CHAPTER 2

OVERVIEW OF CONTINUOUS -TIME SLIDING MODE CONTROL

2.1 INTRODUCTION

The evolution of sliding mode started along with phase-plane technique, a method meant for the analysis of second order nonlinear systems. Though this method is restricted to second-order systems initially, it is extended to multi dimensional MIMO systems, discrete systems etc. The definition of a VSC problem, the basic notions of VSC, design of sliding surface, sliding mode, control structure etc. are discussed in this chapter.

2.2 STATEMENT OF THE VSC PROBLEM

Consider a system represented by its state model

\[
\dot{x}(t) = Ax(t) + Bu(t)
\]  
(2.1)

The objective of variable structure control is to design a switching function \( s(x) \) and a variable structure control input \( u(x, t) \). The switching surface \( s(x) = 0 \) represents the desired dynamics during sliding mode and the purpose of the variable structure control \( u(x, t) \) is to drive the trajectory of the system to reach the switching surface in finite period of time (Young 1978, Wu 2007, Burton and Zinober 1986), when the trajectory lies outside the switching surface, to maintain the trajectory on the sliding surface with
desired dynamics once the surface is intercepted and to slide the trajectory to the steady-state or equilibrium state to make the overall VSC system asymptotically stable (Emelyanov et al 1970, Young 1977). Following are the basic notions of a VSC system (Hung et al 1993).

1. As the origin of the state space represents the equilibrium state or steady-state, the sliding mode represents the transient response of the system i.e. the coefficients of the switching surface $s(x) = 0$ decide the behavior of the system during sliding mode.

2. The order of the switching surface $s(x)$ is lower than that of the original model given in Equation (2.1).

3. The sliding mode trajectory is not inherent to any of the stable or unstable system structures obtained by subjecting the system to different levels of discontinuous input, but a newly defined asymptotically stable structure achieved by properly combining the required part of those structures.

To verify the above notions, two different systems, one limitedly stable (Utkin 1977) and another unstable, are considered (DeCarlo et al 1988). The first system is described as follows.

\[
\dot{x} = -\Psi x 
\]  

(2.2)

The above system is having two structures defined by $\Psi = \alpha^2_1$ and $\Psi = \alpha^2_2$ where $\alpha^2_1 > \alpha^2_2$. In fact, it is a basic bang-bang control problem. The phase portraits of the system given in Equation (2.2) when $\Psi = \alpha^2_1$ and $\Psi = \alpha^2_2$ are given in Figure 2.1 and Figure 2.2 respectively. Obviously those
responses are vortex, and represent limitedly stable systems. The resultant asymptotically stable system obtained by properly combining these two responses is given in Figure 2.3. The asymptotic stability is achieved in the above example by changing the system structure on the coordinate axis, i.e. the switching of structures from one to another, given by

\[
\Psi = \begin{cases} 
\alpha_1^2 & \text{when } xx > 0 \\
\alpha_2^2 & \text{when } xx < 0
\end{cases}
\] (2.3)

Figure 2.1 Response of system given in Equation (2.1) with \( \Psi = \alpha_1^2 \)

Figure 2.2 Response of system given in Equation (2.1) with \( \Psi = \alpha_2^2 \)

In the second example a system with two accessible states with one control input of the form \( u = k_1(x_1, x_2)x_1 + k_2(x_1, x_2)x_2 \), where the gains
$k_i(x_1, x_2)$ take on two possible values, say $\alpha_i$ or $\beta_i$, is considered. The state model of the plant is given by under the variable structure control law $u(t) = k(x_1)x_1(t)$ where $k(x_1)$ can be ‘-2’ or ‘3’. The block diagram of the system is shown in Figure 2.4. With $k = 2$ the system roots are complex conjugate and with $k = -3$ the roots are real. The phase portrait for the former structure is the saddle point and that of the latter is unstable focus as shown in Figure 2.5 and Figure 2.6 respectively. The switching surface selected is $s = s_1(x_1, x_2) = s_1x_1 + x_2$ with $s_1 > 0$. If the feedback is switched according to

$$k(x_1) = \begin{cases} 
-3 & \text{if } s_1(x_1, x_2)x_1 > 0 \\
2 & \text{if } s_1(x_1, x_2)x_1 < 0
\end{cases} \quad (2.4)$$

![Figure 2.3](image.png)

**Figure 2.3** Asymptotically stable VSS of system given in Equation (2.3) consisting of two limitedly stable structures

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (2.3)$$
Figure 2.4  Block diagram of a second order system with variable structure control

Figure 2.5  Unstable focus phase portrait of system given in Equation (2.3) with $k = -3$
Figure 2.6 Unstable saddle point phase portrait of system given in Equation (2.3) with $k = 2$

The resultant behavior of the system is as shown in Figure 2.7. It is clear that the system given in Equation (2.3) is an asymptotically stable VSS when the control is switched in accordance with Equation (2.4). The generalized form of the switching surface for the above problem may be represented as

$$x_2 + cx_1 = 0$$  \hspace{1cm} (2.5)

From Equation (2.5) it is evident that the system response in sliding mode depends only on the constant ‘$c$’ and hence, the system is insensitive to external disturbances and parameter variations (Drazenovic 1969, Emelyanov and Utkin 1963). This invariance property of VSS is of extreme importance when controlling time-varying plants or treating disturbance rejection problems (Emelyanov and Utkin 1967, Thorp and Barmish 1981).
Further, it may also be noted that different choices for the coefficient ‘c’ in Equation (2.5) results in different switching surfaces and hence, different system responses. This allows the designer to choose different control structures at different points of time to meet the specified requirements in sliding mode (Young et al 1977). However, the order of the equation describing the sliding surface is lesser than that of the system.

2.3 CONTROL STRUCTURE

Consider the system represented by Equation (2.1). Let the size of $A$ and $B$ respectively be $n \times n$ and $n \times m$. Associated with the system is $(n - m)$-dimensional discontinuous switching surface defined as follows (Decarlo et al 1999, Utkin and Young 1978)
\[ S = \{ (x, t) \in \mathbb{R}^{n-1} \mid s(x, t) = 0 \} \]  \hspace{1cm} (2.6)

where

\[ s(x, t) = [s_1(x, t), s_2(x, t), \ldots, s_m(x, t)]^T \]  \hspace{1cm} (2.7)

This \((n-m)\) dimensional manifold in the state space \(\mathbb{R}^n\) is determined as the intersection of \(s_i(x, t)\) surfaces of dimension \(m(n-1)\). These surfaces are so designed that the state trajectory, which is restricted to \(S\), has desired response in both disturbance rejection and set-point tracking problems (Emelyanov et al 1966). Though it is possible to have nonlinear, time-variant surfaces for the design of the manifold \(S\), it is convenient and simple to use linear structures (DeCarlo et al 1988, Utkin 1978, Mathews et al 1988, Hung et al 1993, Gao 1993). Two of the standard design procedures for the switching manifold \(S\) specified by Hung et al (1993) are discussed in section 2.6.

In general for an ‘\(n\)’ dimensional system, once the switching surface is properly designed as discussed above, a switched controller, \(u(x, t) = [u_1(x, t), \ldots, u_m(x, t)]\) is constructed of the form

\[ u_i(x, t) = \begin{cases} u_i^+(x, t) & \text{when } s_i(x, t) > 0 \\ u_i^-(x, t) & \text{when } s_i(x, t) < 0 \end{cases} \]  \hspace{1cm} (2.8)

The Equation (2.8) indicates that the control changes value depending on the sign of the switching surface at \(x\) and \(t\). On the switching surface \(s(x)\) the control is undefined. Off the switching surface, the control values \(u_i^\pm\) are chosen so that the tangent vectors of the state trajectory point towards the surface such that the state is driven to and maintained on \(s(x)=0\). Such controllers result in discontinuous closed-loop system.
2.4 SLIDING MODES

The control given in Equation (2.8) is so designed that the state trajectory is attracted to the switching line and once having intercepted it, remains on the switching surface for all subsequent time (Decarlo et al 1996). Only when the tangent or velocity vectors of the state trajectory point toward the sliding surface $s(x)$, a sliding mode can exist (Drakov and Utkin 1990). If the state trajectory intersects the switching surface, the state trajectory remains within a $\varepsilon$ neighborhood of $s(x)$. It is worthy to note that the interception of the switching surface $s(x,t) = 0$ does not guarantee sliding on the surface for all subsequent time as illustrated in Figure 2.8. As discussed in the previous chapter, due to the imperfections in the switched controller such as hysteresis, delay etc., the infinitely fast switching of the switched controller can not be achieved which results in the oscillations of the state trajectory within the neighborhood of the switching surface and this phenomenon is known as chattering.

2.5 CONDITIONS FOR THE EXISTENCE OF SLIDING MODE

The existence of the sliding mode (DeCarlo et al 1988, DeCarlo et al 1999, Utkin 1978, Utkin 1992, Emelyanov et al 1965) is assured only when the system trajectory reaches the switching surface $S = s(x, t) = 0$, in finite time, say $t_1$, and remain in the neighborhood, $\{x | |s(x,t)| < \varepsilon\}$, of $S$ for suitable $\varepsilon > 0$. A domain $D$ of dimension $(n - m)$ in the switching manifold $S$ is a sliding-mode domain if, for each $\varepsilon > 0$ there exists a $\delta > 0$, so that the motion starting with in a $n$ dimensional $\delta$ vicinity of $D$, may leave the $n$ dimensional $\varepsilon$ vicinity of $D$, only through the $\varepsilon$ vicinity of the boundary of $D$ as shown in Figure 2.9. The region of attraction defined by the size of $\varepsilon$ and $\delta$ is the largest subset of the state space from which sliding is available. If the domain of attraction is spread over entire state space then the sliding mode is globally reachable.
Figure 2.8 Existence of sliding mode on the intersection of two surfaces $s_1$ and $s_2$

Figure 2.9 Two dimensional representation of sliding mode domain
2.6 DESIGN OF SLIDING SURFACE

The design procedure entails the construction of switching surfaces so that the system restricted to the switching phase produces a desired behavior. As already discussed, the system behavior in sliding mode depends on the coefficients of the mathematical expression that describes the sliding surface. Several design procedures such as equivalent control (DeCarlo et al 1988), reaching law approach (Hung et al 1993, Gao and Hung 1993, Dorling 1985), have already been proposed. Out of these methods the two widely used methods discussed by Hung et al (1993) and reaching law approach suggested by Gao and Hung (1993) are discussed in this section.

2.6.1 Canonical Form Method

This method requires the system model be specified in controllable canonical form. For a SISO system the canonical form is

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= \sum_{i=1}^{n} a_i x_i + bu
\end{align*}
\]  

(2.9)

Then in the \( n \) dimensional space, a surface of dimension \((n-1)\) is defined by a function

\[
s(\mathbf{x}) = c_1 x_1 + c_2 x_2 + \ldots + x_n
\]  

(2.10)
Substituting the above in Equation (2.9) leads to the following

\[ x_{n-1} = x_n = -c_1 x_1 - c_2 x_2 - \ldots - c_{n-1} x_{n-1} \]  

(2.11)

From the Equation (2.11) it is clear that the coefficients in the switching function given by Equation (2.10) define the characteristic equation of the sliding mode, if the system model is described in controllable canonical form.

### 2.6.2 Linear Transformation Method

Consider a linear time-invariant plant and a switching function described by

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
s(x) &= Cx = 0 
\end{align*}
\]  

(2.12)

where the dimension of \( x \) is \( n \) and dimensions of \( s \) and \( u \) are \( m \). The dynamics of the sliding mode can be easily described if the state vector is composed with \( s \) as \( m \) of the state variables. Therefore the objective is to linearly transform the state model given in Equation (2.12) so that \( s \) and \( m \) are the new state variables in the transformed model (Hung et al. 1993). In the first phase of transformation the plant is transformed by \( y = T_1 x \) to obtain the following.

\[
\begin{align*}
\dot{y}_1 &= \overline{A}_{11} y_1 + \overline{A}_{12} y_2 \\
\dot{y}_2 &= \overline{A}_{21} y_1 + \overline{A}_{22} y_2 + \overline{B} u 
\end{align*}
\]  

(2.13)
where dimension of $y_1$ is $(n-m)$, dimension of $y_2$ is $m$ and $B \neq 0$. This is followed by the transformation

$$
\begin{bmatrix}
y_1 \\
s
\end{bmatrix} = T_2 \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
$$

(2.14)

which brings the system given by Equation (2.13) to the form

$$
\begin{align*}
\dot{y}_1 &= A_{11}y_1 + A_{12}s \\
\dot{s} &= A_{21}y_1 + A_{22}s + Bu
\end{align*}
$$

(2.15)

When the system is in sliding mode the dynamics should satisfy $s = 0$ and so the differential Equation (2.15) for the sliding mode can easily be solved.

Gao and Hung (1993) addressed this linear transformation in a different way as given below. The system given in Equation (2.12) is first partitioned as follows.

$$
x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B \end{bmatrix}
$$

(2.16)

where the dimension of $x_1 = (n-m)$, dimension of $x_2$ is $m$, and other submatrices have appropriate dimensions. Using the following transformation
\[
\begin{bmatrix}
  x_1 \\
  s
\end{bmatrix} = \begin{bmatrix}
  I & 0 \\
  C_1 & C_2
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
\] (2.17)

the system given by Equation (2.14) is transformed into

\[
\begin{aligned}
\dot{x}_1 &= (A_{i1} - A_{i2}C_2^{-1}C_1)x_1 + A_{i2}C_2^{-1}s \\
\dot{s} &= \left[(C_1A_{i1} + C_2A_{i2}) - (C_1A_{i2} + C_2A_{22})C_2^{-1}C_1\right]x_1 \\
&\quad + (C_1A_{i2} + C_2A_{22})C_2^{-1}s + C_2Bu
\end{aligned}
\] (2.18)

On the sliding manifold \( s = 0 \) and hence from (2.18) the sliding mode of the system given by

\[
\begin{aligned}
\dot{x}_1 &= (A_{i1} - A_{i2}C_2^{-1}C_1)x_1 \\
s(x) &= Cx = C_2\begin{bmatrix}
  -K & I
\end{bmatrix}x
\end{aligned}
\] (2.19)

It can be easily proved that if with the pair \((A_{i1}, B)\) is controllable then the pair \((A_{i1}, A_{i2})\) is also controllable and hence, the Equation (2.19) indicates that the problem is reduced to regulator form. Chang and Chen (2000) and Lukyanov and Utkin (1981) have also discussed this problem and suggested the methods to reduce the problem to regulator form. Then by proper choice of the feedback gain matrix, given by \(K = -C_2^{-1}C_1\), which is of size \(m \times (n - m)\), poles of the sliding mode can be placed at desired locations.

2.6.3 Gao’s Reaching Law

According to Gao and Hung (1993), the reaching law is any asymptotically stable differential equation which specifies the dynamics of the switching function \(s(x)\). The dynamics of the differential equation can be
controlled by the proper choice of the parameters of the differential equation. The general form of the reaching law specified by them is

\[
\dot{s} = -q \text{sgn}(s) - K h(s) \tag{2.20}
\]

where

\[
q = \text{diag}[q_1, \ldots, q_m], \quad q_i > 0 \\
\text{sgn}(s) = [\text{sgn}(s_1), \ldots, \text{sgn}(s_m)]^T \\
K = \text{diag}[k_1, \ldots, k_m], \quad k_i > 0 \\
h(s) = [h_1(s_1), \ldots, h_m(s_m)]^T \\
s_i h_i > 0, \quad h_i(0) = 0.
\]

The three different practical cases of the Equation (2.20) are discussed in the following section.

*Constant rate reaching:* This is the simplest of the three and given by

\[
\dot{s} = -q \text{sgn}(s) \tag{2.21}
\]

This law forces the system trajectory from any arbitrary initial condition, towards the switching manifold $S$ at a constant rate given by $|s_i| = -q_i$. The reaching time depends on the magnitude of $q_i$ and so, a small value of it will result in a longer reaching time and a large value cause severe chattering.

*Constant plus proportional rate reaching:* The reaching law in this case is given by
\[ \dot{s} = -q \text{sgn}(s) - Ks \]  \hspace{1cm} (2.22)

The addition of the proportional rate term forces the state trajectory at a rate proportional to \( s \), and hence larger the value of \( s \) is faster the reaching rate.

**Power rate reaching:** In this case the reaching law is so designed that the reaching speed increases when the trajectory is far away from the switching manifold \( S \) and reduces when the trajectory is closer to the manifold. The reaching law is given by

\[ \dot{s}_i = -k_i |s_i|^\alpha \text{sgn}(s_i) \quad 0 < \alpha < 1, \ i = 1 \text{ to } m. \]  \hspace{1cm} (2.23)

This reaching law gives a finite reaching time and due to the absence of the term \(-q\text{sgn}(s)\) on the right-hand side of the Equation (2.23) the chattering is also eliminated.

### 2.7 CONCLUSION

The complete overview of the continuous-time sliding mode control is given in this chapter. Two different examples are considered to describe the basics of sliding mode control. The evolution of sliding mode is also discussed along the necessary conditions for the existence of the sliding mode. The selected design procedures for sliding mode and the control structure are also dealt with.