CHAPTER 2

AN APPLICATION OF FIRST ORDER DIFFERENTIAL SUBORDINATION

2.1 INTRODUCTION

Let $A_p (p \in \mathbb{N} = \{1, 2, 3, \ldots \})$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{m=1}^{\infty} a_{p+m} z^{p+m}$$

which are analytic in the open unit disc $\Delta := \{z : |z| < 1\}$ and $A_1 = A$.

Let $\phi : \mathbb{C}^3 \times \Delta \rightarrow \mathbb{C}$ and let $h(z)$ be univalent in $\Delta$. If $p(z)$ is analytic in $\Delta$ and satisfies the second order differential subordination

$$\phi(p(z), zp'(z), z^2 p''(z); z) \prec h(z),$$

then $p(z)$ is called a solution of the differential subordination. The univalent function $q(z)$ is called a dominant of the differential subordination if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2.2). A dominant $\tilde{q}(z)$ which satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (2.2) is the best dominant.

For $-1 \leq B < A \leq 1$ and $0 < \gamma \leq 1$, a function $f \in A_p$ is said to be in the class $S_p^\gamma(\gamma, A, B)$ if it satisfies

$$\frac{zf'(z)}{f(z)} \prec p \left( \frac{1 + Az}{1 + Bz} \right)^\gamma.$$ 

Also, we write $S_p^\gamma(\gamma, 1, -1) = SS_p^\gamma(\gamma)$, the class of strongly starlike $p$-valent functions of order $\gamma$ in $\Delta$. $S_p^\gamma(1, A, B) = S_p^\gamma(A, B)$, the class of Janowski starlike $p$-valent function, $S_p^\gamma(1, -1) = S_p^\gamma$ the class of $p$-valent starlike function and $S_p^\gamma(1 - 2\gamma, 1) = S_p^\gamma(\gamma)$ $(0 \leq \gamma < 1)$, the class of $p$-valent starlike function of order $\gamma$. 
For Caratheodory functions, Miller (1975) obtained certain sufficient conditions by applying the differential inequalities. In this Chapter, for given \( q(z) \), sufficient conditions are found for the subordination \( p(z) \prec q(z) \) to hold. The results include the results obtained by Nunokawa et al. (2002) and Ravichandran and Jayamala (2004). Some application of the results to obtain sufficient conditions for \( p \)-valent starlikeness and strong starlikeness are also given.

### 2.2 Application of Differential Subordination

By making use of Theorem 1.4.1, we first prove the following theorem.

**Theorem 2.2.1.** Let \( 0 \neq \alpha \in \mathbb{C} \) and \( \lambda \) be positive real number. Let \( q(z) \) be convex univalent in \( \Delta \) and \( \Re \, q(z) \geq \Re \left( \frac{1}{m+1} \right) \), \( m \in \mathbb{N} \setminus \{1\} \), where \( \mathbb{N} \) is the set of all natural numbers. If \( p \in \mathcal{P} \) satisfies

\[
(1 - \alpha)p(z) + \alpha(p(z))^m + \alpha \lambda z p'(z) \prec q(z),
\]

where,

\[
h(z) = (1 - \alpha)q(z) + \alpha(q(z))^m + \alpha \lambda z q'(z),
\]

then,

\[
p(z) \prec q(z)
\]

and \( q(z) \) is the best dominant of (2.3).

**Proof.** Let

\[
\theta(w) = (1 - \alpha)w + \alpha w^m, \phi(w) = \alpha \lambda.
\]

Then clearly \( \theta(w) \) and \( \phi(w) \) are analytic in \( \mathbb{C} \) and \( \phi(w) \neq 0 \). Also let

\[
Q(z) = z q'(z) \phi(q(z)) = \alpha \lambda z q'(z)
\]

and

\[
h(z) = \theta(q(z)) + Q(z)
= (1 - \alpha)q(z) + \alpha(q(z))^m + \alpha \lambda z q'(z).
\]
Since \( q(z) \) is convex univalent, \( zq'(z) \) is starlike univalent. Therefore \( Q(z) \) is starlike univalent in \( \Delta \), and
\[
\Re \left( \frac{zh'(z)}{Q(z)} \right) = \frac{1}{\lambda} \Re \left\{ \frac{1 - \alpha}{\alpha} + m(q(z))^{m-1} + \lambda \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0 \tag{2.5}
\]
for \( z \in \Delta \).

From (2.3)-(2.5) we see that
\[
\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)) = h(z).
\]
Therefore, by applying the Theorem 1.4.1, we conclude that \( p(z) \prec q(z) \) and \( q(z) \) is the best dominant of (2.3). The proof of the theorem is complete. \( \square \)

By taking \( \alpha \) as real and \( q(z) = \left( \frac{1 + Az}{1 + Bz} \right)^{\gamma} \) in Theorem 2.2.1, we get the following corollary.

**Corollary 2.2.1.** Let \(-1 \leq B < A < 1\), \( m \in \mathbb{N}\setminus\{1\}, 0 < \gamma \leq \frac{1}{m-1}\), \( \lambda \) be real number such that \( \lambda > 0 \) and \( 0 < \alpha \leq 1 \). If \( p \in \mathcal{P} \) satisfies
\[
(1 - \alpha)p(z) + \alpha(p(z))^m + \alpha(zp'(z) < h(z), \tag{2.6}
\]
where
\[
h(z) = (1 - \alpha) \left( \frac{1 + Az}{1 + Bz} \right)^{\gamma} + \alpha \left( \frac{1 + Az}{1 + Bz} \right)^{m\gamma} + \frac{\alpha \lambda^\gamma (A - B)z}{(1 + Az)^{1 - \gamma}(1 + Bz)^{1 + \gamma}}. \tag{2.7}
\]
then
\[
p(z) \prec \left( \frac{1 + Az}{1 + Bz} \right)^{\gamma}
\]
and \( \left( \frac{1 + Az}{1 + Bz} \right)^{\gamma} \) is the best dominant of (2.6).

**Corollary 2.2.2.** Let \(-1 \leq B < A < 1\), \( \lambda > 0 \). If \( f \in \mathcal{A}_p \) satisfies \( f(z) \neq 0 \) in \( 0 < |z| < 1 \) and
\[
\frac{zf'(z)}{pf(z)} \left[ 1 - \alpha + \frac{\alpha}{p} (1 - \lambda p) \frac{zf'(z)}{f(z)} + \alpha \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] \prec h(z) \tag{2.8}
\]
where,
\[
h(z) = (1 - \alpha) \left( \frac{1 + Az}{1 + Bz} \right)^{\gamma} + \alpha \left( \frac{1 + Az}{1 + Bz} \right)^{2\gamma} + \frac{\alpha \lambda^\gamma (A - B)z}{(1 + Az)^{1 - \gamma}(1 + Bz)^{1 + \gamma}}. \tag{2.9}
\]
then,
\[
\frac{zf'(z)}{pf(z)} \prec \left( \frac{1 + Az}{1 + Bz} \right)^{\gamma}.
\]
Proof. Let $p(z) = \frac{zf''(z)}{pf(z)}$, then $p \in \mathcal{P}$ and (2.8) can be written as

\[
(1 - \alpha)p(z) + \alpha f^2(z) + \alpha \lambda z f'(z) \\
\leq (1 - \alpha) \left( 1 + \frac{A z}{1 + B z} \right)^{\gamma_1} + \alpha \left( 1 + \frac{A z}{1 + B z} \right)^{\gamma_2} + \frac{\alpha \lambda z (A - B) z}{(1 + A z)^{1-\gamma}(1 + B z)^{1+\gamma}}
\]

(2.10)

Taking $m = 2$ in Corollary 2.2.1 and using (2.10), we have

\[
\frac{zf''(z)}{pf(z)} \leq \left( \frac{1 + A z}{1 + B z} \right)^{\gamma_1}.
\]

By taking $p = \lambda = \gamma = A = 1$ and $B = -1$ in Corollary 2.2.2, we get the following result of Padmanaban (2001).

**Corollary 2.2.3.** Let $f \in \mathcal{A}$ and

\[
\frac{zf''(z)}{f(z)} + \alpha \frac{zf''(z)}{f'(z)} \leq \frac{2\alpha (z^2 + 2z) + 1 - z^2}{(1 - z)^2} \quad (0 < \alpha \leq 1),
\]

(2.11)

then

\[
\Re \left( \frac{zf''(z)}{f(z)} \right) > 0.
\]

(2.12)

For the choices of $A = p = \alpha = \gamma = 1$ and $B = -1$ in Corollary 2.2.2, we get the following result.

**Corollary 2.2.4.** Let $-1 \leq B < A \leq 1$ and $\lambda > 0$. If $f \in \mathcal{A}$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and

\[
\frac{zf''(z)}{f(z)} \left( 1 - \lambda \right) \frac{zf''(z)}{f'(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \leq \left( 1 + \frac{z}{1 - z} \right)^2 + \frac{2\lambda z}{(1 - z)^2}
\]

(2.13)

then, $f$ is starlike univalent.

In particular for $\lambda = 1$, we have

\[
\frac{zf''(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \leq \frac{1 + 4z + z^2}{(1 - z)^2}
\]

then, $f$ is starlike univalent.
Theorem 2.2.2. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$. Let $q(z)$ be convex univalent in $\Delta$ and
\[ \Re(q(z)) \geq \Re \left( \frac{1 - \lambda}{m} \right)^m, m \in \mathbb{N}\setminus\{1\}. \]
If $p \in \mathcal{P}$ satisfies
\[ (\lambda - 1)p(z) + (p(z))^m + zq'(z) < h(z), \quad (2.14) \]
where,
\[ h(z) = (\lambda - 1)q(z) + (q(z))^m + zq'(z), \quad (2.15) \]
then,
\[ p(z) \prec q(z) \quad (2.16) \]
and $q(z)$ is the best dominant of (2.14).

Proof. Let
\[ \theta(w) = (\lambda - 1)w + w^m, \quad \phi(w) = 1. \]
Then clearly $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}$ and $\phi(w) \neq 0$. Also let
\[ Q(z) = zq'(z)\phi(q(z)) = zq'(z) \]
and
\[ h(z) = \theta(q(z)) + Q(z) = (\lambda - 1)q(z) + (q(z))^m + zq'(z). \]
Since $q(z)$ is convex univalent, $zq'(z)$ is starlike univalent. Therefore $Q(z)$ is starlike univalent in $\Delta$, and
\[ \Re \left( \frac{zq'(z)}{\phi(q(z))} \right) = \Re \left\{ (\lambda - 1) + m(q(z))^{m-1} + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0 \quad (2.17) \]
for $z \in \Delta$.

From (2.14)-(2.17) we see that
\[ \theta(p(z)) + zq'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z). \]
Therefore, by applying Theorem 1.4.1, it can be concluded that $p(z) \prec q(z)$ and $q(z)$ is the best dominant of (2.14). The proof of the theorem is complete.

By taking $\lambda$ as real and $q(z) = \left( \frac{1 - \lambda}{1 + \beta z} \right)^2$ in Theorem 2.2.2, we get the following corollary.
Corollary 2.2.5. Let $-1 \leq B < A \leq 1$, $m \in \mathbb{N} \setminus \{1\}$, $0 < \gamma \leq \frac{1}{m-1}$. If $p \in \mathcal{P}$ satisfies
\[(\lambda - 1)p(z) + (p(z))^m + z p'(z) \prec h(z), \quad (2.18)\]
where,
\[h(z) = (\lambda - 1) \left( \frac{1 + Az}{1 + Bz} \right)^\gamma + \left( \frac{1 + Az}{1 + Bz} \right)^{m\gamma} + \frac{\gamma (A - B)z}{(1 + Az)^{1-\gamma}(1 + Bz)^{1+\gamma}},\]
then,
\[p(z) \prec \left( \frac{1 + Az}{1 + Bz} \right)^\gamma\]
and $\left( \frac{1 + Az}{1 + Bz} \right)^\gamma$ is the best dominant of (2.18).

Corollary 2.2.6. Let $-1 \leq B < A \leq 1$, $\lambda > 0$. If $f \in \mathcal{A}_p$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and
\[
\frac{zf''(z)}{pf(z)} \left[ \lambda + \left( \frac{1}{p} - 1 \right) \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \right] \prec h(z), \quad (2.19)
\]
where
\[h(z) = (\lambda - 1) \left( \frac{1 + Az}{1 + Bz} \right)^\gamma + \left( \frac{1 + Az}{1 + Bz} \right)^{2\gamma} + \frac{\gamma (A - B)z}{(1 + Az)^{1-\gamma}(1 + Bz)^{1+\gamma}},\]
then
\[f \in S^\lambda_p(\gamma, A, B).\]

Proof. Let $p(z) = \frac{zf'(z)}{pf(z)}$, then $p \in \mathcal{P}$ and (2.19) can be written as
\[(\lambda - 1)p(z) + p^2(z) + z p'(z) \prec (\lambda - 1) \left( \frac{1 + Az}{1 + Bz} \right)^\gamma + \left( \frac{1 + Az}{1 + Bz} \right)^{2\gamma} + \frac{\gamma (A - B)z}{(1 + Az)^{1-\gamma}(1 + Bz)^{1+\gamma}} \quad (2.20)
\]
Taking $m = 2$ in Corollary 2.2.5 and using (2.20), we have
\[
\frac{zf''(z)}{pf(z)} \prec \left( \frac{1 + Az}{1 + Bz} \right)^\gamma.
\]
By taking $p = \gamma = 1$ and $B = -1$ in corollary 2.2.6, we have the following:
Corollary 2.2.7. If $f \in \mathcal{A}$ and
\[ \frac{z^2 f''(z)}{f'(z)} + \lambda \frac{zf'(z)}{f(z)} < h(z) \]
where
\[ h(z) = \frac{(A^2 + A - A\lambda)z^2 + (2A + \lambda(A - 1) + 2)z + \lambda}{(1 - z)^2}, \tag{2.21} \]
then,
\[ \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 - z} \]
and $\frac{1 + Az}{1 - z}$ is the best dominant.

The investigation of the image of the unit disc under the function $h$ is done as follows. Rewriting $h(z)$, as
\[ h(z) = \frac{(A^2 + A(1 - \lambda))z^2 + (2A + \lambda(A - 1) + 2)z + \lambda}{(1 - z)^2}, \]
we find that $h(0) = \lambda$ and $h(-1) = \frac{2\lambda(1 - A) + A^2 - A - 2}{4}.$

In view of (2.17), it is clear that the function $h$ is close to convex in $\Delta.$
$h(z)$ takes real values for real $z$ and $h(\Delta)$ is symmetric with respect to real axis and it intersects the real axis at one point only. The boundary of the curve is given by
\[ h(e^{i\theta}) = u(\theta) + iv(\theta), \theta \in (-\pi, \pi), \]
where
\[ u(\theta) = \frac{(2A + 2 + \lambda(A - \lambda)) + (\lambda + A^2 + A(1 - \lambda)) \cos \theta}{2(1 - \cos \theta)} \]
and
\[ v(\theta) = \frac{(\lambda(A + 1) - A^2 - A) \sin \theta}{2(1 - \cos \theta)}. \]

Eliminating $\theta$, we get the equation of the boundary curve as
\[ \begin{aligned}
    v^2 &= -\frac{(\lambda(1 + A) - A^2 - A)^2}{A^2 + 3A + 2} \left[ \frac{u - 2\lambda(1 - A) + A^2 - A - 2}{4} \right],
\end{aligned} \]
which is a parabola lies in the left half plane with its vertex at the point \( \left( \frac{2\lambda(1 - A) + A^2 - A - 2}{4}, 0 \right) \) and negative real axis as the axis of parabola. Hence $h(\Delta)$ is the exterior of this parabola and includes the right half plane $(u, v)$ with
\[ u > \frac{2\lambda(1 - A) + A^2 - A - 2}{4}. \]

By taking $A = 1,$ $\lambda = \frac{1}{\alpha}(0 < \alpha \leq 1)$ in Corollary 2.2.7, we get the following result.
Corollary 2.2.8. If \( f \in A \) satisfies \( f(z) \neq 0 \) in \( 0 < |z| < 1 \) and
\[
\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} < h(z),
\]
where
\[
h(z) = \frac{(2\alpha - 1)z^2 + 4\alpha z + 1}{(1 - z)^2},
\] (2.22)
then
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > 0.
\]
That is, \( f \in S^\alpha \).

Remark 2.2.1. For the function \( h(z) \) given by (2.22),
\[
h(\Delta) = \left\{ w = u + iv; v^2 > -\frac{2(1 - \alpha)^2}{3\alpha} \left[ u + \frac{\alpha}{2} \right] \right\},
\]
which properly contains the half plane \( \Re w > -\frac{\alpha}{2} \). Hence Corollary 2.2.8 refines the Theorem by Li and Owa (2002).

Setting \( A = 1 - 2\alpha \), \( 0 \leq \alpha < 1 \) in Corollary 2.2.7, we get the following corollary.

Corollary 2.2.9. If \( f \in A \) satisfies \( f(z) \neq 0 \) in \( 0 < |z| < 1 \) and
\[
\frac{z^2 f''(z)}{f(z)} + \lambda \frac{zf'(z)}{f(z)} < h(z),
\]
where
\[
h(z) = \frac{(1 - 2\alpha)(2 - 2\alpha - \lambda)z^2 + 2(2 - 2\alpha - \alpha\lambda)z + \lambda}{(1 - z)^2},
\]
\( 0 \leq \alpha < 1 \) and \( \lambda \geq 1 - 2\alpha \), then
\[
f \in S^\alpha(\alpha).
\]

We note that if \( h(z) = u + iv \), then \( h(\Delta) \) is the exterior of the parabola given by
\[
v^2 = -\frac{2(1 - \alpha)(\lambda - 1 + 2\alpha)^2}{3 - 2\alpha} \left[ u - \frac{2\alpha^2 - \alpha(1 - 2\lambda) - 1}{2} \right]
\]
with its vertex as \( \left( \frac{2\alpha^2 - \alpha(1 - 2\lambda) - 1}{2\lambda}, 0 \right) \).
Example 2.2.1. Taking \( A = 1, \lambda = \frac{1}{2} \) in Corollary 2.2.7, we obtain
\[
h(z) = \frac{1 + 8z + 3z^2}{2(1 - z)^2}.
\]
In this case, \( h(\Delta) \) is the exterior of the parabola given by
\[
v^2 = \frac{1}{6} \left[ u + \frac{1}{2} \right]
\]
with its vertex at the point \((-\frac{1}{2}, 0)\).

Thus, an application of Corollary 2.2.7, gives

Corollary 2.2.10. If \( f \in \mathcal{A}, f(z) \neq 0 \) in \( 0 < |z| < 1 \) and
\[
2z^2 \frac{f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} < \frac{1 + 8z + 3z^2}{(1 - z)^2},
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{1 + z}{1 - z}.
\]
Region \( h(\Delta) \) has been shown shaded in Figure 2.1.

Corollary 2.2.11. Let \(-1 \leq B < A \leq 1\). If \( f \in \mathcal{A} \) satisfies \( f'(z) \neq 0 \) in \( 0 < |z| < 1 \) and
\[
p \left( \frac{f(z)}{zf'(z)} \right)^2 + \left( 1 - \frac{f(z)}{zf'(z)} - \frac{f(z)f''(z)}{(f'(z))^2} \right) < h(z)
\]
where
\[
h(z) = \frac{1 + A^2z^2 + (3A - B)z}{p(1 + Bz)^2},
\]
then
\[
\frac{pf(z)}{zf'(z)} < \frac{1 + Az}{1 + Bz}.
\]

Proof. If we let \( p(z) = \frac{pf(z)}{zf'(z)} \), then \( p \in \mathcal{P} \) and (2.23) can be expressed as
\[
(p(z))^2 + zf'(z) < \left( \frac{1 + Az}{1 + Bz} \right)^2 + \frac{(A - B)z}{(1 + Bz)^2}.
\]
Hence, by taking \( \lambda = \gamma = 1, m = 2 \) in Corollary 2.2.5, we have \( p(z) < \frac{1 + Az}{1 + Bz} \). So, \( f \in S_1^+(B, A) \).

Setting \( p = 1 \) and \( B = -1 \) in Corollary 2.2.11, we get
Corollary 2.2.12. Let \(-1 < A \leq 1\). If \(f \in \mathcal{A}\) satisfies \(f'(z) \neq 0\) in \(0 < |z| < 1\) and
\[
\left( \frac{f(z)}{zf'(z)} \right)^2 + \left( 1 - \frac{f(z)}{zf'(z)} + \frac{f(z)f''(z)}{(f'(z))^2} \right) < h(z),
\]
where
\[
h(z) = \frac{1 + A^2 z^2 + (1 + 3.1)^2}{(1 - z)^2},
\]
then
\[
\frac{f(z)}{zf'(z)} < \frac{1 + Az}{1 - z}.
\]

Remark 2.2.2. For the function \(h(z)\) given by (2.24), we have
\[
h(\Delta) = \{ w = u + iv; \quad i^2 > a_0 [u - b_0] \},
\]
which properly contains the half plane \(\Re w > b_0\), where
\[
a_0 = \frac{(1 - A^2)}{A^2 + 3A + 2},
\]
\[
b_0 = \frac{(A^2 - 3A)}{4}.
\]
Theorem 2.2.3. Let \( \alpha, \beta, \xi, \eta \in \mathbb{C} \) and \( \eta \neq 0 \). Let \( q(z) \) be convex univalent in \( \Delta \) and satisfy
\[
\Re \left[ \frac{1}{\eta} (\beta + 2\xi q(z)) \right] > 0. \tag{2.25}
\]
If \( \rho \in \mathcal{P} \) satisfies
\[
\alpha + \beta \rho(z) + \xi \rho^2(z) + \eta \rho(z) \rho'(z) \prec \alpha + \beta q(z) + \xi q^2(z) + \eta q(z) q'(z) = h(z), \tag{2.26}
\]
then
\[
\rho(z) \prec q(z) \tag{2.27}
\]
and \( q(z) \) is the best dominant of (2.26).

Proof. By setting \( \Theta(w) := \alpha + \beta w + \xi w^2 \) and \( \phi(w) := \eta \) it can be easily observed that \( \Theta(w) \) and \( \phi(w) \) are analytic in \( \mathbb{C} \) and that \( \phi(w) \neq 0 \) \( (w \in \mathbb{C}\setminus\{0\}) \).

Also, by letting
\[
Q(z) = zq'(z)\phi(q(z)) = \eta zq'(z) \tag{2.28}
\]
\[
h(z) = \Theta(q(z)) + Q(z)
\]
\[
\quad = \alpha + \beta q(z) + \xi q^2(z) + \eta q(z) q'(z), \tag{2.29}
\]
we find that \( Q(z) \) is starlike univalent in \( \Delta \) and that
\[
\Re \left( \frac{z h'(z)}{Q(z)} \right) = \Re \left[ \frac{1}{\eta} (\beta + 2\xi q(z)) + \left( 1 + \frac{z q''(z)}{q'(z)} \right) \right] > 0.
\]
The differential subordination
\[
\alpha + \beta \rho(z) + \xi \rho^2(z) + \eta \rho(z) \rho'(z) \prec \alpha + \beta q(z) + \xi q^2(z) + \eta q(z) q'(z)
\]
becomes
\[
\Theta(\rho(z)) + z q'(z) \phi(\rho(z)) \prec \Theta(q(z)) + z q'(z) \phi(q(z)).
\]
Now, the result follows as an application of Theorem 1.4.1.

Theorem 2.2.4. Let \( \alpha, \beta, \xi, \eta \) and \( \delta \) be complex numbers, \( \delta \neq 0 \). Let \( 0 \neq q(z) \) be univalent in \( \Delta \) and satisfy the following conditions for \( z \in \Delta \):
(i) Let \( Q(z) = \delta z f'(z)/q(z) \) be starlike,

\[
\Re \left\{ \frac{4}{z} q(z) + \frac{2s}{z} q^2(z) - \frac{\eta}{q(z)} + \frac{\eta Q'(z)}{Q(z)} \right\} > 0.
\]

If \( p \in \mathcal{P} \) satisfies

\[
\alpha + \beta p(z) + \xi(p(z))^2 + \frac{\eta}{p(z)} + \delta \frac{z p'(z)}{p(z)} \prec \alpha + \beta q(z) + \xi(q(z))^2 + \frac{\eta}{q(z)} + \delta \frac{z q'(z)}{q(z)}
\]

then

\[
p(z) \prec q(z)
\]

and \( q(z) \) is the best dominant.

**Proof.** The proof of this theorem is similar to that of Theorem 2.2.3 and hence omitted. \( \square \)

**Remark 2.2.3.** By taking \( \alpha = \beta = 0, \xi = \frac{1}{\mu}, \mu > 0, \lambda = \frac{\eta}{\mu}, \eta = 1 \) and \( q(z) = \frac{1+z}{1-z} \) in Theorem 2.2.3 we get the result of Nunokawa et al. (2002) which was proved by a different method.

**Remark 2.2.4.** For the choices of \( \alpha = \beta = 0 \) in Theorem 2.2.3, we get the result of (Thm. 1, p. 192, Ravichandran and Jayamala (2004)) and for \( \alpha = \xi = \eta = 0 \) in Theorem 2.2.4 we get (Thm. 2, p. 194, Ravichandran and Jayamala (2004)).

**Corollary 2.2.13.** Let \( -1 \leq B < A \leq 1, 0 < \gamma \leq 1 \) and \( \lambda > 0 \). If \( f \in \mathcal{A}_p \) satisfies \( f(z)f''(z) \neq 0 \) in \( 0 < |z| < 1 \), then

\[
(1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf'''(z)}{f'(z)} \right) \prec \left( 1 + Az \right)^\gamma + \frac{\lambda z(A-B)z}{(1+Az)(1+Bz)}
\]

(2.30)

implies

\[
f \in S_p^\gamma(\gamma, A, B).
\]

Also,

\[
1 + \frac{zf'''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{\gamma(A-B)z}{(1+Az)(1+Bz)},
\]

(2.31)

implies

\[
f \in S_p^\gamma(\gamma, A, B).
\]
Proof. By taking \( \alpha = \xi = \eta = 0, \beta = \xi, \delta = 1, \phi(z) = \frac{zf'(z)}{f(z)} \) and \( \phi(z) = (\frac{1+A\bar{z}}{1+B\bar{z}})^{\gamma} \) in Theorem 2.2.4 we get the first part.

Proof of the second part follows by setting \( \alpha = \beta = \xi = \eta = 0, \delta = 1, \phi(z) = \frac{zf'(z)}{f(z)} \) and \( \phi(z) = (\frac{1+A\bar{z}}{1+B\bar{z}})^{\gamma} \). \( \square \)

For \( \alpha = \xi = 0, \beta = 1, \phi(z) = \frac{zf'(z)}{f(z)} \) and \( \phi(z) = \frac{1+A\bar{z}}{1-B\bar{z}}, -1 < A \leq 1 \) in Theorem 2.2.3, we have the following result.

**Corollary 2.2.14.** If \( f \in \mathcal{A} \) satisfies \( f(z) \neq 0, z \in \Delta \) and

\[
\frac{zf'(z)}{f(z)} \left[ \left( 1 - \eta \frac{zf'(z)}{f(z)} \right) + \eta \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < \bar{h}(z),
\]

where

\[
\bar{h}(z) = \frac{1 + Az}{1-z} + \eta \frac{(1 + A)z}{(1-z)^2},
\]

then

\[
\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1-z}.
\]

The investigation of the image of the unit disc under the function \( h(z) \) is done as follows. Rewriting \( h(z) \), as

\[
h(z) = \frac{1 - Az^2 + (\eta + \eta A + A - 1)z}{(1-z)^2},
\]

we find that \( h(0) = 1 \) and \( h(-1) = \frac{2 - 2A + (1+A)}{4} \). In view of Theorem 2.2.3, it is clear that the function \( h \) is close to convex in \( \Delta \). \( h(z) \) takes real values for real \( z \) and \( h(\Delta) \) is symmetric with respect to the real axis and it intersects the real axis at one point only. The boundary of the curve is given \( h(e^{i\theta}) = u(\theta) + iv(\theta), \theta \in (-\pi, \pi) \), where

\[
u(\theta) = \frac{(1 + A)\sin \theta}{2(1 - \cos \theta)}
\]

and

\[
u(\theta) = \frac{(1 + A)\sin \theta}{2(1 - \cos \theta)}
\]

Eliminating \( \theta \), we get the equation of the boundary curve as

\[
u^2 = \frac{1 + A}{\eta} \left[ u - \frac{2 - 2A - \eta(1 + A)}{4} \right]
\]
which is a parabola lies in the left half plane, with its vertex at the point \( \left( \frac{2 - 2A - \eta(1 + A)}{4}, 0 \right) \) and the negative real axis as the axis of parabola. Hence \( h(\Delta) \) is the exterior of this parabola and includes the right half plane \((u, v)\) with

\[
u > \frac{2 - 2A - \eta(1 + A)}{4}.
\]

Setting \( A = 1 - 2\gamma, 0 \leq \gamma < 1 \) in Corollary 2.2.14, we obtain the following result.

**Corollary 2.2.15.** If \( f \in \mathcal{A} \) satisfies \( f(z) \neq 0, z \in \Delta \) and

\[
\frac{zf'(z)}{f(z)} \left[ \left( 1 - \frac{zf'(z)}{f(z)} \right) + \eta \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right] < h(z),
\]

where

\[
h(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} + \eta \frac{(1 + (1 - 2\gamma))z}{(1 - z)^2},
\]

then

\[
f \in S^h(\gamma).
\]

It is noted that if \( h(z) = u + iv \), then \( h(\Delta) \) is the exterior of the parabola given by

\[
v^2 = -\frac{(2 - 2\gamma)}{\eta} \left[ u - \frac{2 - 2(1 - 2\gamma) - \eta(2 - 2\gamma)}{4} \right]
\]

with its vertex as \( \left( \frac{2 - \eta(4 - 2\gamma)}{2}, 0 \right) \).

**Example 2.2.2.** Taking \( \eta = A = 1 \) in Corollary 2.2.14, we obtain

\[
h(z) = \frac{1 + 2z - z^2}{(1 - z)^2}.
\]

In this case, \( h(\Delta) \) is the exterior of the parabola given by

\[
v^2 = -2 \left[ u + \frac{1}{2} \right]
\]

with its vertex at the point \( \left( -\frac{1}{2}, 0 \right) \).

Thus, an application of Corollary 2.2.14, gives
Corollary 2.2.16. If $f \in \mathcal{A}$ satisfies $f(z) \neq 0$, $z \in \Delta$ and

$$\frac{zf'(z)}{f(z)} \left[ 2 - \frac{zf''(z)}{f'(z)} + \frac{zf'''(z)}{f''(z)} \right] < \frac{1 + 2z - z^2}{(1 - z)^2},$$

then

$$\frac{zf'(z)}{f(z)} < \frac{1 + z}{1 - z}.$$  

Region $h(\Delta)$ has been shown shaded in Figure 2.2.

Letting $\alpha = \beta = 0$, $\xi = \eta = 1$, $p(z) = \frac{zf'(z)}{f(z)}$ and $q(z) = \frac{1 + (1 - z_0)z}{1 - z}$ in Theorem 2.2.3, we get the following.

Corollary 2.2.17. If $f \in \mathcal{A}$ satisfies $f(z) \neq 0$, $0 < |z| < 1$ and

$$\frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < h(z)$$

where

$$h(z) = \frac{(1 - 2\gamma)^2z^2 + 2(2 - 3\gamma)z + 1}{(1 - z)^2} \quad (2.37)$$
for some \( \gamma \) (0 \( \leq \gamma < 1 \)), then

\[
\Re \left( \frac{zf''(z)}{f(z)} \right) > \gamma.
\]

For the univalent function \( h(z) \) given by (2.37), we now find the image
\( h(\Delta) \) of the unit disc \( \Delta \). Let \( h = u + iv \), where \( u \) and \( v \) are real. We have

\[
\begin{align*}
u &= \frac{(2 - 3\gamma) + (1 + 2\gamma - 2\gamma \cos \theta)}{(1 - \cos \theta)} \quad \text{and} \\
v &= \frac{2\gamma(1 - \gamma)}{1 - \cos \theta}.
\end{align*}
\]

Elimination of \( \theta \) yields

\[
v^2 = \frac{-8\gamma^2(1 - \gamma)}{3 - 2\gamma} \left[ u - \frac{2\gamma^2 + \gamma - 1}{2} \right].
\]

Therefore, we conclude that

\[
h(\Delta) = \left\{ w = u + iv : v^2 > \frac{-8\gamma^2(1 - \gamma)}{3 - 2\gamma} \left[ u - \frac{2\gamma^2 + \gamma - 1}{2} \right] \right\},
\]

which properly contains the half plane \( \Re w > \frac{2\gamma^2 + \gamma - 1}{2} \).

**Corollary 2.2.18.** Let \( -1 \leq B < A \leq 1 \) and \( \Re \beta \geq 0 \). If \( f \in A_p \) satisfies \( f'(z) \neq 0 \) in \( 0 < |z| < 1 \) and

\[
(1 - \beta) \frac{f(z)}{zf'(z)} + \frac{f(z)f''(z)}{(f'(z))^2} < h(z)
\]

where

\[
h(z) = \frac{B(\beta B - \beta A)z^2 + ((2p + 1 - \beta)B - (1 + \beta)A)z + p - \beta}{p(1 + Bz)^2},
\]

then

\[
f \in S_p^\beta(B, A).
\]

**Proof.** If we let \( p(z) = \frac{f(z)}{zf'(z)} \), then \( p \in P \) and (2.38) can be expressed as

\[
\beta p(z) + zp'(z) < \beta \left( \frac{1 + Az}{1 + Bz} \right) + \frac{(A - B)z}{(1 + Bz)^2}.
\]

Hence, by taking \( \alpha = \xi = 0 \), \( \eta = 1 \), \( q(z) = \frac{1 + Az}{1 + Bz} \) and \( \Re \beta \geq 0 \) in Theorem 2.2.3, we have \( p(z) < \frac{1 + Az}{1 + Bz} \). So, \( f \in S_p^\beta(B, A) \).

Setting \( p = 1 \) and \( B = -1 \) in Corollary 2.2.18, we get
Corollary 2.2.19. Let $-1 < A \leq 1$ and $\Re \beta \geq 0$. If $f \in A$ satisfies $f'(z) \neq 0$ in $0 < |z| < 1$ and

$$(1 - \beta) \frac{f(z)}{zf'(z)} + \frac{f(z)f''(z)}{(f'(z))^2} < h(z),$$

where

$$h(z) = \frac{(1 + \beta A)z^2 + ((\beta - 3) - (1 + \beta)A)z + 1 - \beta}{(1 - z)^2},$$

then

$$\frac{f(z)}{zf'(z)} < \frac{1 + Az}{1 - z}.$$\hspace{1cm} (2.40)

Remark 2.2.5. For the function $h(z)$ given by (2.40), we have

$$h(\Delta) = \{w = u + iv; v^2 > a_0[u - b_0]\},$$

which properly contains the half plane $\Re w > b_0$, where

$$a_0 = (1 + A)\beta^2,$$

$$b_0 = \frac{5 + A + 2\beta(a - 1)}{4}.$$\hspace{1cm} (2.41)

By putting $p = A = \beta = 1$ and $B = -1$ in Corollary 2.2.18, we get the following result of Tuneski (2000).

Corollary 2.2.20. If $f \in A$ and

$$\frac{f(z)f''(z)}{(f'(z))^2} < \frac{2z(z - 2)}{(1 - z)^2},$$

then

$$\left(\frac{f(z)}{zf'(z)}\right) < \frac{1 + z}{1 - z},$$

that is

$$\Re \left(\frac{f(z)}{zf'(z)}\right) > 0.$$\hspace{1cm} (2.42)

Remark 2.2.6. By putting $0 = B < A \leq 1$, $p = 1$ and $\beta = 0$ in Corollary 2.2.18, we get the result obtained by Singh (2001), which refines the result of Silverman (1999).

By taking $p(z) = \frac{1}{p} \left(\frac{f(z)}{zf'(z)}\right)^{\nu}$ in Theorem 2.2.3, we have the following corollary.
Corollary 2.2.21. Let $0 \neq \eta$ and $q(z)$ be convex univalent in $\Delta$ with $q(0) = 1$ and satisfy (2.25).

Let $f \in A_p$ and

$$\psi(z) := \alpha + \beta \left( \frac{f(z)}{z^p} \right)^{2\mu} + \frac{\zeta}{2\mu} \left( \frac{f(z)}{z^p} \right)^{2\mu} + \eta \left( \frac{f(z)}{z^p} \right)^{2\mu} \left[ \frac{zf'(z)}{pzf(z)} - 1 \right],$$

then

$$\frac{1}{p} \left( \frac{f(z)}{z^p} \right)^{2\mu} \prec q(z)$$

and $q(z)$ is the best dominant.

Corollary 2.2.22. Let $0 \neq \lambda \in \mathbb{C}$ and $q(z)$ be convex univalent in $\Delta$ with $q(0) = 1$ and satisfy

$$\Re \left( \frac{\mu}{\lambda} \right) > 0.$$

(i) If $f \in A$ satisfies

$$(1 - \lambda) \left( \frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} \prec q(z) + \lambda \frac{1}{\mu} zf'(z),$$

then

$$\left( \frac{f(z)}{z} \right)^{\mu} \prec q(z).$$

(ii) If $f \in A$ satisfies

$$f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} - \left( \frac{f(z)}{z} \right)^{\mu} < \frac{1}{\mu} zf'(z)$$

then

$$\left( \frac{f(z)}{z} \right)^{\mu} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Proof of the first part follows from Corollary 2.2.21, by taking $\beta = p = 1$, $\alpha = \xi = 0$ and $\eta = \frac{\lambda}{\mu}$. \hfill \box
The proof of the second part follows from Corollary 2.2.21, by taking \( \alpha = \beta = \xi = 0, \ p = 1 \) and \( \eta = \frac{1}{p} \).

By taking \( \lambda = \mu = n \) where \( n \) is a positive integer and
\[
q(z) = A + (1 - A)[1 - \frac{2}{z} \log(1 - z)], \ (A < 1), \ 
\]
in the first part of the Corollary 2.2.22, we get the following result of Ponmuasa (1988).

**Corollary 2.2.23.** Let \( f \in \mathcal{A} \), then for a positive integer \( n \), we have
\[
\Re \left\{ (1 - n) \left( \frac{f(z)}{z} \right)^n + n f'(z) \left( \frac{f(z)}{z} \right)^{n-1} \right\} > A
\]
implies
\[
\left( \frac{f(z)}{z} \right)^n < A + (1 - A) \left( 1 - \frac{2}{z} \log(1 - z) \right)
\]
and \( A + (1 - A)[1 - \frac{2}{z} \log(1 - z)] \) is the best dominant.

**Remark 2.2.7.** By taking \( \mu = 1 \) and \( q(z) = 1 + \left( \frac{A}{1 + A} \right) z \) in first part of Corollary 2.2.22 and \( \mu = \lambda = 1 \) and \( q(z) = \frac{A}{B} + \left( 1 - \frac{A}{B} \right) \frac{\log(1 + Bz)}{Bz} \) in second part of Corollary 2.2.22 we get the result of Ponmuasa and Juneja (1990).

By taking \( \beta = \xi = \eta = 0, \ \alpha = p = 1, \ \delta = \frac{1}{p}, \ p(z) = \frac{1}{p} \left( \frac{f(z)}{z^p} \right)^\mu \) and \( q(z) = e^{\mu A z} \) with \( |\mu A| < \pi \) in Theorem 2.2.4, we get the following result obtained by Owa and Obradovic (1990).

**Corollary 2.2.24.** Let \( f \in \mathcal{A} \) and
\[
\frac{zf'(z)}{f(z)} < 1 + A z,
\]
then
\[
\left( \frac{f(z)}{z} \right)^\mu < e^{\mu A z}
\]
and \( e^{\mu A z} \) is the best dominant.

We remark here that \( q(z) = e^{\mu A z} \) is univalent if and only if \( |\mu A| < \pi \).

**Remark 2.2.8.** For a special case when \( p(z) = \frac{1}{p} \left( \frac{f(z)}{z^p} \right)^\mu, \ q(z) = \frac{1}{(1 - z)^\beta} \) where \( b \in \mathbb{C} \setminus \{0\} \) and \( \beta = \xi = \eta = 0, \ \alpha = \mu = p = 1 \) and \( \delta = \frac{1}{p} \) in Theorem 2.2.4, we have the result obtained by Srivastava and Lashin (2005).
Corollary 2.2.25. If \( f \in \mathcal{A} \) satisfies
\[
(1 + \lambda) \left( \frac{z}{f(z)} \right)^\mu - \lambda f'(z) \left( \frac{z}{f(z)} \right)^{\mu+1} \prec q(z) + \frac{\lambda}{\mu} z q'(z),
\]
then
\[
\left( \frac{z}{f(z)} \right)^\mu \prec q(z)
\]
and \( q \) is the best dominant.

Proof. By taking \( \rho(z) = \frac{1}{p} \left( \frac{z}{f(z)} \right)^\mu \) and \( \alpha = \zeta = 0, \beta = p = 1 \) and \( \eta = \frac{1}{p} \) in Theorem 2.2.3, we get the above corollary. \( \square \)

In the next Chapter, certain subordination results for \( \phi \)-like functions in a sector are discussed. Also, using subordination and superordination principle, sandwich type results for \( \phi \)-like functions which involves best dominants and best subordinants are obtained.