Chapter 10

Mixed convective flow of a couple stress fluid with Robin boundary conditions

10.1 Introduction

Mixed convection heat transfer in vertical channels occurs in many industrial processes and natural phenomena. It has therefore been the subject of many detailed, mostly numerical studies for different flow configurations. Most of the interest in this subject is due to its applications, for instance, in the design of cooling systems for electronic devices and in the field of solar energy collection. Some of the published papers on this topic, such as Aung and Worku (1986a), Cheng et al. (1990), Barletta (1998, 1999), Barletta and Zanchini (1999), Chamkha (2002), Barletta et al. (2008b), El-Din (2003) deal with the evaluation of the temperature and velocity profiles for the vertical parallel flow fully developed regime. As is well known, heat exchanger technology involves convective flows in vertical channels. In most cases, these flows imply conditions of uniform heating of a channel, which can be modeled either by uniform wall temperature
Mixed convective flow of a couple stress fluid with Robin boundary conditions. Analytical solution for fully developed mixed convection flow between two parallel vertical plates with heat and mass transfer was presented by Boulama and Galanis (2004). Mokni et al. (2009) investigated numerically the mixed convection in a vertical heated channel. Kumar et al. (2011a,b) have studied the effect of mixed convection of immiscible viscous fluids in the presence and absence of applied magnetic field in a vertical channel.

All the above quoted analyses of mixed convection flow in vertical channels are based on the hypothesis that the fluids are Newtonian. Recently, interest in problems of non-Newtonian fluids has grown considerably because of more and more applications in chemical process industries, food preservation techniques, petroleum production and in power engineering. Many industrial fluids are non-Newtonian in nature and their characteristics are considered rheological. More particularly, slurries (china clay and coal in water, sewage sludge, etc.), multiphase mixtures (oil–water emulsions, gas–liquid dispersions such as froths and foams, butter etc.) are non-Newtonian fluids. Further more examples displaying a variety of non-Newtonian fluid characteristics include pharmaceutical formulations, cosmetics and toiletries, paints, synthetic lubricants, biological fluids (blood, synovial fluid saliva etc.), and foodstuffs (jams, jellies, soaps, marmalades etc.). Several models have been proposed to explain the non-Newtonian behavior of fluids. Some of the published papers on different non-Newtonian fluids, such as Brain (1983); Ellahi et al. (2010); Fetecau et al. (2009); Kaloni and Lou (2005); Kaloni and Siddiqui (1983); Rudraiah et al. (1990) and Sajid et al. (2010). Among these, couple stress fluids introduced by Stokes (1966) have distinct features, such as the presence of couple stresses, body couples and non-symmetric stress tensor. The couple stress fluid theory presents models for fluids whose microstructure is mechanically significant. The effect of very small microstructure in a fluid can be felt if the characteristic geometric dimension of the problem considered is of the same order of magnitude as the size of the microstructure. The main feature of couple stresses is
to introduce a size dependent effect. Classical continuum mechanics neglects the size effect of material particles within the continua. This is consistent with ignoring the rotational interaction among particles, which results in symmetry of the force-stress tensor. However, in some important cases such as fluid flow with suspended particles, this cannot be true and a size dependent couple-stress theory is needed. The spin field due to microrotation of freely suspended particles set up an antisymmetric stress, known as couple-stress, and thus forming couple-stress fluid. These fluids are capable of describing various types of lubricants, blood, suspension fluids etc. The study of couple-stress fluids has applications in a number of processes that occur in industry such as the extrusion of polymer fluids, solidification of liquid crystals, cooling of metallic plate in a bath, and colloidal solutions etc. Stokes (1966) discussed the hydromagnetic steady flow of a fluid with couple stress effects. A review of couple stress (polar) fluid dynamics was reported by Stokes (1984). Umavathi and Malashetty (1999) have analyzed the flow and heat transfer characteristics of Oberbeck convection of a couple stress fluid in a vertical porous stratum. Flow and heat transfer of a couple stress fluid sandwiched between viscous fluid layers was also analyzed by Umavathi et al. (2005a). Finally, the most recent paper by Srinivasacharya and Kaladhar (2012a) who have investigated the Hall and Ion-slip effects on fully developed electrically conducting couple stress fluid flow between vertical parallel plates in the presence of a temperature dependent heat source. Employing similarity transformations the governing nonlinear partial differential equations are transformed into nonlinear ordinary differential equations. These equations are then solved analytically using the homotopy analysis method.

Shah and London (1971) have summarized the laminar forced convection heat transfer results for various channel cross sections, which were reported in literature until 1970. In the past, the laminar forced convection heat transfer in the thermal entrance region of a rectangular channel has been analyzed either for the temperature boundary condition of the first kind, characterized by prescribed wall temperature (Javeri, 1976;
Mixed convective flow of a couple stress fluid with Robin boundary ···

Lyczkowski et al., 1969; Wibulswas, 1966), or for the boundary condition of the second kind, expressed by the prescribed wall heat flux (Sparrow and Siegal, 1960). A more realistic condition in many applications, however, will be temperature boundary condition of the third kind also known as Robin boundary conditions: the local wall heat flux is a linear function of the local wall temperature. Heat transfer in the laminar region of a flat channel for the temperature boundary condition of the third kind was explored by Javeri (1977). Zanchini (1998) analyzed the effect of viscous dissipation on mixed convection in a vertical channel with boundary conditions of the third kind. Umavathi and Sultana (2011) did the analytical study of mixed convection of a micropolar fluid in a vertical channel with boundary conditions of the third kind in the presence of heat source/sink. The effects of viscous dissipation and heat source/sink on fully developed mixed convection for the laminar flow in a parallel-plate vertical channel were investigated by Umavathi et al. (2012b). Recently, Umavathi and Veershetty (2012) is analyzed the combined free and forced convection flow in a parallel plate vertical channel filled with porous matrix in the fully developed region with boundary conditions of third kind.

The motivation of this work is to extend the DTM for the problem solved by Zanchini (1998) replacing the Newtonian fluid by a non-Newtonian fluid (couple stress fluid).

10.2 Mathematical Formulation

We consider the steady, laminar flow of an incompressible couple stress fluid in the fully developed region of a parallel plate vertical channel. The $X$-axis coincides with the axial plane of the channel and directed against gravity. The $Y$-axis is perpendicular to the walls. The channel is assumed to occupy the region of space $-b/2 \leq Y \leq b/2$. The flow regime is subjected to the influence of couple stress fluid is as shown in Figure 10.1.
The physical properties characterizing the fluid, namely, the thermal conductivity, the thermal diffusivity, the dynamic viscosity and the thermal expansion coefficient of the fluid are assumed to be constants. As customary, the Boussinesq approximation and the equation of state

\[ \rho = \rho_0 [1 - \beta (T - T_0)] \]  

will be adopted. Moreover, it will be assumed that the only non-zero component of the velocity field \( U \) is the \( X \)-component \( U \). Thus, since \( \nabla U = 0 \), one has

\[ \frac{\partial U}{\partial X} = 0. \]  

so that \( U \) depends only on \( Y \). The momentum equations governing the motion of an incompressible couple stress fluid in the absence of body couple are given by (Stokes,
Mixed convective flow of a couple stress fluid with Robin boundary ···

1966)

\[ \beta g \rho_0 (T - T_0) - \frac{\partial P}{\partial X} + \mu \frac{d^2 U}{dY^2} - \eta_c \frac{d^4 U}{dY^4} = 0, \]  \hspace{1cm} (10.3)

\[ \frac{\partial P}{\partial Y} = 0. \]  \hspace{1cm} (10.4)

where \( P = p + \rho_o g X \). Since on account of equation (10.4) \( P \) depends only on \( X \), equation (10.3) can be rewritten as

\[ T - T_0 = \frac{1}{\beta g \rho_0} \frac{dP}{dX} - \frac{\nu}{\beta g} \frac{d^2 U}{dY^2} + \frac{\eta_c}{\beta g \rho_0} \frac{d^4 U}{dY^4}. \]  \hspace{1cm} (10.5)

In case of polar fluids the action of one part of the body on its neighborhood cannot be represented by a force alone but rather by a force and couple. The last term on the left-hand side of equation (10.3) represents the effects of couple stresses in an incompressible fluid. Here \( \eta_c \) is a material constant responsible for the couple stress property and has the dimension of momentum \((MLT^{-1})\). According to Stokes’ theory, the rheological flow properties for an incompressible viscous couple stress fluid are characterized by the two constants \( \mu \) and \( \eta_c \). Since the dimension of \( \mu \) is \( ML^{-1}T^{-1} \) and that of \( \eta_c \) is \( MLT^{-1} \), the ratio \((\eta_c/\mu)^{1/2}\) has the dimension of length.

The effect of couple stress are quite large for large values of the non-dimensional number \( m = l/d \), where \( d \) is the typical dimension of the flow geometry and \( l \) is the material constant \( l = (\eta_c/\mu)^{1/2} \). If \( l \) is a function of the molecular dimensions of the liquid, it will vary greatly for different liquids. For example, the length of a polymer chain may be a million times the diameter of water molecule. Therefore, one may expect that couple stresses appear in noticeable magnitudes in liquids with large molecules.

From equation (10.5), one obtains,

\[ \frac{\partial T}{\partial X} = \frac{1}{\beta g \rho_0} \frac{d^2 P}{dX^2}. \]  \hspace{1cm} (10.6)
Mixed convective flow of a couple stress fluid with Robin boundary ···

\[ \frac{\partial T}{\partial Y} = -\frac{\nu}{\beta g} \frac{d^3 U}{dY^3} + \frac{\eta_c}{\beta g \rho_0} \frac{d^5 U}{dY^5}, \]  \hspace{1cm} (10.7)  

\[ \frac{\partial^2 T}{\partial Y^2} = -\frac{\nu}{\beta g} \frac{d^4 U}{dY^4} + \frac{\eta_c}{\beta g \rho_0} \frac{d^6 U}{dY^6}. \]  \hspace{1cm} (10.8)  

Both the walls of the channel will assumed to have a negligible thickness and to exchange heat by convection with an external fluid. In particular, at \( Y = -b/2 \) the external convection coefficient will be considered as uniform with the value \( h_1 \) and the fluid in the region \( Y < -b/2 \) will be assumed to have a uniform reference temperature \( T_1 \). At \( Y = b/2 \) the external convection coefficient will be considered as uniform with the value \( h_2 \) and the fluid in the region \( Y > b/2 \) will be assumed to have a uniform reference temperature \( T_2 \) (\( T_2 \geq T_1 \)). Therefore, the boundary conditions on the temperature field can be expressed as

\[ -K \left. \frac{\partial T}{\partial Y} \right|_{Y=-b/2} = h_1 [T_1 - T(X,-b/2)], \]  \hspace{1cm} (10.9)  

\[ -K \left. \frac{\partial T}{\partial Y} \right|_{Y=b/2} = h_2 [T(X,b/2) - T_2]. \]  \hspace{1cm} (10.10)  

On account of equation (10.7), equations (10.9) and (10.10) can be rewritten as

\[ \left( \frac{d^3 U}{dY^3} - \frac{\eta_c}{\rho_0 \nu} \frac{d^5 U}{dY^5} \right) \bigg|_{Y=-b/2} = \frac{\beta g h_1}{K \nu} [T_1 - T(X,-b/2)], \]  \hspace{1cm} (10.11)  

\[ \left( \frac{d^3 U}{dY^3} - \frac{\eta_c}{\rho_0 \nu} \frac{d^5 U}{dY^5} \right) \bigg|_{Y=b/2} = \frac{\beta g h_2}{K \nu} [T(X,b/2) - T_2]. \]  \hspace{1cm} (10.12)  

It is easily verified that the equations (10.11) and (10.12) imply that \( \partial T/\partial X \) is zero both at \( Y = -b/2 \) and at \( Y = b/2 \). Since equation (10.6) ensures that \( \partial T/\partial X \) does not depend on \( Y \), one is led to the conclusion that \( \frac{\partial T}{\partial X} = 0 \) everywhere. Therefore, the temperature \( T \) depends only on \( Y \) i.e., \( T = T(Y) \). Thus, on account of equation

230
Mixed convective flow of a couple stress fluid with Robin boundary · · ·

(10.6), there exists a constant $A$ such that

\[
\frac{dP}{dX} = A. \tag{10.13}
\]

For the problem under exam, the energy balance equation in the presence of viscous dissipation can be written as,

\[
\frac{d^2T}{dY^2} = -\frac{\nu}{\alpha C_p} \left( \frac{dU}{dY} \right)^2. \tag{10.14}
\]

Equations (10.8) and (10.14) yield a differential equation for $U$, namely

\[
\eta_c \frac{d^6U}{dY^6} = \mu \frac{d^4U}{dY^4} - \frac{\mu \rho_0 g \beta}{K} \left( \frac{dU}{dY} \right)^2. \tag{10.15}
\]

The boundary conditions on $U$ are

\[
U = 0, \quad \frac{d^2U}{dY^2} = 0 \quad \text{at} \quad Y = \pm \frac{b}{2} \tag{10.16}
\]

together with the equations (10.11) and (10.12), which on account of equation (10.5), can be rewritten as

\[
\left. \left( \frac{d^3U}{dY^3} - \frac{h_1}{K} \frac{d^2U}{dY^2} \right) - \frac{\eta_c}{\mu} \left( \frac{d^6U}{dY^6} + \frac{h_1 \eta_c}{K} \frac{d^4U}{dY^4} \right) \right|_{Y=-b/2} = -\frac{Ah_1}{\mu K} - \frac{\beta g h_1}{K \nu} (T_0 - T_1). \tag{10.17}
\]

\[
\left. \left( \frac{d^3U}{dY^3} + \frac{h_2}{K} \frac{d^2U}{dY^2} \right) - \frac{\eta_c}{\mu} \left( \frac{d^6U}{dY^6} + \frac{h_2 \eta_c}{K} \frac{d^4U}{dY^4} \right) \right|_{Y=b/2} = \frac{Ah_2}{\mu K} - \frac{\beta g h_2}{K \nu} (T_2 - T_0). \tag{10.18}
\]

Equations (10.15)–(10.18) determine the velocity distribution, they can be written in a dimensionless form by means of the following dimensionless parameters

\[
u = \frac{U}{U_0}, \quad \theta = \frac{T - T_0}{\Delta T}, \quad y = \frac{Y}{D}, \quad Gr = \frac{g \beta \Delta T D^3}{\nu^2}, \quad Re = \frac{U_0 D}{\nu}, \quad Br = \frac{\mu U_0^2}{K \Delta T}, \quad Pr = \frac{\nu}{\alpha},
\]

231
Mixed convective flow of a couple stress fluid with Robin boundary 

\[
GR = \frac{Gr}{Re}, \quad R_t = \frac{T_2 - T_1}{\Delta T}, \quad a_c^2 = \frac{\eta_c}{\mu D^2}, \quad Bi_1 = \frac{h_1 D}{K}, \quad Bi_2 = \frac{h_2 D}{K},
\]

(10.19)

\[
S = \frac{Bi_1 Bi_2}{Bi_1 Bi_2 + 2Bi_1 + 2Bi_2},
\]

where \( D = 2b \) is the hydraulic diameter. The reference velocity \( U_0 \) and the reference temperature \( T_0 \) are given by

\[
U_0 = -\frac{A D^2}{48\mu}, \quad T_0 = \frac{T_1 + T_2}{2} + S \left( \frac{1}{Bi_1} - \frac{1}{Bi_2} \right) (T_2 - T_1).
\]

(10.20)

The reference temperature difference \( \Delta T \) is given by

\[
\Delta T = T_2 - T \quad \text{if } T_1 < T_2
\]

\[
\Delta T = \frac{\nu^2}{C_p D^2} \quad \text{if } T_1 = T_2.
\]

(10.21)

Therefore, as in Barletta (1998), the value of the dimensionless parameter \( R_t \) can be either 0 or 1. More precisely, \( R_t \) equals 1 for asymmetric fluid temperatures \( T_1 < T_2 \), and equals 0 for symmetric fluid temperatures \( T_1 = T_2 \).

The dimensionless mean velocity \( \bar{u} \) is given by

\[
\bar{u} = 2 \int_{-1/4}^{1/4} u \, dy,
\]

(10.22)

On account of equation (10.13), for upward flow \( A < 0 \), so that \( Re \) and \( GR \) are positive. For downward flow \( A > 0 \), while \( U_0, Re \) and \( GR \) negative. By employing the dimensionless quantities defined in equation (10.19), equations (10.15)-(10.18) can be rewritten as

\[
a_c^2 \frac{d^6 u}{dy^6} = \frac{d^4 u}{dy^4} - GR Br \left( \frac{du}{dy} \right)^2,
\]

(10.23)

\[
u = 0, \quad \frac{d^2 u}{dy^2} = 0 \quad \text{at} \quad y = \pm \frac{1}{4}
\]

(10.24)
Mixed convective flow of a couple stress fluid with Robin boundary ···

\[
\left( \frac{d^2 u}{dy^2} - \frac{1}{Br_1} \frac{d^3 u}{dy^3} - a_c^2 \frac{d^4 u}{dy^4} + \frac{a_c^2}{Br_1} \frac{d^5 u}{dy^5} \right) \bigg|_{y=-1/4} = -48 + \frac{R_tGRS}{2} \left( 1 + \frac{4}{Br_1} \right),
\]

\[
\left( \frac{d^2 u}{dy^2} + \frac{1}{Br_2} \frac{d^3 u}{dy^3} - a_c^2 \frac{d^4 u}{dy^4} - \frac{a_c^2}{Br_2} \frac{d^5 u}{dy^5} \right) \bigg|_{y=1/4} = -48 - \frac{R_tGRS}{2} \left( 1 + \frac{4}{Br_2} \right),
\]

when \( a = 0 \), i.e., in the absence of couple stresses, we get the classical results for clear viscous fluid (see e.g., Zanchini, 1998).

Similarly equation (10.14), using equation (10.19) yield

\[
\frac{d^2 \theta}{dy^2} + Br \left( \frac{du}{dy} \right)^2 = 0,
\]

while from equation (10.5) and (10.19) one obtains

\[
\theta = -\frac{1}{GR} \left( 48 + \frac{d^2 u}{dy^2} - a_c^2 \frac{d^4 u}{dy^4} \right),
\]

Equations (10.23)-(10.27) show that the dimensionless velocity profile and the dimensionless temperature profile depend on six parameters: the mixed convection parameter \( GR \), the Brinkman number \( Br \), couple stress parameter \( a_c \), the temperature difference ratio \( R_t \) and the Biot numbers \( Bi_1 \) and \( Bi_2 \).

A Nusselt number can be defined at each boundary, as follows:

\[
Nu_1 = \frac{D}{R_t \left[ T(b/2) - T(-b/2) \right] + (1 - R_t) \Delta T} \frac{dT}{dY} \bigg|_{y=-b/2},
\]

\[
Nu_2 = \frac{D}{R_t \left[ T(b/2) - T(-b/2) \right] + (1 - R_t) \Delta T} \frac{dT}{dY} \bigg|_{y=b/2}.
\]

By employing equation (10.19), equation (10.28) can be written as

\[
Nu_1 = \frac{1}{R_t \left[ \theta(1/4) - \theta(-1/4) \right] + (1 - R_t) \frac{d\theta}{dy}} \bigg|_{y=-1/4},
\]

\[
Nu_2 = \frac{1}{R_t \left[ \theta(1/4) - \theta(-1/4) \right] + (1 - R_t) \frac{d\theta}{dy}} \bigg|_{y=1/4}.
\]
10.3 Method of Solutions

10.3.1 Solution using DTM

Now we apply the Differential Transform method into equation (10.23). Taking the differential transform of equation (10.23) with respect to $k$, according to Table 1.1, gives:

\[
\bar{U}(k+6) = \frac{1}{a_2^2(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)} [(k+1)(k+2)(k+3)(k+4) \bar{U}(k+4) - GRBr \sum_{r=0}^{k} (k-r+1)(r+1) \bar{U}(k-r+1) \bar{U}(r+1)]
\]

(10.30)

where $\bar{U}(k)$ is the transformed functions of $u(y)$.

The transformed initial conditions are as follows

\[
\bar{U}(0) = a_1, \quad \bar{U}(1) = a_2, \quad \bar{U}(2) = \frac{a_3}{2!}, \quad \bar{U}(3) = \frac{a_4}{3!}, \quad \bar{U}(4) = \frac{a_5}{4!}, \quad \bar{U}(5) = \frac{a_6}{5!}
\]

(10.31)

Using the conditions as given in equations (10.24) and (10.25), one can evaluate the unknowns $a_1, a_2, a_3, a_4, a_5$ and $a_6$. By using the DTM and the transformed boundary conditions, above equations that finally leads to the solution of a system of algebraic equations.

The temperature distribution in the channel is evaluating by substituting the velocity solution in equation (10.27).

10.3.2 Solution using PM

10.3.2.1 Case-1: Combined effects of buoyancy forces and viscous dissipation

In this section, equations (10.23)–(10.25) are solved by regular perturbation series method. Then the dimensionless temperature field is determined by means of equation
Mixed convective flow of a couple stress fluid with Robin boundary · · ·

(10.28). Let us consider the dimensionless parameter

$$
\epsilon = GR Br = Re Pr \frac{\beta g D}{c_p}.
$$

(10.32)

which is independent of the reference temperature difference $\Delta T$. For fixed values of $R_t$, $GR$, $Bi_2$ and $Bi_2$, the solution of equation (10.23) can be expressed by the perturbation expansion

$$
u(y) = \nu_0(y) + \epsilon \nu_1(y) + \epsilon^2 \nu_2(y) + \ldots = \sum_{n=0}^{\infty} \epsilon^n \nu_n(y).
$$

(10.33)

Solution to zeroth order equation

The boundary value problem for $n = 0$ is

$$
a_c^2 \frac{d^6 \nu_0}{dy^6} = \frac{d^4 \nu_0}{dy^4},
$$

(10.34)

with the boundary conditions

$$
u_0(\pm 1/4) = 0, \quad \frac{d^2 \nu_0}{dy^2} = 0 \quad \text{at} \quad y = \pm \frac{1}{4},
$$

(10.35)

$$
\left( \frac{d^2 \nu_0}{dy^2} - \frac{1}{Bi_1} \frac{d^3 \nu_0}{dy^3} - a_c^2 \frac{d^4 \nu_0}{dy^4} + a_c^2 \frac{d^5 \nu_0}{dy^5} \right) \bigg|_{y=-1/4} = -48 + \frac{R_t GR S}{2} \left( 1 + \frac{4}{Bi_1} \right),
$$

$$
\left( \frac{d^2 \nu_0}{dy^2} + \frac{1}{Bi_2} \frac{d^3 \nu_0}{dy^3} - a_c^2 \frac{d^4 \nu_0}{dy^4} + a_c^2 \frac{d^5 \nu_0}{dy^5} \right) \bigg|_{y=1/4} = -48 - \frac{R_t GR S}{2} \left( 1 + \frac{4}{Bi_2} \right).
$$

(10.36)

The solution of the equations (10.34)–(10.36) is given by

$$
u_0(y) = C_1 + C_2 y + C_3 y^2 + C_4 y^3 + C_5 \cosh \left( \frac{y}{a} \right) + C_6 \sinh \left( \frac{y}{a} \right)
$$

(10.37)

$$
\theta_0 = 2SR_t y.
$$

(10.38)
Mixed convective flow of a couple stress fluid with Robin boundary ···

The boundary value problem for every integer \( n > 0 \) is

\[
a_2^2 \frac{d^6u_n}{dy^6} = \frac{d^4u_n}{dy^4} - \sum_{j=0}^{n-1} \frac{du_j}{dy} \frac{du_{n-j-1}}{dy},
\]

(10.39)

with the boundary conditions

\[
u_n(-1/4) = u_n(1/4) = 0, \quad \frac{d^2u_n}{dy^2} = 0 \text{ at } y = \pm \frac{1}{4} \quad (10.40)
\]

\[
\left. \left( \frac{d^2u_n}{dy^2} - \frac{1}{Bi_1} \frac{d^3u_n}{dy^3} - a_c^2 \frac{d^4u_n}{dy^4} + \frac{a_c^2}{Bi_1} \frac{d^5u_n}{dy^5} \right) \right|_{y=-1/4} = 0,
\]

\[
\left. \left( \frac{d^2u_n}{dy^2} + \frac{1}{Bi_2} \frac{d^3u_n}{dy^3} - a_c^2 \frac{d^4u_n}{dy^4} + \frac{a_c^2}{Bi_2} \frac{d^5u_n}{dy^5} \right) \right|_{y=1/4} = 0. \quad (10.41)
\]

Since \( u_0(y) \) is the known function given by equation (10.37) an iterative solution of equations (10.39)–(10.41) is possible and yields the functions \( u_n(y), \ n > 0 \). As a consequence of equations (10.27), (10.38) and (10.39), the dimensionless temperature \( \theta \) can be written in the form

\[
\theta (y) = 2SR_ty + \frac{1}{GR} \sum_{n=1}^{\infty} \epsilon^n \left[ \frac{d^4u_n}{dy^4} - a_c^2 \frac{d^2u_n}{dy^2} \right].
\]

(10.42)

Equations (10.29) and (10.42) yield the following expressions of \( Nu_1 \) and \( Nu_2 \).

\[
Nu_1 = \frac{2SR_t + \sum_{n=1}^{\infty} a_n \epsilon^n}{R_t [SR_t - 1 + \sum_{n=1}^{\infty} c_n \epsilon^n] + 1},
\]

(10.43)

\[
Nu_2 = \frac{2SR_t + \sum_{n=1}^{\infty} b_n \epsilon^n}{R_t [SR_t - 1 + \sum_{n=1}^{\infty} c_n \epsilon^n] + 1}.
\]

(10.44)

where the coefficients \( a_n, b_n \) and \( c_n \) are given by

\[
a_n = \frac{1}{a_c^2 GR} \left. \left[ \frac{d^5u_n}{dy^5} - a_c^2 \frac{d^3u_n}{dy^3} \right] \right|_{y=-1/4}, \quad b_n = \frac{1}{a_c^2 GR} \left. \left[ \frac{d^5u_n}{dy^5} - a_c^2 \frac{d^3u_n}{dy^3} \right] \right|_{y=1/4},
\]

(10.45)
Mixed convective flow of a couple stress fluid with Robin boundary

\[ c_n = \frac{1}{a_c^2 GR} \left[ \left( \frac{d^4 u_n}{dy^4} - a_c^2 \frac{d^2 u_n}{dy^2} \right) \bigg|_{y=1/4} - \left( \frac{d^4 u_n}{dy^4} - a_c^2 \frac{d^2 u_n}{dy^2} \right) \bigg|_{y=-1/4} \right]. \quad (10.46) \]

Equations (10.22) and (10.37) yield the expression of the mean dimensionless velocity

\[ \bar{u} = C_1 + \frac{C_3}{48} + 4a_cC_5 \sinh \left( \frac{1}{4a_c} \right) + \sum_{n=1}^{\infty} d_n e^n. \quad (10.47) \]

where \( C_1, C_3 \) are given in appendix and the coefficient \( d_n \) is given by

\[ d_n = 2 \int_{-1/4}^{1/4} u_n(y) dy. \quad (10.48) \]

10.3.2.2 Case-2: Separated effects of buoyancy forces and viscous dissipation

Case of negligible viscous dissipation \((Br = 0)\).

The case of negligible viscous dissipation can be obtained by setting \( Br = 0 \) in the dimensionless energy equation (10.26) as a consequence, the dimensionless temperature field \( \theta \) is independent of the dimensionless velocity field \( u \). The equations (10.23)–(10.25) reduces to

\[ a_c^2 \frac{d^6 u}{dy^6} = \frac{d^4 u}{dy^4}. \quad (10.49) \]

The corresponding boundary conditions are given in equations (10.24) and (10.25).

Taking the Differential Transform of equation (10.49) according to Table 1.1, gives:

\[ \bar{U} (k + 6) = \frac{\bar{U} (k + 4)}{a_c^2 (k + 5)(k + 6)}. \quad (10.50) \]

Substituting the initial conditions (10.31), evaluate the unknowns \( a_1, a_2, a_3, a_4, a_5 \) and \( a_6 \) by using the boundary conditions (10.24) and (10.25).

The exact solution of equation (10.49) is

\[ u(y) = C_1 + C_2 y + C_3 y^2 + C_4 y^3 + C_5 \cosh \left( \frac{y}{a_c} \right) + C_6 \sinh \left( \frac{y}{a_c} \right). \quad (10.51) \]
Mixed convective flow of a couple stress fluid with Robin boundary 

The exact solution of equation (10.27) using equation (10.51) become

$$\theta = 2S R_t y.$$  \hspace{1cm} (10.52)

By substituting equation (10.51) in equation (10.22), one obtains

$$\bar{u} = 1 - 48a_c (1 - \tanh (1/4a_c)).$$  \hspace{1cm} (10.53)

**Case of negligible buoyancy force** ($GR = 0$).

Let us now consider the case of negligible buoyancy forces with a relevant viscous dissipation, which corresponds to $GR = \frac{Gr}{Re} \approx 0$. Since a purely forced convection occurs in this case, the Hagen-Poiseuille velocity profile is present within the channel.

The equations for velocity and temperature are

$$ \frac{a_c^2 \frac{d^4 u}{dy^4} - \frac{d^2 u}{dy^2}}{4} - 48 = 0, \tag{10.54}$$

$$ \frac{d^2 \theta}{dy^2} + Br \left( \frac{du}{dy} \right)^2 = 0, \tag{10.55}$$

The corresponding boundary conditions on velocity and temperature are

$$ u (-1/4) = u (1/4) = 0, \quad \frac{d^2 u}{dy^2} = 0 \text{ at } y = \pm \frac{1}{4} \tag{10.56}$$

$$ \left. \frac{d\theta}{dy} \right|_{y=-1/4} = B i_1 \left[ \theta (-1/4) + \frac{R_i S}{2} \left( 1 + \frac{4}{B_i} \right) \right],$$

$$ \left. \frac{d\theta}{dy} \right|_{y=1/4} = B i_2 \left[ -\theta (1/4) + \frac{R_i S}{2} \left( 1 + \frac{4}{B_i} \right) \right]. \tag{10.57}$$

Taking the Differential Transform of equations (10.54) and (10.55) according to Table 1.1, gives:

$$ \bar{U} (k + 4) = \frac{(k + 1) (k + 2) \bar{U} (k + 2) + 48 \delta (k)}{a_c^2 (k + 1) (k + 2) (k + 3) (k + 4)}, \tag{10.58}$$
Mixed convective flow of a couple stress fluid with Robin boundary ···

\[ \Theta (k + 2) = \frac{-Br}{(k + 1)(k + 2)} \sum_{r=0}^{k} (k - r + 1)(r + 1) \bar{U}(k - r + 1) \bar{U}(r + 1), \]  \hspace{1cm} (10.59)

The transformed initial conditions are as follows

\[ \bar{U}(0) = a_1, \quad \bar{U}(1) = a_2, \quad \bar{U}(2) = \frac{a_3}{2!}, \quad \bar{U}(3) = \frac{a_4}{3!}, \]  \hspace{1cm} (10.60)

Using the conditions as given in equations (10.56) and (10.57), one can evaluate the unknowns \( a_1, a_2, a_3, a_4, b_1 \) and \( b_2 \).

Using the boundary conditions (10.56) and (10.57), the exact solution of equations (10.54) and (10.55) are

\[ u = f_1 + f_2 y + f_3 \cosh \left( \frac{y}{a_c} \right) + f_4 \sinh \left( \frac{y}{a_c} \right) - 24y^2 \]  \hspace{1cm} (10.61)

\[ \theta = L_1 y^2 + L_2 y^3 + L_3 y^4 + L_4 \cosh \left( \frac{y}{a_c} \right) + L_5 \sinh \left( \frac{y}{a_c} \right) + L_6 \cosh \left( \frac{2y}{a_c} \right) \\
+ L_7 \sinh \left( \frac{2y}{a_c} \right) + L_8 y \cosh \left( \frac{y}{a_c} \right) + L_9 y \sinh \left( \frac{y}{a_c} \right) + Ay + B. \]  \hspace{1cm} (10.62)

The constants appeared in the above solutions are listed in the appendix.
Stokes' 10.2

Different values of mixed convection parameter GR are chosen to present velocity range of values of the parameters affecting the flow and heat transfer phenomena. We pressible, couple stress fluid in a vertical channel. Here we present our findings in

The theory of couple stress fluid due to 10.4 Results and Discussion

Appendix

\[ D_1 = \frac{1}{2} (C_3 C_5 + 24 C_5) \sinh (1/4 a_c) + \frac{3a_c}{2} C_6 C_4 \cosh (1/4 a_c) - 6a_c^2 C_6 C_4 \sinh (1/4 a_c) \]
\[ + \frac{1}{2} (C_1 C_3 + 24 C_1), \]
\[ C_1 = \frac{3}{2} - 48 a_c^2, C_2 = R_t G S \left( \frac{1}{48} - 2 a_c^2 \right), C_3 = -24, C_4 = \frac{-R_t G S}{3}, \]
\[ C_5 = \frac{48 a_c^2}{\cosh(1/4 a_c)}, C_6 = \frac{R_t G S a_c^2}{2 \sinh(1/4 a_c)}, \]
\[ Z_1 = \frac{-L_1}{16} - \frac{L_2}{2} - \frac{L_3 \cosh(1/4 a_c)}{4 a_c} - L_3 \sinh (1/4 a_c) - \frac{L_4 \sinh(1/4 a_c)}{a_c} + A, \]
\[ Z_2 = \frac{L_1}{16} + \frac{L_2}{2} + \frac{L_3 \cosh(1/4 a_c)}{4 a_c} + L_3 \sinh (1/4 a_c) + \frac{L_4 \sinh(1/4 a_c)}{a_c} + A, \]
\[ A = 2 S \left( \frac{1}{B_{i1}} - \frac{1}{B_{i2}} \right) \left( \frac{L_1}{16} + \frac{L_2}{2} + \frac{L_3 \cosh(1/4 a_c)}{4 a_c} + L_3 \sinh (1/4 a_c) + \frac{L_4 \sinh(1/4 a_c)}{a_c} \right) + 2 R_t S, \]
\[ B = A \left( 1 - \frac{1}{B_{i1}} - \frac{1}{B_{i2}} \right) - \frac{L_1}{256} - \frac{L_2}{16} - \frac{L_3 \sinh(1/4 a_c)}{4} - L_4 \cosh (1/4 a_c) - \frac{R_t S}{2} \left( 1 + \frac{1}{B_{i2}} \right) \]
\[ - \frac{1}{B_{i1}} \left( \frac{L_1}{16} + \frac{L_2}{2} + \frac{L_3 \cosh(1/4 a_c)}{4 a_c} \right) \sinh (1/4 a_c), \]
\[ L_1 = \frac{B r f_3^2}{2} - \frac{B r f_2^2}{4 a_c^2} + \frac{B r f_1^2}{4 a_c^2}, L_2 = -16 B r f_2, L_3 = 192 B r, L_4 = 192 a_c^2 B r f_3 + 2 a_c B r f_2 f_4, \]
\[ L_5 = 192 a_c^2 B r f_4 + 2 a_c B r f_2 f_3, L_6 = \frac{B r f_1^2}{2} + \frac{B r f_2^2}{8}, L_7 = \frac{B r f_3 f_4}{4}, L_8 = 96 a_c B r f_4, \]
\[ L_9 = 96 a_c B r f_3. \]

10.4 Results and Discussion

The theory of couple stress fluid due to Stokes (1966) has been made use to formulate a set of basic equations for the mixed convective flow and heat transfer of an incompressible, couple stress fluid in a vertical channel. Here we present our findings in graphical forms in order to investigate the important features of the solution for a range of values of the parameters affecting the flow and heat transfer phenomena. We choose to present velocity \( u \), temperature \( \theta \), mass flow rate and Nusselt number for different values of mixed convection parameter GR, Brinkman number \( B r \), couple stress parameter \( a_c \) and Biot numbers \( B i_1, B i_2 \) respectively for both asymmetric and symmetric wall boundary conditions. The mixed convection parameter \( G R = 0, 500, 1000 \), the Brinkman number \( B r = 0, 0.5, 1.0 \) and the couple stress parameter \( a_c = 0.1, 0.2, 0.5 \) are chosen arbitrarily in order to study the influence on the flow behavior and the results are shown graphically in Figures 10.2-10.14.
Mixed convective flow of a couple stress fluid with Robin boundary ···

Equation (10.51) for velocity field $u$ is evaluated for different values of mixed convection parameter $GR$ is the ratio of Grashof number to Reynolds number, and couple stress parameter $a_c$ are shown in Figure 10.2 when $Br = 0$ (case of negligible viscous dissipation) with $Bi_1 = Bi_2 = 10$ and $R_t = 1$. It is seen that the velocity profile for $GR = 0, 500, 1000$ shows decrease in nature on the left half of the channel ($y = 0$) and increase in nature on the right half of the channel for all values of couple stress parameter $a_c$. It is seen that the flow reversal is observed for $GR = 1000$ for small values of couple stress parameter near the cold wall $y = -\frac{1}{4}$. With $U_0 > 0$ i.e., for upward flow it is expected that for sufficiently large values of Grashof number a flow reversal induced by the buoyancy forces occurs at the cold wall. By performing a reflection of the $y$-axis of Figure 10.2, plots of $u$ for $GR = -100, -300$ can be obtained. Physically, $GR = 0$ corresponds to the no free convection currents and for other values the physical situation corresponds to the case of cooling of the walls. It is also noticed that couple stress parameter $a_c$ decreases the velocity and minimize the flow reversal near the boundaries.

Equation (10.26) for the non-dimensional temperature field $\theta$ is evaluated for different values of Brinkman number $Br$ and $a_c$ for both equal and unequal Biot numbers with $GR = 0$ (case of negligible buoyancy force) and $R_t = 1$. We notice that the temperature field is linear indicating that the heat transfer is dependent basically on conduction in the absence of viscous dissipation ($Br = 0$) for both equal and unequal Biot numbers. Further there is no effect of couple stress parameter on the temperature in the absence of viscous dissipation. Also, the temperature field increases with increase in the value of $Br$ for both equal and unequal Biot numbers. Here also, the effect of couple stress parameter is more operative for smaller values and is negligible effect for higher values as seen in Figures 10.3a,b. The effect of $Br$ for equal and unequal Biot numbers is similar to the results obtained by Zanchini (1998). That is when $Bi_2 > Bi_1$ and $Br > \frac{Bi_1 Bi_2}{12(Bi_2 - Bi_1)}$ (for $a_c \to 0$), one obtains $T\left(-\frac{L}{2}\right) > T\left(\frac{L}{2}\right)$ even if $T_1 < T_2$. 

241
The effect of mixed convection parameter $GR$ on velocity $u(y)$ and temperature $\theta(y)$ are shown through Figures 10.4 and 10.5 respectively for $Br = 0.009$, $Bi_1 = Bi_2 = 10$ and $R_t = 1$. It is observed from Figure 10.4 that the velocity increases with increase of $GR$ caused by dominance of buoyancy forces over the ratio of inertial forces to viscous forces. We observed from Figure 10.5 that the mixed convection parameter $GR$ is also increases the temperature field. It is also observed from Figures 10.4 and 10.5 that the effect of the couple stress parameter $a_c$ decreases the velocity $u$ and temperature $\theta$.

Plots of velocity $u$ and temperature $\theta$ for different values of Brinkman number $Br$ and couple stress parameter $a_c$ with $Bi_1 = Bi_2 = 10$ and $R_t = 1$ is reported in Figures 10.6 and 10.7. Figure 10.6 show that, the velocity, at each position, are increasing functions of $Br$ for $a_c = 0.1$. The temperature $\theta$ increases with increase in Brinkman number for $a_c = 0.1$ as seen in Figure 10.7. Moreover the velocity and temperature are more operative for $a_c = 0.1$ and the is less operative as $a_c \geq 0.5$. To enhance the velocity of a couple stress fluid, viscous dissipation must be taken into account in the energy balance equation. The effect of viscous dissipation may become very important in several flow configurations occurring in the engineering practice. In fact viscous dissipation affects strongly the heat transfer process whenever the operating fluid a low thermal conductivity and a high viscosity.

Figure 10.8 illustrates the effect of couple stress parameter and Brinkman number on temperature for unequal Biot numbers $Bi_1 = 0.1$, $Bi_2 = 10$. It is observed that the effect of couple stress parameter is invariant in the absence of viscous dissipation ($Br = 0$) whereas the couple stress parameter is to decrease the temperature in the presence of viscous dissipation ($Br = 0.1$). It is noticed that Brinkman number is more effective near the cold wall for unequal Biot numbers.

Figures 10.9 and 10.10 shows the effect of mixed convection parameter and couple stress parameter for symmetric wall temperature ($R_t = 0$) with $Br = 0.1$ and $Bi_1 =$
Mixed convective flow of a couple stress fluid with Robin boundary ···

$Bi_2 = 1$. The mixed convection parameter increases the velocity and temperature field whereas couple stress parameter decreases the velocity and temperature field. It is observed that velocity profiles are parabolic for larger values of mixed convection parameter and the parabolic profiles become flat for larger values of couple stress parameter. It is also observed that the effect of couple stress parameter is negligible for higher values of $a_c$.

The effect of Brinkman number and couple stress parameter on velocity and temperature for symmetric wall temperature with $GR = 10$ and $Bi_1 = Bi_2 = 1$ are shown in Figures 10.11 and 10.12. It is clear from these figures that velocity and temperature field increases as Brinkman increases whereas couple stress parameter decreases the velocity and temperature field.

The mass flow rate for different values of mixed convection parameter $GR$ and Brinkman number $Br$ are plotted against couple stress parameter $a_c$ in Figures 10.13 and 10.14 respectively. It is observed that mass flow rate increases as mixed convection parameter $GR$ and Brinkman number $Br$ increases while decreases with increase in couple stress parameter and it invariant after the value $a_c = 0.4$ (approximately).

Finally, Figures 10.15 and 10.16 display the influence of mixed convection parameter $GR$, Brinkman number $Br$ and couple stress parameter on the Nusselt number at both the walls with $R_t = 1$ and $Bi_1 = Bi_2 = 1$. It is observed that Nusselt number increases at the right wall (hot wall) for increasing the values of mixed convection parameter $GR$, Brinkman number $Br$ and couple stress parameter while it decreases at the left wall (cold wall). Here also the couple stress parameter shows the negligible effect after the values $a_c = 0.5$ (approximately).
10.4.1 Validation of results

Before concluding the discussion of the new results obtained in this study, it is useful to establish the accuracy and reliability of the present solution methodology. This objective is accomplished here by performing a few benchmark comparisons as detailed in the ensuing section.

In order to verify the accuracy of the present DTM solutions, we have compared our results with numerical method called Runge-Kutta shooting method. The values of velocity $u$ and temperature $\theta$ obtained by DTM, PM and numerical shooting solutions are shown in Table 10.1. From Table 10.1 we found that an excellent agreement has been achieved between the DTM, PM and the numerical solutions.

Finally, the present results for the limiting case of couple stress parameter $a_c \to 0$ the DTM solution used herein has been validated by comparing the present values with the previous results on mixed convection heat transfer from an asymmetrically confined vertical channel in Newtonian fluid with boundary conditions of the third kind (Tables 10.2 and 10.3). The present results are seen to be virtually indistinguishable from that of the corresponding literature values. Table 10.3 shows a similar comparison with the recent results of Umavathi and Veershetty (2012). The results obtained for $Bi = 10^5$ in Table 10.2 are also in perfect agreement with those reported in Barletta (1998) for the boundary condition of prescribed wall temperatures. Once again the correspondence is seen to be excellent. Such close correspondence between the previous and present results lends further credibility to the present methodology employed in this work. Thus, in summary, based on the aforementioned comparisons reveal that the DTM can achieve more suitable results in predicting the solution of mixed convection flow and heat transfer of non-Newtonian fluid problems in a vertical channel.
Mixed convective flow of a couple stress fluid with Robin boundary

Figure 10.2: Plots of $u$ versus $y$ for different values of $GR$ and $a_c$ with $Br = 0$, $Bi_1 = Bi_2 = 20$ and $R_t = 1$.

Figure 10.3: Plots of $\theta$ versus $y$ for different values of $Br$ and $a_c$ with $GR = 0$, $R_t = 1$ (a) $Bi_1 = Bi_2 = 10$ and (b) $Bi_1 = 1$, $Bi_2 = 10$. 
Figure 10.4: Plots of $u$ versus $y$ for different values of $GR$ and $a_c$.

Figure 10.5: Plots of $\theta$ versus $y$ for different values of $GR$ and $a_c$. 
Mixed convective flow of a couple stress fluid with Robin boundary 

Figure 10.6: Plots of $u$ versus $y$ for different values of $Br$ and $a_c$.

Figure 10.7: Plots of $\theta$ versus $y$ for different values of $Br$ and $a_c$ for $Bi_1 = Bi_2 = 10$. 

247
Figure 10.8: Plots of $\theta$ versus $y$ for different values of $Br$ and $a_c$ for $Bi_1 = 0.1$, $Bi_2 = 10$

Figure 10.9: Plots of $u$ versus $y$ for different values of $GR$ and $a_c$ for $R_t = 0$. 
Figure 10.10: Plots of $\theta$ versus $y$ for different values of $GR$ and $a_c$ for $R_t = 0$.

Figure 10.11: Plots of $u$ versus $y$ for different values of $Br$ and $a_c$ for $R_t = 0$. 
Mixed convective flow of a couple stress fluid with Robin boundary

Figure 10.12: Plots of $\theta$ versus $y$ for different values of $Br$ and $a_c$ for $R_t = 0$.

Figure 10.13: Mass flow rate versus couple stress parameter $a_c$ for different values of $GR$. 

250
Mixed convective flow of a couple stress fluid with Robin boundary····

**Figure 10.14:** Mass flow rate versus couple stress parameter $a_c$ for different values of $Br$.

**Figure 10.15:** Nusselt number versus couple stress parameter $a_c$ for different values of $GR$.

251
Mixed convective flow of a couple stress fluid with Robin boundary···

Figure 10.16: Nusselt number versus couple stress parameter $a_c$ for different values of $Br$.  

$\text{Br} = 0$  $\text{Br} = 0.5$  $\text{Br} = 1.0$  $GR = 10$  
$R_i = 1$  $Bi_1 = Bi_2 = 10$
Table 10.1: Comparison of velocity and temperature with $GR = 10$, $Br = 0.1$, $Bi_1 = Bi_2 = 10$.

<table>
<thead>
<tr>
<th>$y$</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DTM</td>
</tr>
<tr>
<td></td>
<td>$a_c = 0.1$</td>
</tr>
<tr>
<td>0.25</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3657171</td>
</tr>
<tr>
<td>0.15</td>
<td>0.6840009</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9255779</td>
</tr>
<tr>
<td>0.05</td>
<td>1.0742373</td>
</tr>
<tr>
<td>0</td>
<td>1.1222737</td>
</tr>
<tr>
<td>-0.05</td>
<td>1.0680755</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.9152585</td>
</tr>
<tr>
<td>-0.15</td>
<td>0.6731308</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.3586012</td>
</tr>
<tr>
<td>-0.25</td>
<td>0</td>
</tr>
<tr>
<td>0.25</td>
<td>0.4211175</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3748209</td>
</tr>
<tr>
<td>0.15</td>
<td>0.3165714</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2502854</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1800108</td>
</tr>
<tr>
<td>0</td>
<td>0.1085927</td>
</tr>
<tr>
<td>-0.05</td>
<td>0.0369991</td>
</tr>
<tr>
<td>-0.1</td>
<td>-0.0358390</td>
</tr>
<tr>
<td>-0.15</td>
<td>-0.1127518</td>
</tr>
<tr>
<td>-0.2</td>
<td>-0.1976006</td>
</tr>
<tr>
<td>-0.25</td>
<td>-0.2940129</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$y$</th>
<th>Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.4211174</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3748209</td>
</tr>
<tr>
<td>0.15</td>
<td>0.3165714</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2502854</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1800108</td>
</tr>
<tr>
<td>0</td>
<td>0.1085927</td>
</tr>
<tr>
<td>-0.05</td>
<td>0.0369991</td>
</tr>
<tr>
<td>-0.1</td>
<td>-0.0358390</td>
</tr>
<tr>
<td>-0.15</td>
<td>-0.1127518</td>
</tr>
<tr>
<td>-0.2</td>
<td>-0.1976006</td>
</tr>
<tr>
<td>-0.25</td>
<td>-0.2940129</td>
</tr>
</tbody>
</table>
Mixed convective flow of a couple stress fluid with Robin boundary ···

Table 10.2: Values of $GR Nu$ as a function of $\epsilon$ and $Bi$ for $R_t = 0$ and $Bi_1 = Bi_2 = Bi$ (completely symmetric case).

<table>
<thead>
<tr>
<th>$\epsilon = GR Br$</th>
<th>$Bi = 10^4$, $GR Nu$</th>
<th>$Bi = 20$, $GR Nu$</th>
<th>$Bi = 7$, $GR Nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4.0</td>
<td>-43.216</td>
<td>-39.858</td>
<td>-35.0267</td>
</tr>
<tr>
<td>-3.5</td>
<td>-38.274</td>
<td>-35.603</td>
<td>-31.6604</td>
</tr>
<tr>
<td>-1.5</td>
<td>-17.2688</td>
<td>-16.6855</td>
<td>-15.7203</td>
</tr>
<tr>
<td>-1.0</td>
<td>-11.6693</td>
<td>-11.3976</td>
<td>-10.9321</td>
</tr>
<tr>
<td>-0.5</td>
<td>-5.91582</td>
<td>-5.8452</td>
<td>-5.7158</td>
</tr>
<tr>
<td>0.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.5</td>
<td>6.08735</td>
<td>6.16637</td>
<td>6.32023</td>
</tr>
<tr>
<td>1.0</td>
<td>12.3562</td>
<td>12.6899</td>
<td>13.3754</td>
</tr>
<tr>
<td>1.5</td>
<td>18.817</td>
<td>19.6123</td>
<td>21.3465</td>
</tr>
<tr>
<td>2.0</td>
<td>25.4825</td>
<td>26.9825</td>
<td>30.4914</td>
</tr>
<tr>
<td>2.5</td>
<td>32.3648</td>
<td>34.8597</td>
<td>41.196</td>
</tr>
<tr>
<td>3.0</td>
<td>39.4784</td>
<td>43.3153</td>
<td>54.0785</td>
</tr>
<tr>
<td>3.5</td>
<td>46.8391</td>
<td>52.4371</td>
<td>70.2307</td>
</tr>
<tr>
<td>4.0</td>
<td>54.4641</td>
<td>62.3349</td>
<td>91.9063</td>
</tr>
</tbody>
</table>

Table 10.3: Comparison of velocity and temperature with $GR = 100$, $a_c = 0.0$, $Br = 0.0001$, $Bi_1 = Bi_2 = 10$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Velocity</td>
<td>Temperature</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.25</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-0.15</td>
<td>0.81752468</td>
<td>0.81752468</td>
<td>0.81752469</td>
</tr>
<tr>
<td>-0.05</td>
<td>1.36916041</td>
<td>1.36916041</td>
<td>1.36916046</td>
</tr>
<tr>
<td>0.05</td>
<td>1.51202553</td>
<td>1.51202553</td>
<td>1.51202559</td>
</tr>
<tr>
<td>0.15</td>
<td>1.10325604</td>
<td>1.10325604</td>
<td>1.10325610</td>
</tr>
<tr>
<td>0.25</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-0.25</td>
<td>-0.35703534</td>
<td>-0.35703534</td>
<td>-0.35703534</td>
</tr>
<tr>
<td>-0.15</td>
<td>-0.21410690</td>
<td>-0.21410690</td>
<td>-0.21410689</td>
</tr>
<tr>
<td>-0.05</td>
<td>-0.07122816</td>
<td>-0.07122816</td>
<td>-0.07122816</td>
</tr>
<tr>
<td>0.05</td>
<td>0.07163487</td>
<td>0.07163487</td>
<td>0.07163488</td>
</tr>
<tr>
<td>0.15</td>
<td>0.21449135</td>
<td>0.21449135</td>
<td>0.21449136</td>
</tr>
<tr>
<td>0.25</td>
<td>0.35728440</td>
<td>0.35728440</td>
<td>0.35728441</td>
</tr>
</tbody>
</table>