CHAPTER 3

SECTORIAL SUBORDINATION OF $\Phi$-LIKE FUNCTIONS

3.1 INTRODUCTION

Let $\mathcal{A}$ denote the class of all analytic functions $f(z)$ of the form as defined in (1.1). Let $\phi$ be an analytic function in a domain containing $f(\Delta)$, $\phi(0) = 0$ and $\phi'(0) > 0$. The function $f \in \mathcal{A}$ is called $\phi$-like if

$$\Re \left( \frac{zf'(z)}{\phi(f(z))} \right) > 0 \quad (z \in \Delta). \quad (3.1)$$

This concept was introduced by Brickman (1973) and established that an analytic function $f \in \mathcal{A}$ is univalent if and only if $f$ is $\phi$-like for some $\phi$. When $\phi(w) = w$ and $\phi(w) = \lambda w$, the function $f$ is starlike and spirallike of type $\arg \lambda$ respectively.

Ruscheweyh (1976) introduced and studied the following general class of $\phi$-like functions.

**Definition 3.1.1.** Let $\phi$ be analytic in a domain containing $f(\Delta)$, $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(\Delta) - \{0\}$. Let $q(z)$ be a fixed analytic function in $\Delta$, $q(0) = 1$. The function $f \in \mathcal{A}$ is called $\phi$-like with respect to $q$ if

$$\frac{zf'(z)}{\phi(f(z))} \sim q(z) \quad (z \in \Delta). \quad (3.2)$$

When $\phi(w) = w$ the class of all $\phi$-like functions with respect to $q$ by $S^\phi(q)$ is denoted.

Recently, many important properties of sectorial subordination were

Motivated by the above work, in the second section of this Chapter, $\Phi$-like functions in a sector are discussed. In particular, for $0 < \beta < 1$, the largest $C(a, b, \alpha, \beta, \eta)$ is found such that

$$
\left( \frac{zf'(z)}{\Phi(f(z))} \right)^{\alpha} \left\{ a \left( \frac{zf'(z)}{\Phi(f(z))} \right) + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z\Phi(f(z))'}{\Phi(f(z))} \right) \right\}^{\beta} \eta
$$

implies

$$
\frac{zf'(z)}{\Phi(f(z))} \prec \left( \frac{1 + z}{1 - z} \right)^{\beta},
$$

where $a > 0$, $b \geq 0$, $\alpha$ and $\eta$ are real numbers such that $0 < \eta \leq 1$, $\alpha + \eta \geq 0$ and $\Phi(w)$ be an analytic function in a domain containing $f(\Delta)$, $\Phi(0) = \Phi'(0) = 1 = 0$ and $\Phi(w) \neq 0$ in $f(\Delta) \setminus \{0\}$.

Some application of the results are given to obtain sufficient conditions for starlikeness. The results include the recent results obtained by Darus (2003, 2005) and Marjono and Thomas (2001).

In the third section of this Chapter, sufficient condition for normalized analytic function $f$ to satisfy

$$
q_1(z) \prec \frac{zf'(z)}{\Phi(f(z))} \prec q_2(z)
$$

is obtained. Here $q_1$ and $q_2$ are univalent in $\Delta$.

In order to prove the main result, the following Lemma due to Miller and Mocanu (1981) is needed, which is a version of Jack’s Lemma.

**Lemma 3.1.1 (Miller and Mocanu (1981)).** Let $p(z)$ be analytic function in $\Delta$ and $q(z)$ be analytic and univalent in $\Delta$ with $p(0) = q(0)$. If $p(z) \not\prec q(z)$, then there is a point $z_0 \in \Delta$ and $\zeta_0 \in \partial \Delta$ such that

$$
p(|z| < |z_0|) \subset q(U)
$$
and
\[ p(z_0) = q(z_0) \]
and
\[ z_0 \beta' = k \zeta_0 \delta'(z_0) \]
for \( k \geq 1 \).

### 3.2 SECTORIAL SUBORDINATION OF \( \Phi \)-LIKE FUNCTIONS

By making use of Lemma 3.1.1 we prove the following theorem.

**Theorem 3.2.1.** Let \( f(z) \) be an analytic function in \( \Delta \) with \( f(0) = f'(0) = 1 = 0 \). Let \( a > 0, b \geq 0, \alpha \) and \( \eta \) be fixed real numbers, with \( 0 < \eta \leq 1, \alpha + \eta \geq 0 \). Let
\[ \Psi(a, b, \alpha, \eta, z) := \left( \frac{zf''(z)}{\Phi(f(z))} \right)^{\alpha} \left\{ a \left( \frac{zf'(z)}{\Phi(f(z))} \right) + b \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z\Phi(f(z))'}{\Phi(f(z))} \right) \right\}^{\eta}. \]
For \( 0 < \beta < 1 \), if
\[ \Psi(a, b, \alpha, \eta, z) < \left( \frac{1 + z}{1 - z} \right)^{C(a, b, \alpha, \beta, \eta)} \]
then
\[ \frac{zf'(z)}{\Phi(f(z))} < \left( \frac{1 + z}{1 - z} \right)^{\beta}, \tag{3.3} \]
for \( z \in \Delta \), where
\[ C(a, b, \alpha, \beta, \eta) = \]
\[ \alpha \beta + \frac{2 \eta}{\pi} \arctan \left\{ \tan \frac{\pi \beta}{2} + \frac{b \beta}{a (1 + \beta)^{\frac{1 + \beta}{2}} (1 - \beta)^{\frac{1 - \beta}{2}} \cos \frac{\pi \beta}{2} \right\}, \tag{3.4} \]
is the largest number such that (3.3) holds.

**Proof.** Let \( p(z) = \frac{zf'(z)}{\Phi(f(z))} \), so that \( p(z) \) is analytic in \( \Delta \) and \( p(0) = 1 \). We need to show that
\[ (p(z))^\alpha \left( \alpha p(z) + \frac{i z\Phi'(z)}{p(z)} \right)^\eta \left( \frac{1 + z}{1 - z} \right)^{C(a, b, \alpha, \beta, \eta)} \)
implies
\[ p(z) < \left( \frac{1 + z}{1 - z} \right)^{\beta}. \]
We now write
\[ h(z) = \left( \frac{1 + z}{1 - z} \right)^{C(a, b, \alpha, \beta, \gamma)} \text{ and } q(z) = \left( \frac{1 + z}{1 - z} \right)^\beta, \]
so that \( |\arg h(z)| < \frac{\pi}{2}C(a, b, \alpha, \beta, \eta) \) and \( |\arg q(z)| < \frac{\pi\beta}{2} \). Suppose that \( p(z) \neq q(z) \), then from Lemma 3.1.1, there is a point \( z_0 \in \Delta \) and \( \zeta_0 \in \partial\Delta \) such that \( p(|z| < |z_0|) \subset q(\Delta) \) and \( p(z_0) = q(\zeta_0) \) and \( z_0p'(z_0) = k\zeta_0q'(\zeta_0) \), for \( k \geq 1 \).

Since \( p(z_0) = q(\zeta_0) = \left( \frac{1 + \zeta_0}{1 - \zeta_0} \right)^\beta \neq 0 \), it follows that \( \zeta_0 \neq \pm 1 \). Thus, we can write
\[ \frac{1 + \zeta_0}{1 - \zeta_0} = ri, \]
for \( r \in \mathbb{R} - \{0\} \). Consequently,
\[ \zeta_0 = \frac{ri - 1}{ri + 1}. \]

Next assume that \( r > 0 \), then
\[
\begin{align*}
(p(z_0))^\alpha \left( ap(z_0) + b\frac{z_0p'(z_0)}{p(z_0)} \right)^\eta \\
= \left( \frac{ri - 1}{ri + 1} \right)^\alpha \left( ar^\beta \right)^\alpha \left( \frac{b\frac{z_0p'(z_0)}{p(z_0)}}{2r} \right)^\eta \\
= \left( \frac{ri - 1}{ri + 1} \right)^\alpha \left( ar^\beta \cos \frac{\beta\pi}{2} + i \left( ar^\beta \sin \frac{\beta\pi}{2} + \frac{b\frac{z_0p'(z_0)}{p(z_0)}}{2r} \right) \right)^\eta.
\end{align*}
\]

Since \( k \geq 1 \), an elementary argument shows that
\[
\arg \left\{ (p(z_0))^\alpha \left( ap(z_0) + b\frac{z_0p'(z_0)}{p(z_0)} \right)^\eta \right\}
= \alpha \frac{\beta\pi}{2} + \eta \arctan \left\{ \tan \frac{\beta\pi}{2} + \frac{b\frac{z_0p'(z_0)}{p(z_0)}}{2ar^\beta + \frac{b\frac{z_0p'(z_0)}{p(z_0)}}{2r}} \right\}
\geq \alpha \frac{\beta\pi}{2} + \eta \arctan \left\{ \tan \frac{\beta\pi}{2} + \frac{b\frac{z_0p'(z_0)}{p(z_0)}}{2ar^\beta + \frac{b\frac{z_0p'(z_0)}{p(z_0)}}{2r}} \right\}, \quad (3.5)
\]

Let \( g(r) = \frac{1 + r^2}{r^{\beta + 1}} \), then \( g'(r) = \frac{2r^2 - (\beta + 1)(1 + r^2)}{r^{\beta + 2}} \). Therefore \( g'(r) = 0 \) and \( g''(r) > 0 \) if \( r = \sqrt{\frac{1 + \beta}{1 - \beta}} \). Hence \( g(r) \) attains its minimum at \( r = \sqrt{\frac{1 + \beta}{1 - \beta}} \).

Now,
\[
\frac{\beta (1 + r^2)}{2r^{\beta + 1} \cos \frac{\beta\pi}{2}} \geq \frac{\beta}{(1 + \beta)^{\frac{\beta}{2}} (1 - \beta)^{\frac{\beta}{2}} \cos \frac{\beta\pi}{2}}. \quad (3.6)
\]
Using (3.6) in (3.5), we have
\[ \arg \left \{ (p(z_0))^\alpha \left( a p(z_0) + \frac{b z_0 p'(z_0)}{p(z_0)} \right)^\eta \right \} \]
\[ \geq \alpha \beta \frac{\pi}{2} + \eta \arctan \left\{ \tan \frac{\beta \pi}{2} + \frac{b \beta}{a (1 + \beta)} \frac{\sin \frac{\pi}{2}}{(1 - \beta)^{-rac{1}{2}}} \cos \frac{\pi}{2} \right\} \]
\[ = \frac{\pi}{2} C(a, b, \alpha, \beta, \eta). \]
Hence
\[ \frac{\pi}{2} C(a, b, \alpha, \beta, \eta) \leq \arg \left \{ (p(z))^\alpha \left( a p(z) + \frac{b z p'(z)}{p(z)} \right)^\eta \right \} \leq \frac{\pi}{2}, \]
which contradicts the fact that
\[ \arg \left \{ (p(z))^\alpha \left( a p(z) + \frac{b z p'(z)}{p(z)} \right)^\eta \right \} \leq \frac{\pi}{2} C(a, b, \alpha, \beta, \eta), \]
provided that (3.4) holds.

To show that \( C(a, b, \alpha, \beta, \eta) \) given by (3.4) is the largest value such that (3.3) holds.

Let
\[ p(z) = \left( \frac{1 + z}{1 - z} \right)^\beta, \]
then from the maximum modulus principle for harmonic functions, it follows that
\[ \inf_{|z|<1} \arg \left \{ (p(z))^\alpha \left( a p(z) + \frac{b z p'(z)}{p(z)} \right)^\eta \right \} \]
is attained at some point when \( z = e^{i\theta} \), for \( 0 < \theta < 2\pi \). Thus
\[ (p(z))^\alpha \left( a p(z) + \frac{b z p'(z)}{p(z)} \right)^\eta \]
\[ = \left( \frac{1 + z}{1 - z} \right)^{\alpha \beta} \left( a \left( \frac{1 + z}{1 - z} \right)^\beta + b \frac{2 \beta z}{1 - z^2} \right)^\eta \]
\[ = \left( \frac{\sin \theta}{1 - \cos \theta} \right)^{\alpha \beta} e^{\alpha \frac{2\pi}{2}} \left( a \left( \frac{\sin \theta}{1 - \cos \theta} \right)^\beta e^{i\frac{2\pi}{2}} + b \frac{i \beta}{\sin \theta} \right)^\eta. \] (3.7)
Letting \( t = \cos \theta \) in (3.7), we have
\[(p(z))^\eta \left( a p(z) + t \frac{z p'(z)}{p(z)} \right) \]
\[= \left( \frac{1 + t}{1 - t} \right)^{\frac{\alpha \beta \pi}{2}} e^{\frac{\pi i}{2}} \left( a \left( \frac{1 + t}{1 - t} \right)^{\frac{\alpha \beta \pi}{2}} e^{\frac{\pi i}{2}} + b \frac{i \beta}{\sqrt{1 - t^2}} \right)^\eta. \]

Hence
\[\arg(p(z))^\alpha \left( a p(z) + t \frac{z p'(z)}{p(z)} \right) \]
\[= \frac{\alpha \beta \pi}{2} + \eta \arctan \left\{ \frac{\tan \frac{\beta \pi}{2} + \frac{b \beta}{a (1 + t) \frac{1 + \beta}{2} (1 - t) \frac{1 + \beta}{2} \cos \frac{\beta \pi}{2}}}{a (1 + t) \frac{1 + \beta}{2} (1 - t) \frac{1 + \beta}{2}} \right\}. \]

Let \( v(t) = \frac{1}{(1 + t) \frac{1 + \beta}{2} (1 - t) \frac{1 + \beta}{2}} \),
then
\[v'(t) = (1 + t)^{-\frac{\alpha \beta \pi}{2}} (1 - t)^{\frac{\alpha \beta \pi}{2}} \left[ \left( \frac{1 - \beta}{2} \right) (1 + t) - \left( \frac{1 + \beta}{2} \right) (1 - t) \right]. \]
Therefore \( v'(\beta) = 0 \) and \( v''(\beta) > 0 \). Hence \( v(t) \) attains its minimum at \( t = \beta \).
Hence
\[\arg(p(z))^\alpha \left( a p(z) + t \frac{z p'(z)}{p(z)} \right) \]
\[\geq \frac{\alpha \beta \pi}{2} + \eta \arctan \left\{ \frac{\tan \frac{\beta \pi}{2} + \frac{b \beta}{a (1 + \beta) \frac{1 + \beta}{2} (1 - \beta) \frac{1 + \beta}{2} \cos \frac{\beta \pi}{2}}}{a (1 + \beta) \frac{1 + \beta}{2} (1 - \beta) \frac{1 + \beta}{2}} \right\} \]
\[= \frac{\pi}{2} C(a, b, \alpha, \beta, \eta). \]

Thus \( C(a, b, \alpha, \beta, \eta) \) is the largest value such that \( (3.3) \) holds and this complete the proof of theorem.

In the remaining part of this chapter unless otherwise specified \( f \in \mathcal{A} \) means \( f(z) \) be analytic in \( \Delta \) with \( f(0) = f'(0) = 1 = 0 \) and \( \frac{f(z)}{z} f'(z) \neq 0 \).

From the Theorem 3.2.1, a number of interesting results are obtained, which are listed below, as corollaries.

Taking \( \alpha = \eta = 1 \) and \( \Phi(\psi) = \psi \) in Theorem 3.2.1, provides the following result of Darus (2005).
Corollary 3.2.1. Let \( f \in \mathcal{A} \). Suppose \( a > 0, b \geq 0 \). Then for \( 0 < \beta < 1 \),
\[
\left\{ (a - b) \left( \frac{zf'(z)}{f(z)} \right) + b \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \left( \frac{zf'(z)}{f(z)} \right) \prec \left( \frac{1 + z}{1 - z} \right)^{C(a, b, 1, \beta, 1)}
\]
implies
\[
\frac{zf'(z)}{f(z)} \prec \left( \frac{1 + z}{1 - z} \right)^{\beta}, \quad z \in \Delta. \tag{3.8}
\]

Where
\[
C(a, b, 1, \beta, 1) = \beta + \frac{2}{\pi} \arctan \left\{ \frac{\tan \frac{\pi \beta}{2} + \frac{b \beta}{a (1 + \beta) \frac{1 + \beta}{2} (1 - \beta) \frac{1 - \beta}{2} \cos \frac{\pi \beta}{2}}}{2} \right\} \tag{3.9}
\]
and \( C(a, b, 1, \beta, 1) \) given by (3.9) is the largest number such that (3.8) holds.

Writing \( \alpha = \eta = b = 1 \) and \( \Phi(w) = w \) in Theorem 3.2.1, the following result of Darus (2005) is obtained.

Corollary 3.2.2. Let \( f \in \mathcal{A} \). Then for \( a > 0, 0 < \beta < 1 \),
\[
\left\{ (a - 1) \left( \frac{zf'(z)}{f(z)} \right) + \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \left( \frac{zf'(z)}{f(z)} \right) \prec \left( \frac{1 + z}{1 - z} \right)^{C(a, 1, 1, \beta, 1)}
\]
implies
\[
\frac{zf'(z)}{f(z)} \prec \left( \frac{1 + z}{1 - z} \right)^{\beta}, \quad z \in \Delta. \tag{3.10}
\]

Where
\[
C(a, 1, 1, \beta, 1) = \beta + \frac{2}{\pi} \arctan \left\{ \frac{\tan \frac{\pi \beta}{2} + \frac{\beta}{a (1 + \beta) \frac{1 + \beta}{2} (1 - \beta) \frac{1 - \beta}{2} \cos \frac{\pi \beta}{2}}}{2} \right\} \tag{3.11}
\]
and \( C(a, 1, 1, \beta, 1) \) given by (3.11) is the largest number such that (3.10) holds.

Writing \( \alpha = \eta = a = 1 \) and \( \Phi(w) = w \) in Theorem 3.2.1, the following result of Darus (2005) is obtained.

Corollary 3.2.3. Let \( f \in \mathcal{A} \). Then for \( b \geq 0, 0 < \beta < 1 \),
\[
\left\{ (1 - b) \left( \frac{zf'(z)}{f(z)} \right) + b \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \left( \frac{zf'(z)}{f(z)} \right) \prec \left( \frac{1 + z}{1 - z} \right)^{C(1, b, 1, \beta, 1)}
\]
implies
\[
\frac{zf'(z)}{f(z)} \prec \left( \frac{1 + z}{1 - z} \right)^{\beta}, \quad z \in \Delta. \tag{3.12}
\]
Where
\[ C(1, b, 1, \beta, 1) = \beta + \frac{2}{\pi} \arctan \left\{ \tan \frac{\pi \beta}{2} + \frac{b \beta}{(1 + \beta) \frac{1}{2} (1 - \beta) \frac{1}{2} \cos \frac{\pi \beta}{2} \right\} \] (3.13)
and \( C(1, b, 1, \beta, 1) \) given by (3.13) is the largest number such that (3.12) holds.

When \( \alpha = \eta = \alpha = b = 1 \) and \( \Phi(w) = w \) in Theorem 3.2.1, the result of Darus (2005) is obtained.

**Corollary 3.2.4.** Let \( f \in \mathcal{A} \). Then for \( 0 < \beta < 1 \),
\[
\left( 1 + \frac{z f''(z)}{f'(z)} \right) \left\{ \frac{z f'(z)}{f(z)} \right\} < \left( \frac{1 + z}{1 - z} \right)^{C(1, 1, 1, 1, 1)}
\]
implies
\[
\frac{z f'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta}, \quad z \in \Delta,
\] (3.14)

Where
\[ C(1, 1, 1, \beta, 1) = \beta + \frac{2}{\pi} \arctan \left\{ \tan \frac{\pi \beta}{2} + \frac{\beta}{(1 + \beta) \frac{1}{2} (1 - \beta) \frac{1}{2} \cos \frac{\pi \beta}{2} \right\} \] (3.15)
and \( C(1, 1, 1, \beta, 1) \) given by (3.15) is the largest number such that (3.14) holds.

For \( a = \mu + \lambda, \quad b = \lambda, \quad \alpha = 0, \quad \eta = 1 \) and \( \Phi(w) = w \), Theorem 3.2.1, the following result of Darus (2003) is obtained.

**Corollary 3.2.5.** Let \( f \in \mathcal{A} \). Suppose \( \mu + \lambda > 0, \lambda \geq 1 \). Then for \( 0 < \beta < 1 \),
\[
\mu \left\{ \frac{z f''(z)}{f'(z)} \right\} + \lambda \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} < \left( \frac{1 + z}{1 - z} \right)^{C(\mu + \lambda, 1, 0, \beta, 1)}
\]
implies
\[
\frac{z f'(z)}{f(z)} < \left( \frac{1 + z}{1 - z} \right)^{\beta}, \quad z \in \Delta.
\] (3.16)

Where
\[ C(\mu + \lambda, 1, 0, \beta, 1) = \frac{2}{\pi} \arctan \left\{ \tan \frac{\pi \beta}{2} + \frac{\lambda \beta}{(\mu + \lambda) (1 + \beta) \frac{1}{2} (1 - \beta) \frac{1}{2} \cos \frac{\pi \beta}{2} \right\} \] (3.17)
and \( C(\mu + \lambda, 1, 0, \beta, 1) \) given by (3.17) is the largest number such that (3.16) holds.
By setting $\alpha = 1$ and $\Phi(w) = w$ in Theorem 3.2.1, the following result is obtained.

**Corollary 3.2.6.** Let $f(z) \in \mathcal{A}$ and $b \geq 0$, $\alpha$ and $\eta$ be fixed real numbers, with $0 < \eta \leq 1$, $\alpha + \eta \geq 0$. For $0 < \beta < 1$,

$$
\left( \frac{zf''(z)}{f(z)} \right)^\alpha \left\{ (1 - b) \left( \frac{zf''(z)}{f(z)} \right) + b \left( 1 + \frac{zf''(z)}{f(z)} \right) \right\}^\eta \prec \left( \frac{1 + z}{1 - z} \right)^{C(1, b, \alpha, \beta, \eta)}
$$

implies

$$
\frac{zf'(z)}{f(z)} \prec \left( \frac{1 + z}{1 - z} \right)^\beta,
$$

for $z \in \Delta$, where

$$
C(1, b, \alpha, \beta, \eta) = \alpha\beta + \frac{2\eta}{\pi} \arctan \left\{ \tan \frac{\pi \beta}{2} + \frac{\beta^2}{(1 + \beta)^2 \left( 1 - \beta \right)^2 \cos \frac{\pi \beta}{2} (1 - \beta) \frac{1 - \beta}{2} \cos \frac{\pi \beta}{2} \right\},
$$

and $C(1, b, \alpha, \beta, \eta)$ given by (3.19) is the largest number such that (3.18) holds.

The above corollary can also written as:

**Corollary 3.2.7.** Let $f(z)$ be analytic function in $\Delta$ with $f(0) = f'(0) - 1 = 0$. Let $b \geq 0$, $\alpha$ and $\eta$ be fixed real numbers, with $0 < \eta \leq 1$, $\alpha + \eta \geq 0$. For a real number $0 < \gamma < 1$, the function $f \in \mathcal{A}$ satisfies the differential subordination

$$
\left( \frac{zf''(z)}{f(z)} \right)^\alpha \left\{ (1 - b) \left( \frac{zf''(z)}{f(z)} \right) + b \left( 1 + \frac{zf''(z)}{f(z)} \right) \right\}^\eta \prec \left( \frac{1 + z}{1 - z} \right)^\beta \left\{ \left( \frac{1 + z}{1 - z} \right)^\gamma + 2\beta \gamma \right\},
$$

then $f \in \mathcal{S}^I(\gamma)$.

By taking $\alpha = 1 - \eta$ and $b = 1$ in Corollary 3.2.6, the following result essentially proved by Darus and Thomas (1990) is obtained.

**Corollary 3.2.8.** Let $f(z)$ be analytic function in $\Delta$ with $f(0) = f'(0) - 1 = 0$. Let $\eta$ be fixed real numbers, with $0 < \eta \leq 1$, $\alpha + \eta \geq 0$. Let $0 < \beta < 1$ and $f \in \mathcal{A}$ satisfies

$$
\left( \frac{zf''(z)}{f(z)} \right)^{1-\eta} \left( 1 + \frac{zf''(z)}{f(z)} \right)^\eta \prec \left( \frac{1 + z}{1 - z} \right)^{C(1, 1, 1-\eta, \beta, \gamma)},
$$
then
\[ \frac{zf''(z)}{f(z)} < \left( \frac{1+z}{1-z} \right)^\beta, \quad (3.20) \]
for \( z \in \Delta \), where \( C(1, 1, 1 - \eta, \beta, \eta) \)
\[ = (1 - \eta) \beta + \frac{2\eta}{d} \pi \arctan \left\{ \tan \frac{\pi \beta}{2} + \frac{\beta}{(1 + \beta) \frac{14\pi}{13} (1 - \beta) \frac{14\pi}{13} \cos \frac{\pi \beta}{2} \right\} \quad (3.21) \]
and \( C(1, 1, 1 - \eta, \beta, \eta) \) given by (3.21) is the largest number such that (3.20) holds.

For the choices of \( \alpha = -1, \alpha = b = \eta = 1 \) and \( \Phi(w) = w \) in Theorem 3.2.1, we get the following result:

**Corollary 3.2.9.** For \( 0 < \beta < 1 \). Let \( f \in \mathcal{A} \) satisfies the differential subordination
\[ \frac{1 + zf''(z)'}{zf'(z)} < \left( \frac{1 + z}{1 - z} \right)^{C(1, 1, -1, \beta, 1)}, \]
then
\[ \frac{zf''(z)}{f(z)} < \left( \frac{1+z}{1-z} \right)^\beta, \quad (3.22) \]
for \( z \in \Delta \), where
\[ C(1, 1, -1, \beta, 1) = \frac{2}{\pi} \arctan \left\{ \tan \frac{\pi \beta}{2} + \frac{\beta}{(1 + \beta) \frac{14\pi}{13} (1 - \beta) \frac{14\pi}{13} \cos \frac{\pi \beta}{2} \right\} - \beta \quad (3.23) \]
and \( C(1, 1, -1, \beta, 1) \) given by (3.23) is the largest number such that (3.22) holds.

By taking \( \alpha = \eta = b = 1 \) in Corollary 3.2.6, the following subordination result is obtained.

**Corollary 3.2.10.** For a real number \( 0 < \gamma < 1 \), the function \( f \in \mathcal{A} \) satisfies the differential subordination
\[ \frac{zf''(z)}{f'(z)} + \frac{zf'''(z)}{f'(z)} < \left( \frac{1+z}{1-z} \right)^\gamma \left\{ \left( \frac{1+z}{1-z} \right)^\gamma + \frac{2b\gamma}{1-\gamma^2} \right\}, \quad (3.24) \]
then \( f \in \mathcal{S}^\gamma(\gamma) \).
Taking $a = \eta = 1$, $\alpha = 0$, and $\Phi(w) = w$ in Theorem 3.2.1, the following result of Marjono and Thomas (2001) is obtained.

**Corollary 3.2.11.** Let $f \in \mathcal{A}$. Then for $0 < \beta \leq 1$ and $\delta > 0$, there exists $C(\beta, b)$ such that

$$(1 - b) \frac{zf'(z)}{f(z)} + b \left(1 + \frac{zf''(z)}{f'(z)}\right) < \left(\frac{1 + z}{1 - z}\right)^{C(1, b, 0, \beta, 1)},$$

implies

$$\frac{zf''(z)}{f'(z)} < \left(\frac{1 + z}{1 - z}\right)^{\beta}, \quad z \in \Delta. \quad (3.25)$$

In particular, we may choose

$$C(1, b, 0, \beta, 1) = \frac{2}{\pi} \arctan \left[\tan \frac{\pi \beta}{2} + \frac{b \beta}{(1 + \beta) \frac{\pi}{2} (1 - \beta) \frac{\pi}{2} \cos \frac{\pi \beta}{2}}\right], \quad (3.26)$$

and $C(1, b, 0, \beta, 1)$ given by (3.26) is the largest number such that (3.25) holds.

By taking $a = b = \eta = 1$, $\alpha = 0$ and $\Phi(w) = w$ in Theorem 3.2.1, the following result of Nunokawa and Thomas (1996) is obtained.

**Corollary 3.2.12.** Let $f \in \mathcal{A}$. Then for $0 < \beta \leq 1$, there exists $C(\alpha, \beta)$, such that

$$1 + \frac{zf''(z)}{f'(z)} < \left(\frac{1 + z}{1 - z}\right)^{C(1, 1, 0, \beta, 1)},$$

implies

$$\frac{zf''(z)}{f'(z)} < \left(\frac{1 + z}{1 - z}\right)^{\beta}, \quad z \in \Delta. \quad (3.27)$$

In particular, we may choose

$$C(1, 1, 0, \beta, 1) = \frac{2}{\pi} \arctan \left[\tan \frac{\pi \beta}{2} + \frac{\beta}{(1 + \beta) \frac{\pi}{2} (1 - \beta) \frac{\pi}{2} \cos \frac{\pi \beta}{2}}\right], \quad (3.28)$$

and $C(1, 1, 0, \beta, 1)$ given by (3.28) is the largest number such that (3.27) holds.

When the choice of $\alpha = 0$ and $\eta = 1$ in Theorem 3.2.1, the result of Shanmugam and Ramachandran (preprint) is obtained.

**Corollary 3.2.13.** Let $f \in \mathcal{A}$. Suppose $a > 0$, $b \geq 0$. For $0 < \beta < 1,$

$$a \left\{ \frac{zf'(z)}{\Phi(f(z))} \right\} + b \left\{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(f(z))}{\Phi(f(z))}\right\} < \left(\frac{1 + z}{1 - z}\right)^{C(a, b, 0, \beta, 1)}$$

...
implies
\[ \frac{zf'(z)}{\Phi(f(z))} < \left( \frac{1+z}{1-z} \right)^{\beta}, \quad z \in \Delta, \]  
(3.29)

where
\[ C(a, \beta, 0, \beta, 1) = \frac{2}{\pi} \arctan \left\{ \tan \frac{\pi\beta}{2} + \frac{b\beta}{a(1+\beta) \frac{1+\beta}{2} (1-\beta) \frac{1-\beta}{2} \cos \frac{\pi\beta}{2} \right\} \]  
(3.30)

and \( C(a, \beta, 0, \beta, 1) \) given by (3.30) is the largest number such that (3.29) holds.

### 3.3 SUFFICIENT CONDITIONS FOR \( \Phi \)-LIKE FUNCTIONS

In this section sandwich type results and related best dominants and best subordinants are obtained for \( \Phi \)-like functions, using subordination and superordination principles.

Recently Miller and Mocanu (2003) obtained conditions on \( h, q \) and \( \phi \) for which the following implication holds:
\[ h(z) \prec \phi(p(z), z^p'(z), z^p''(z); z) \Rightarrow q(z) \prec p(z). \]  
(3.31)

Using the result of Miller and Mocanu (2003), Bulboacă (2000a) has considered certain class of first order differential superordination as well as superordination preserving integral operator (2002).

In the present investigation, sufficient conditions for a normalized analytic functions \( f \) to satisfy
\[ q_1(z) \prec \frac{zf'(z)}{\Phi(f(z))} \prec q_2(z) \]

is obtained.

**Theorem 3.3.1.** Let \( \alpha \) and \( \beta \) be complex numbers. Let \( 0 \neq q(z) \) be univalent in \( \Delta \) and \( \frac{zq'(z)}{q(z)} \) be starlike univalent in \( \Delta \). Further assume that
\[ \Re \left[ 1 + \frac{zq''(z)}{q(z)} - \frac{2zq'(z)}{q(z)} \right] > 0. \]  
(3.32)
If $f \in \mathcal{A}$ satisfies
\[
\alpha + \beta \left\{ 1 + \frac{zf''(z) - z[\Phi(f(z))]'}{\Phi(f(z))} \right\} \prec \alpha + \frac{\beta zf'(z)}{q^2(z)},
\]
then,
\[
\frac{zf'(z)}{\Phi(f(z))} \prec q(z)
\]
that is, $f$ is $\Phi$-like with respect to $q$ and $q$ is the best dominant.

Proof. Let the function $p(z)$ be defined by
\[
p(z) := \frac{zf'(z)}{\Phi(f(z))}.
\]
By a straightforward computation, we have
\[
\frac{zp(z)}{p^2(z)} = \frac{1 + \frac{zf''(z)}{f'(z)} - \frac{z[\Phi(f(z))]'}{\Phi(f(z))}}{\frac{zf'(z)}{\Phi(f(z))}}.
\]
By setting $\theta(\omega) := \alpha$ and $\phi(\omega) := \frac{\beta}{\omega^2}$, it can be easily observed that $\theta(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C} - \{0\}$ and $\theta(\omega) \neq 0$. Also, by letting
\[
Q(z) = zf'(z)\phi(q(z)) = \frac{\beta zf'(z)}{q^2(z)}
\]
and
\[
h(z) = \theta(q(z)) + Q(z) = \alpha + \frac{\beta zf'(z)}{q^2(z)},
\]
we find that $Q(z)$ is starlike univalent in $\Delta$ and that
\[
\Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left\{ 1 + \frac{zf''(z)}{q'q(z)} - \frac{2zf'(z)}{q^2(z)} \right\} > 0. \quad (q(z) \neq 0).
\]
The assertion (3.33) of Theorem 3.3.1 now follows by an application of Theorem 1.4.1.

By taking $\alpha, \beta$ as real and $\alpha = \beta = 1$ and $\theta(\omega) = \omega$ in Theorem 3.3.1, then we have the following:
Corollary 3.3.1. Let $0 \neq q(z)$ be univalent in $\Delta$ and $q(0) = 1$. Further assuming that (3.32) holds. If $f \in \mathcal{A}$ satisfies
\[
\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} < 1 + \frac{zq'(z)}{q^2(z)},
\]
then
\[
\frac{zf'(z)}{f(z)} < q(z)
\]
and $q$ is the best dominant.

For the choice of $q(z) = \frac{1 + Az}{1 + Bz}$, $(-1 \leq B < A \leq 1)$, in corollary 3.3.1, the following result of Ravichandran and Darus (2003) (Theorem 6, p. 123) is obtained.

Corollary 3.3.2. If $f \in \mathcal{A}$, and
\[
\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} < 1 + \frac{(A - B)z}{(1 + Az)^2},
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}.
\]

By taking $A = 0$, $B = b$ in corollary 3.3.2, the following result of Obradovic and Tuneski (2000) is obtained.

Corollary 3.3.3. If $f \in \mathcal{A}$, and
\[
\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} < 1 - bz,
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{1}{1 + bz}.
\]

By taking $A = 1, B = -1$ in corollary 3.3.2, the following result of Obradovic and Tuneski (2000) is obtained.

Corollary 3.3.4. If $f \in \mathcal{A}$, and
\[
\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} < 1 + \frac{2z}{(1 + z)^2},
\]
then
\[f \in \mathcal{S}^1.\]
For the choice of $q(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha$ in Corollary 3.3.1, we get the following corollary.

**Corollary 3.3.5.** If $f \in \mathcal{A}$, and

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} < 1 + \frac{2\alpha z}{(1 + z)^{1+\alpha}(1 - z)^{1-\alpha}},$$

then

$$\left|\arg \left(\frac{zf'(z)}{f(z)}\right)\right| \leq \frac{\pi \alpha}{2}.$$

**Theorem 3.3.2.** Let $\alpha$ and $\beta$ be complex numbers. Let $0 \neq q(z)$ be convex univalent in $\Delta$ and $\frac{zf'(z)}{f(z)}$ be starlike univalent in $\Delta$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$, and

$$\alpha + \beta \left(1 + \frac{zf''(z)/f'(z)}{zf'(z)/f(z)} - \frac{z[\Phi(f(z))]''}{\Phi(f(z))}\right)$$

is univalent in $\Delta$, then

$$\alpha + \frac{\beta zf'(z)}{q^2(z)} \prec \alpha + \beta \left(1 + \frac{zf''(z)/f'(z)}{zf'(z)/f(z)} - \frac{z[\Phi(f(z))]''}{\Phi(f(z))}\right),$$

implies

$$q(z) \prec \frac{zf'(z)}{\Phi(f(z))}$$

and $q$ is the best subordinant.

**Proof.** Define the function $p(z)$ by

$$p(z) := \frac{zf'(z)}{\Phi(f(z))}.$$ 

Then a computation shows that

$$\alpha + \frac{\beta \left(1 + \frac{zf''(z)/f'(z)}{zf'(z)/f(z)} - \frac{z[\Phi(f(z))]''}{\Phi(f(z))}\right)}{zf'(z)/f(z)} = \alpha + \frac{\beta zf'(z)}{p^2(z)}. \tag{3.39}$$

By using (3.38) in (3.39), we have

$$\alpha + \frac{\beta zf'(z)}{q^2(z)} \prec \alpha + \frac{\beta zf'(z)}{p^2(z)} \tag{3.40}$$

The superordination (3.40) is same as (1.24) with $\theta(\omega) := \alpha$, $\phi(\omega) := \frac{\beta}{\omega}$. Now the result follows by an application of Theorem 1.4.2.
By combining Theorem 3.3.1 and Theorem 3.3.2 we get the following Sandwich theorem.

**Theorem 3.3.3.** Let \( q_1 \) be convex univalent in \( \Delta \) satisfying (3.32) and \( q_2 \) be convex univalent in \( \Delta \). Suppose that \( \frac{zf'(z)}{q_i(z)} \) be starlike univalent in \( \Delta \) for \( i = 1, 2 \). Let \( f \in \mathcal{A}, \frac{zf'(z)}{\Phi(f(z))} \in \mathcal{H}[1, 1] \cap \mathcal{Q}, \) and \( \alpha + \beta \left\{ 1 + \frac{zf''(z) - z\Phi(f(z))'}{\Phi(f(z))} \right\} \) is univalent in \( \Delta \), then

\[
\alpha + zq_1(z) < \alpha + \beta \left\{ 1 + \frac{zf''(z) - z\Phi(f(z))'}{\Phi(f(z))} \right\} < \alpha + zq_2(z),
\]

implies

\[
q_1(z) \prec \frac{zf'(z)}{\Phi(f(z))} \prec q_2(z)
\]

and \( q_1 \) and \( q_2 \) are respectively the best subordinant and best dominant.

For \( q_1(z) = \frac{1 + A_1 z}{1 + B_1 z}, q_2(z) = \frac{1 + A_2 z}{1 + B_2 z} \) \((-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1)\) in Theorem 3.3.3 we have the following:

**Corollary 3.3.6.** If \( f \in \mathcal{A}, \frac{zf'(z)}{\Phi(f(z))} \in \mathcal{H}[1, 1] \cap \mathcal{Q}, \) then \( \alpha + \beta \left\{ 1 + \frac{zf''(z) - z\Phi(f(z))'}{\Phi(f(z))} \right\} \) is univalent in \( \Delta \), then

\[
\alpha + (A_1 - B_1)z \prec \alpha + \beta \left\{ 1 + \frac{zf''(z) - z\Phi(f(z))'}{\Phi(f(z))} \right\} \prec \alpha + (A_2 - B_2)z,
\]

implies

\[
\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{zf'(z)}{\Phi(f(z))} \prec \frac{1 + A_2 z}{1 + B_2 z}.
\]

The functions \( \frac{1 + A_1 z}{1 + B_1 z} \) and \( \frac{1 + A_2 z}{1 + B_2 z} \) are respectively the best subordinant and best dominant.

The results in section 3.2 of this Chapter has been accepted for publication in “Southeast Asian Bull. Math. Journal”.

In the next Chapter, certain subordination and superordination results for univalent and multivalent analytic functions in the open unit disc are derived. Functions defined through the Dziok-Srivastava operator and Multiplier transformation are discussed.