr/Re \) and the Brinkman number was used as a perturbation parameter. In this section, we choose to present velocity \( u \), temperature \( \theta \), mass flow rate and Nusselt number for different values of mixed convection parameter \( GR \), Brinkman number \( Br \), couple stress parameter \( a_c \), heat source/sink parameter \( \phi_h \) and Biot numbers \( Bi_1, Bi_2 \) for both asymmetric and
Flow and heat transfer of couple stress fluid in a vertical channel ···

symmetric wall boundary conditions in graphical and tabular form.

Figure 11.2 illustrates the velocity field $u$ for different values of mixed convection parameter $GR$ is the ratio of Grashof number to Reynolds number, couple stress parameter $a_c$ and heat generation/absorption parameter $\phi_h$ are shown in Figure 11.2 when $Br = 0$ (case of negligible viscous dissipation) with $Bi_1 = Bi_2 = 10$ and $R_t = 1$. It is seen that the velocity profile for $GR = 100, 500, 1000$ shows decrease in nature on the left half of the channel ($y = 0$) and increase in nature on the right half of the channel for all values of couple stress parameter $a_c$ and heat generation/absorption $\phi_h$. It is seen that the flow reversal is observed for $GR = 1000$ for small values of couple stress parameter near the cold wall $y = -\frac{1}{4}$. With $U_0 > 0$ i.e., for upward flow it is expected that for sufficiently large values of Grashof number a flow reversal induced by the buoyancy forces occurs at the cold wall. It is also noticed that couple stress parameter $a_c$ decreases the velocity and minimize the flow reversal near the boundaries.

The dimensionless temperature field $\theta$ evaluated for different values of the Brinkman number ($Br$) and couple-stress parameter ($a_c$) is shown in Figures 11.3(a,b) for equal and unequal Biot numbers, respectively with $GR = 0$ (case of negligible buoyancy force) and $R_t = 1$. These figures show that the temperature field increases with increasing values of $Br$ for both equal and unequal Biot numbers as can be seen from Figures 11.3(a,b). Further, these figures also show that as the heat generation/absorption parameter $\phi_h$ increases, temperature field increases. The couple stress parameter $a_c$ decreases the temperature field for both equal and unequal Biot numbers and it is observed that the temperature field is more operative for small values of couple stress parameter. From these figures it is noticed that the heat generation is more operative when compared to heat absorption also for unequal Biot numbers the temperature is more effective near the cold wall as seen in Figure 11.3b.

The effect of the couple stress parameter $a_c$ on the velocity profiles in the channel is shown in Figure 11.4. It is observed from Figure 11.4 that as the couple-stress
Flow and heat transfer of couple stress fluid in a vertical channel ···

parameter increases, the velocity decreases. Therefore flow can be stopped by choosing a large couple-stress parameter and on the other hand, the maximum flow rate can be achieved by choosing a small couple stress parameter $a_c$. Since $a_c$ indicates the relation between the chain length of the molecules and distance between the plates. That is $a_c = \frac{l}{D}$ where $l = \sqrt{\frac{2}{\mu}}$ is the material constant. If $l$ is a function of the molecular dimensions of the liquid, it will vary greatly for different liquids. For example, the length of a polymer chain may be a million times the diameter of water molecule (Stokes, 1984). Therefore, there are all the reasons to expect that couple-stresses appear in noticeable magnitudes in liquids with large molecules. It is also seen from Figure 11.4 that small values of couple stress parameter $a_c$ will lead to the flow nature being the same as that of viscous fluid and large values of $a_c$ will lead to retard the flow. We can also notice from Figure 11.4 that the effect of heat generation $\phi_h$ is more effective when compared to heat absorption for small values of $a_c$. Plots of temperature $\theta$ for different values of couple stress parameter $a_c$ and heat generation/absorption parameter $\phi_h$ with $Bi_1 = Bi_2 = 10$ and $R_t = 1$ is reported in Figure 11.5. As the couple stress parameter increases the temperature decreases for both heat generation and absorption. It is seen from Figure 11.5 that the heat generation effect is more whereas heat absorption effect is less.

The effect of heat generation/absorption parameter $\phi_h$ on the flow field is shown in Figures 11.6 and 11.7. It is observed that velocity increases with increase of heat generation and velocity decreases with increase of heat absorption as seen in Figure 11.6. Similar effect of heat generation/absorption is observed on temperature as seen in Figure 11.7. That is, temperature increases for heat generation and decreases for heat absorption.

Figures 11.8 and 11.9 shows the variation of velocity $u$ and temperature $\theta$ for different values of mixed convection parameter $GR$ and heat generation/absorption parameter $\phi_h$ with equal Biot numbers $Bi_1 = Bi_2 = 10$. It is seen that, mixed
convection parameter increases the velocity and temperature fields. This is because that enhancement of viscous dissipation results in higher values of temperature which in turn enhances the values of the buoyancy force. The increase of the buoyancy force yields an increase of the fluid velocity. Here also velocity and temperature are more for heat generation when compared to heat absorption.

Plots of velocity $u$ and temperature $\theta$ for different values of Brinkman number $Br$ and heat generation/absorption parameter $\phi_h$ with $Bi_1 = Bi_2 = 10$ and $R_t = 1$ is reported in Figures 11.10 and 11.11. Figure 11.10 show that, the velocity, at each position, are increasing functions of $Br$. The temperature $\theta$ is also increases with increase in Brinkman number as seen in Figure 11.11. Moreover the velocity and temperature are more effective for hear generation when compared to heat absorption. To enhance the velocity of a couple stress fluid, viscous dissipation must be taken into account in the energy balance equation. The effect of viscous dissipation may become very important in several flow configurations occurring in the engineering practice. In fact viscous dissipation affects strongly the heat transfer process whenever the operating fluid a low thermal conductivity and a high viscosity.

Figures 11.12 and 11.13 depict the velocity $u$ and temperature $\theta$ for different values of heat generation and absorption parameter $\phi_h$ with unequal Biot numbers. As the heat generation parameter $\phi_h$ increases both velocity and temperature increases and reversal effect is observed for heat absorption parameter. It is noted that the velocity and temperature are more for $\phi_h = 0$ and the flow fields are more effective for heat absorption when compared to heat generation.

Figures 11.14 and 11.15 illustrates the effect of Brinkman number $Br$ and heat generation/absorption parameter $\phi_h$ on velocity and temperature fields respectively for unequal Biot numbers $Bi_1 = 0.1$, $Bi_2 = 10$. As the Brinkman number increases both velocity and temperature increase which is the similar result observed in the case of equal Biot numbers. It is observed that the effect of generation/absorption
parameter $\phi_h$ is invariant in the absence of viscous dissipation ($Br = 0$) whereas the
temperature is more for heat generation and less for heat absorption for $Br \neq 0$. Here
also the temperature is more effective near the cold wall.

The mass flow rate for different values of mixed convection parameter $GR$ and
Brinkman number $Br$ are plotted against couple stress parameter $a_c$ in Figures 11.16
and 11.17 respectively for both heat generation and absorption. It is observed that
mass flow rate increases as mixed convection parameter $GR$ and Brinkman number $Br$
increases while decreases with increase in couple stress parameter and it invariant
after the value $a_c = 0.4$ (approximately). It is seen from Figure 11.16 that the effect
of mixed convection parameter is small for both heat generation and absorption. The
effect of heat generation parameter is more effective when compared to heat absorption
as seen in Figure 11.17.

Finally, Figures 11.18 and 11.19 display the influence of mixed convection parameter
$GR$, Brinkman number $Br$, heat generation/absorption parameter $\phi_h$ and couple stress
parameter $a$ on the Nusselt number at both the walls with $Re = 1$ and $Bi_1 = Bi_2 = 10$.
It is observed that Nusselt number increases at the right wall (hot wall) for increasing
the values of mixed convection parameter $GR$, Brinkman number $Br$ and couple stress
parameter while it decreases at the left wall (cold wall) for both heat generation and
absorption. Here also the couple stress parameter shows the negligible effect after
the values $a_c = 0.8$ (approximately) for mixed convection parameter and $a_c = 0.5$
(approximately) for Brinkman number.

**11.4.1 Validity of DTM solution**

In order to verify the accuracy of the present method, we have compared our
results with those of numerical method using the Runge-Kutta shooting method and
analytical method using the perturbation method. The graphical results show accuracy
of the DTM method. In particular, Figures 11.20 and 11.21 shows the velocity and
temperature distribution for different values of Brinkman number $Br$. As it can be seen, this method leads to tremendously accurate results. Moreover, it can be clearly seen from these figures that, as $Br$ increases, the velocity and temperature increases for both heat generation and absorption. From these figures we find out that DTM leads to acceptable results between DTM and RKSM when compared to PM and RKSM even for large $Br$. Table 11.1 also reveals the comparison results, it is seen that DTM and RKSM are agree upto four decimal places for $a_c = 0.2$ whereas it agree very well for $a_c = 0.5$. But PM and RKSM are agree upto one decimal place for $a_c = 0.2$ whereas it agree three decimal place only. It is clearly seen that DTM gives more acceptable result. Other sources of validity of our results are listed in Table 11.2, in which the present results for the velocity and temperature are compared with the results of Zanchini (1998) and Umavathi and Veershetty (2012) and also compare the results of Nusselt number $GRNu$ (Table 11.3) for the limiting case of $a_c = 0$ and $\phi_h = 0$. The results show good agreement.
Flow and heat transfer of couple stress fluid in a vertical channel.

Figure 11.2: Plots of $u$ versus $y$ for different values of $GR$ and $a_c$ with $Br = 0$, $Bi_1 = Bi_2 = 10$ and $R_t = 1$.

Figure 11.3: Plots of $\theta$ versus $y$ for different values of $Br$ and $a_c$ with $GR = 0$, $R_t = 1$ (a) $Bi_1 = Bi_2 = 10$ and (b) $Bi_1 = 1$, $Bi_2 = 10$.
Flow and heat transfer of couple stress fluid in a vertical channel

Figure 11.4: Plots of $u$ versus $y$ for different values of $a_c$.

Figure 11.5: Plots of $\theta$ versus $y$ for different values of $a_c$. 

275
Flow and heat transfer of couple stress fluid in a vertical channel ...
Flow and heat transfer of couple stress fluid in a vertical channel ··

Figure 11.8: Plots of $u$ versus $y$ for different values of $GR$.

Figure 11.9: Plots of $\theta$ versus $y$ for different values of $GR$.
Figure 11.10: Plots of $u$ versus $y$ for different values of $Br$.

Figure 11.11: Plots of $\theta$ versus $y$ for different values of $Br$. 

Flow and heat transfer of couple stress fluid in a vertical channel ...
Flow and heat transfer of couple stress fluid in a vertical channel

Figure 11.12: Plots of $u$ versus $y$ for different values of $\phi_h$.

Figure 11.13: Plots of $\theta$ versus $y$ for different values of $\phi_h$. 

279
Flow and heat transfer of couple stress fluid in a vertical channel...

Figure 11.14: Plots of $u$ versus $y$ for different values of $Br$.

Figure 11.15: Plots of $\theta$ versus $y$ for different values of $Br$. 
Flow and heat transfer of couple stress fluid in a vertical channel

Figure 11.16: Mass flow rate versus couple stress parameter $a_c$ for different values of $GR$.

Figure 11.17: Mass flow rate versus couple stress parameter $a_c$ for different values of $Br$. 
Flow and heat transfer of couple stress fluid in a vertical channel...

Figure 11.18: Nusselt number versus couple stress parameter $a_c$ for different values of $GR$.

Figure 11.19: Nusselt number versus couple stress parameter $a_c$ for different values of $Br$. 

282
Flow and heat transfer of couple stress fluid in a vertical channel

Figure 11.20: Plots of $u$ versus $y$ for different values of $Br$.

Figure 11.21: Plots of $\theta$ versus $y$ for different values of $Br$. 
Table 11.1: Comparison of velocity and temperature with $GR = 10$, $Br = 0.1$, $\phi_h = 5$, $R_t = 1$ and $Bi_1 = Bi_2 = 10$.

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Flow and heat transfer of couple stress fluid in a vertical channel

Table 11.2: Comparison of velocity and temperature with $GR = 100$, $\phi_h = 0$, $a_c = 0.0$, $Br = 0.0001$, $Bi_1 = Bi_2 = 10$.

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Table 11.3: Values of $GR Nu$ as a function of $\epsilon$ and $Bi_1 = Bi_2 = Bi$ for $R_t = 0$, $\phi_h = 0$, $a_c = 0.0$ (completely symmetric case).

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