CHAPTER 3

MODEL REDUCTION ALGORITHMS FOR REDUNDANCY REMOVAL

3.1 INTRODUCTION

There had been many methods in the past for obtaining optimal solution for a given LPP/SP problems. These methods are generally in terms of Model Reduction Algorithms (MRAs) or hybrid heuristic algorithms. In this chapter, the focus is on new MRAs. The general concept of the MRA is to detect redundant constraint(s)/variable(s) and/or objective function(s) of a given problem and thereby reduce the dimension to save computational effort and memory of a machine. Automatic detection and exploitation of structural redundancies in LPPs/SPs have been the subject of continuing research over the past few decades. Though some enhanced MRAs in the form of heuristic algorithms have been proposed by earlier researchers, not all redundancies could be cracked down *apriori* leaving scope for further improvements. Some of the contributions made by earlier researchers in the *apriori* identification of redundancies and improvements over them have been referred here.

The organisation of the chapter is as follows. The first part introduces a simple “Direct method” of solving LPPs. The proposed Direct method gives an Advanced starting basis (i.e., a better initial basic feasible solution) near to the optimal point for the Simplex method. In the process it finds redundant constraints and redundant variables, which will have no role to play in deciding the optimal solution. In the second part, a redundancy algorithm is proposed for finding redundant objective function(s), if any, in

MONLSFP problems. Finally, an integrated redundancy algorithm is developed for finding redundant objective function(s) and redundant constraint(s) in MONLSFP problems.

3.2 A DIRECT METHOD OF SOLVING LPPs

The Direct method takes advantage of the Maximum change criterion in the objective function instead of the Rate of change of criterion which the Simplex algorithm adopts. The starting point in this method is the construction of a Matrix of intercepts of the decision variables. This matrix is used to identify an advanced basis which accelerates the rate of convergence to the optimal solution. The matrix of intercepts keeps some constraints and variables off the bay and thereby reduces the original problem to a smaller dimension. "Dimensionality is a curse" in large scale linear and nonl-inear mathematical programming problems. This curse is curbed to some extent. The proposed method is compared with the Simplex method through illustrative examples.

3.2.1 Motivation

The problem of detection of redundant constraints and redundant variables is very difficult. Unfortunately, all the methods proposed so far for finding redundant constraints and/or variables do not guarantee complete detection. Therefore, there is a scope for further improvement of some of the existing algorithms or a development of new MRA. Stojkovic and Stanimirovic (2001) used the method of minimum angles in the preparation of the starting point for the simplex method. Luh and Tsaih (2002) presented an auxiliary algorithm, which helps the Simplex method for commencing a better initial basic feasible solution. The idea used in their method is the construction of an interior search direction towards an optimal point. Numerical results of this algorithm show only a saving of 40% in terms of the number of iterations. Even though these two methods provide a better starting point for the Simplex method, there is no reduction in the problem size. This became a motivating reason for a new method, which not only constructs a better starting point for the Simplex method but also finds redundant constraints and redundant variables, resulting in a reduced problem size.

3.2.2 Problem Formulation

The problem here is to develop a new method in helping the Simplex method for starting at a better initial basic feasible solution. This is achieved through the construction of the 'Matrix of intercept', which helps to identify and remove redundant constraints and variables from the following LPP.

$$Max\ Z(X) = \sum_{j=1}^{n} c_j x_j = \langle C / X \rangle \tag{3.1}$$

subject to linear constraints

$$\sum_{j=1}^{n} a_{ij} x_j = \langle r_i / X \rangle \le b_i;\ i = 1,2,\ldots,m$$
$$x_j \ge 0;\ j = 1,2,\ldots,n \tag{3.2}$$

where $\langle r_i / X \rangle$; $i = 1,2,\ldots,m$ are the scalar products of vectors r_i and X and $C = (c_1, c_2, \ldots, c_j, \ldots, c_n)$, $X = (x_1, x_2, \ldots, x_j, \ldots, x_n)$

$r_i = (a_{1j}, \ldots, a_{ij}, \ldots, a_{in})$ for $i = 1,2,\ldots,m$.

3.2.3 Terminologies used in the Proposed Method

(i) Matrix of intercept

It is an $n \times m$ matrix with rows specifying decision variables and columns specifying slack/surplus variables.

$$\begin{array}{c} \begin{matrix} S_1 & S_2 & & S_j & & S_m \end{matrix} \\ \begin{bmatrix} \frac{b_1}{a_{11}} & \frac{b_2}{a_{21}} & \cdots & \frac{b_i}{a_{i1}} & \cdots & \frac{b_m}{a_{m1}} \\ . & . & & . & & . \\ . & . & & . & & . \\ \frac{b_1}{a_{1j}} & \frac{b_2}{a_{2j}} & \cdots & \frac{b_i}{a_{ij}} & \cdots & \frac{b_m}{a_{mj}} \\ . & . & & . & & . \\ . & . & & . & & . \\ \frac{b_1}{a_{1n}} & \frac{b_2}{a_{2n}} & \cdots & \frac{b_i}{a_{in}} & \cdots & \frac{b_m}{a_{mn}} \end{bmatrix} \end{array}$$

(ii) Maximum change criterion

Simplex procedure identifies the most promising variable to enter the *basis* on the basis of the values of the contribution/cost coefficients of the objective function and tie if any is resolved arbitrarily using the Lexicographic rule. A scrutiny of the objective function will reveal that it is the product of a term like $c_j x_j$ which contributes to the objective function and not the coefficient alone. An alternate criterion by name "maximum change criterion" is used to identify the entering variable. It considers the product of the variable and its coefficient. This is achieved by constructing a matrix of intercepts from which the variable which introduces the maximum change in the objective function is drawn into the basis.

3.2.4 The Proposed Direct Method

Step 1 : Construct a matrix of intercepts.

Step 2 : Take the transpose of the matrix of intercepts.

Step 3 : Identify the smallest of the intercepts in each column and box it.

Step 4 : Multiply the boxed elements (pivot elements) with the corresponding cost coefficients.

Step 5 : Using maximum change criterion score out the columns and rows containing the elements identified in step 3.

The un-scored columns and rows in the matrix of intercepts are identified tentatively as redundant variables and constraints of the problem.

It may be observed that the above method is an improvement over the application of the Primal-Dual properties for redundancy identification. The primal dual properties relate the constraints of one model to the variable in the other and vice versa. Hence it is not a very efficient method to identify redundancies.

3.2.5 Validation of Constraint Redundancy

The following steps are used to validate the identified redundant constraints.

Step 1 : Substitute the optimal solution of the apparent model in the identified redundancies. Flag the violating constraints.

Step 2 : Update the violating constraints using the sensitivity analysis.

Step 3 : Repeat the steps until there exists no violating constraints.

The pictorial representation of the validation procedure is given in Figure 3.1.

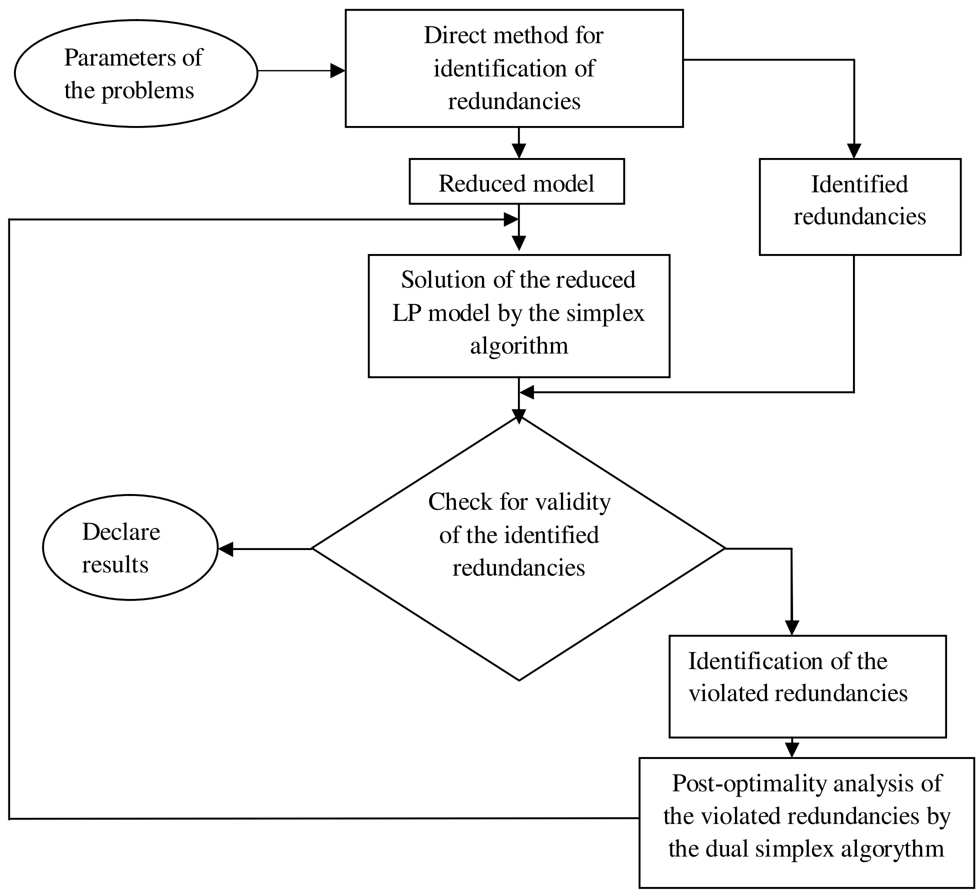

Figure 3.1 Schematic Representation of Validation Procedure

The apparent model is the one identified by the model reduction algorithm. The reduced model is finalized only after the apparent model is validated. Computational experiences have revealed that the Direct method does not use the Simplex method to find the optimal solution for some of the problems. In case the Direct method fails to yield the optimal solution, Simplex procedure is invoked with the advanced basis generated by the Direct method as the starting point. Worked examples are given below to illustrate the method of direct solution.

3.2.6 Illustrative Examples

Example 3.1

$$\text{Max } Z = 0.4x_1 + 0.28x_2 + 0.32x_3 + 0.72x_4 + 0.64x_5 + 0.6x_6$$

subject to

$$0.01x_1 + 0.01x_2 + 0.01x_3 + 0.03x_4 + 0.03x_5 + 0.03x_6 \leq 850$$
$$0.02x_1 + 0.05x_4 \leq 750$$
$$0.02x_2 + 0.05x_5 \leq 100$$
$$0.03x_3 + 0.08x_6 \leq 900$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

	0.4	0.28	0.32	0.72	0.64	0.6
	x_1	x_2	x_3	x_4	x_5	x_6
S_1	85000	85000	85000	85000/3	85000/3	85000/3
S_2	37500	----	----	15000	----	----
S_3	----	5000	----	----	2000	----
S_4	----	----	30000	----	----	11250
c_jx_j	15000	1400	9600	10800	1280	6750

Applying the algorithm, variables x_4, x_5 and x_6 are identified as redundant besides the first constraint. The reduced model is

$$\text{Max } Z = 0.4x_1 + 0.28x_2 + 0.32x_3$$

$$\begin{aligned} \text{subject to}\quad 0.02x_1 &\leq 750 \\ 0.02x_2 &\leq 100 \\ 0.03x_3 &\leq 900 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

The problem is self solving and the optimal solution is Z=26,000; x_1=37,500; x_2=5000; x_3=30,000.

The Simplex procedure took six iterations allowing x_4, x_5 and x_6 to pop out and then introduce x_1, x_2 and x_3 into the basis. The maximum change criterion took three iterations eradicating popping variables, whereas the redundancy identification method took zero simplex iteration.

Example 3.2

Consider the maximization of the objective function $11x_1 + 18x_2 + 29x_3 - 5x_4$ subject to the following constraints:

$$\begin{aligned} x_1 + 4x_2 + 6x_3 + 2x_4 &\leq 60 \\ 4x_1 + x_2 + 5x_3 + 9x_4 &\leq 91 \\ 6x_1 + 5x_2 + 8x_3 + 5x_4 &\leq 102 \\ 2x_1 + 5x_2 + 7x_3 + 7x_4 &\leq 110 \\ -x_1/2 + x_2/5 - x_3/3 - x_4/2 &\leq 1 \end{aligned}$$

$$x_1,\ x_2,\ x_3,\ x_4 \geq 0$$

	11	18	29	-0.8
	x_1	x_2	x_3	x_4
S_1	60	15	10	30
S_2	91/4	91	91/5	91/9
S_3	17	102/5	102/8	102/5
S_4	55	21	110/7	110/7
S_5	----	5	----	----
c_jx_j	187	90	290	-8.09

Active constraints in two variables are

$$x_1 + 6x_3 = 60$$
$$6x_1 + 8x_3 = 102$$

After dropping the variables x_2, x_4 and the constraints 2, 4, 5, the reduced model is then solved giving x_1=33/7, x_3=129/147 and Max Z=319.07. Using the primal dual relationship the constraints 2, 4, 5 and variables x_2, x_4 are validated as redundant.

The Simplex method gives the solution in 3 iterations, whereas the Direct method finds the solution immediately.

Example 3.3

Max $Z = 3x_1 + 4x_2 + 5x_3$ under the constraints

$$2x_1 + x_2 + x_3 \leq 5$$
$$3x_2 + 5x_3 \leq 11$$
$$4x_1 + x_2 + x_3 \leq 12$$
$$x_1 + 2x_2 + 4x_3 \leq 15$$
$$x_1,\ x_2,\ x_3 \geq 0$$

	3	4	5
	x_1	x_2	x_3
S_1	2.5	5	5
S_2	---	3.67	2.2
S_3	3	12	12
S_4	15	7.5	3.75
c_jx_j	7.5	14.68	11

By solving constraint equations 1 and 2 the Direct method gives the solution Max Z = 16.67, x_1=0.67, x_2=3.67, whereas the Simplex methods takes four iterations.

Example 3.4

Max $Z = x_1 + x_2$ subject to the constraints

$$x_1/1000 + x_2/9 \leq 1$$
$$x_1/5 + x_2/70 \leq 1$$
$$x_1/4.9 + x_2/111 \leq 1$$
$$x_1,\ x_2 \geq 0$$

	1	1
	x_1	x_2
S_1	1000	9
S_2	5	70
S_3	4.9	111
c_jx_j	4.9	9

By solving constraint equations 1 and 3 the Direct method gives the solution Max Z = 13.32; x_1 = 4.36; x_2=8.96, whereas the Simplex method takes three iterations. Constraint 2 is critically redundant.

Example 3.5

Maximize $4x_1 + 3x_2 + 5x_3 + 2x_4 + 5x_5$

subject to $3x_1 + 2x_2 - x_3 - 2x_4 + 4x_5 \leq 1$

$2x_1 + x_2 + 3x_3 + x_4 + 2x_5 \leq 1$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

	4	3	5	2	5
	x_1	x_2	x_3	x_4	x_5
S_1	1/3	1/2	----	----	1/4
S_2	1/2	1	1/3	1	1/2
c_jx_j	4/3	3/2	5/3	2	5/4

This algorithm identifies x_1, x_3, x_5 as redundant and the model is reduced to

$2x_2 - 2x_4 = 1$

$x_2 + x_4 = 1$

The optimal solution is maximum value 2.75 and x_2= 0.75, x_4= 0.25. The Simplex method solves this problem in four iterations.

Example 3.6

Maximize $240x_1 + 104x_2 + 60x_3 + 19x_4$

subject to $20x_1 + 9x_2 + 6x_3 + x_4 \leq 20$

$10x_1 + 4x_2 + 2x_3 + x_4 \leq 10$

$x_1, x_2, x_3, x_4 \geq 0$

	240	104	60	19
	x_1	x_2	x_3	x_4
S_1	1	20/9	10/3	20
S_2	1	5/2	5	10
c_jx_j	240	234	200	190

Avoiding a degenerated solution which increases the computation, x_2 and x_4 enter the basis.

$9x_2 + x_4 = 20$

$4x_2 + x_4 = 10$

$x_2 = x_4 = 2$ and maximum value is 246. The Simplex method takes three iterations.

3.2.7 Observations

It may be observed from the illustrative examples that the proposed Direct method straight away identifies redundant variables and redundant constraints in one shot without going in for further computation. In all the above six examples solved, popping variables are arrested which improves the computational efficiency. Table 3.1 gives in a nutshell the results of the computation.

Table 3.1 Comparison with Simplex Method

Example No.	1	2	3	4	5	6
Number of Simplex iterations	6	3	4	3	4	3
Size of the problem	4x6	5x4	4x3	3x2	2x5	2x4

The attraction of the proposed algorithm lies in its simplicity of operation. It requires very little computational effort for identification of unattractive decision variables and irrelevant constraints.

3.3 AN APPROACH TO FIND REDUNDANT OBJECTIVE FUNCTIONS(S) IN MONLSFP PROBLEMS

Redundancy in constraints and variables are usually studied in linear, integer and non-linear programming problems. The main emphasis has so far been given only to LPPs. Meanwhile the literature on FP has kept growing. To take care of uncertainties in FP, SFP assumes greater importance in a wide variety of applications. But there are only a few articles reported in the literature which explain redundancy in MOSFP problems and none in the area of MONLSFP problems. Therefore in the present work, an algorithm that identifies redundant objective function(s) in MONLSFP problems is provided. The basic idea used in this approach is that in the first stage, stochastic fractional objective functions are converted into constraints forms, which are then linearised using the SLP method. In the second stage, redundancy algorithm is applied to these linearised constraints for finding redundant objective function(s). The entire solution procedure is illustrated with numerical examples. The proposed algorithm reduces the number of non-linear fractional objective function(s) in cases where redundant objective function(s) exist.

3.3.1 Motivation

In the modelling of the real world problems like financial and corporate planning, market and media selection, university planning and student admission, health care and hospital planning, air force maintenance units, bank branches, etc, frequently one may be faced with a decision to optimize the department/equity ratio, profit/cost, inventory/sales, actual cost/staff cost, output/employee, nurse/patient ration etc., with respect to some constraints (Lai and Hwang 1996).

In the literature, different approaches appear to solve linear FP problems easily. But in the large scale decision problems, there is more than one objective, which must be satisfied at the same time as possible. Moreover, most of these are non-linear fractional objectives. Some of these can be redundant also. If some of the objective functions are redundant, then they need more computational effort. This is because each one of the redundant objective function adds an extra constraint in the constraint section and these constraints have no definite role to play in deciding the optimal solution. It is difficult to talk about the optimal solution of these problems. This has given motivation to develop a new approach which not only removes redundant objective function(s) but also solves the problems efficiently.

3.3.2 Problem Formulation

A MONLSFP problem in a criterion space is defined as follows:

Max $R(X) = [\, R_1(X), R_2(X), \ldots, R_k(X) \,]$,

where $$R_y(X) = \frac{N_y(X) + \alpha_y}{D_y(X) + \beta_y}, \quad y = 1, 2, \ldots, n \tag{3.3}$$

subject to $\Pr[\sum_{j=1}^{n} t_{ij}x_j \le b_i] \ge 1-p_i$ $\quad i = 1, 2, ..., m \quad (3.4)$

$\sum_{j=1}^{n} t_{ij}^{(1)}x_j \le b_i^{(1)}$ $\quad i = m+1, ..., h \quad (3.5)$

where, $0 \le X_{nx1} = \| x_j \| \subset \mathbb{R}^n$ is a feasible set, and R: $\mathbb{R}^n \to \mathbb{R}^k$, $T_{mxn} = \| t_{ij} \|$, $b_{mx1} = \| b_i \|$, $i = 1, 2, ..., m,$ $j = 1, 2, ..., n,$ α_y, β_y are scalars. $N_y(X) = \sum_{j=1}^{n} c_{yj}x_j^2$ and $D_y(X) = \sum_{j=1}^{n} d_{yj}x_j^2$.

In this model, out of $N_y(X)$, $D_y(X)$, T and b, at least one may be a random variable and S = {X | equations (3.4) - (3.5), $X \ge 0$, $X \subset \mathbb{R}^n$ } is non-empty, convex and compact set in $\mathbb{R}^n$.

The problem here is to develop a redundancy algorithm to identify redundant objective function(s), if any, in the MONLSFP problem and to provide a general algorithm for solving it.

3.3.3 Deterministic Equivalents of Probabilistic Constraints

Let T be a random variable in equation (3.4) and it follows $N(u_{ij}, s_{ij}^2)$, $i = 1, 2, ..., m,$ $j = 1, 2, ..., n,$ where u_{ij} is the mean and s_{ij}^2 is the variance. Let $l_i = \sum_{j=1}^{n} t_{ij}x_j$, $i = 1, 2, ..., m$.

$$E(l_i) = \sum_{j=1}^{n} u_{ij}x_j \; ; \; V(l_i) = X' V_i X = \sum_{j=1}^{n} s_{ij}^2 x_j^2,$$

where V_i - i^{th} covariance matrix. When T is independent, the covariance terms become zero. The i^{th} deterministic constraint for equation (3.4) is obtained from Charles and Dutta (2001, 2003) as follows:

$$\Pr(l_i \le b_i) \ge 1-p_i \text{ (or) } \Pr(Z_i \le z_i) \ge 1-p_i,$$

where $Z_i = (l_i - E(l_i))/\sqrt{V(l_i)}$ follows standard normal distribution and $z_i = (b_i - E(l_i))/\sqrt{V(l_i)}$. Thus, $\phi(z_i) \geq \phi(Kq_i)$, where $1-p_i = q_i = \phi(Kq_i)$, is the cumulative distribution function of a standard normal distribution. Clearly, $\phi(.)$ is a non-decreasing continuous function, hence $z_i \geq Kq_i$. Substituting in this equation the values of $E(l_i)$ and $V(l_i)$,

$$\sum_{j=1}^{n} u_{ij}x_j + Kq_i \sqrt{\sum_{j=1}^{n} s_{ij}^2 x_j^2} \leq b_i \tag{3.6}$$

If b_i is a random variable in equation. (3.4), i.e., $b_i \sim N(u_{bi}, s_{bi}^2)$, $i = 1, 2, ..., m,$ where u_{bi}, s_{bi}^2 are the mean and variance respectively. With the similar argument that led to the inequality in (3.6), one can obtain inequality (3.7), the i^{th} deterministic constraint for equation (3.4) as follows:

$$\sum_{j=1}^{n} t_{ij}x_j \leq u_{bi} + Kp_i s_{bi} \tag{3.7}$$

Suppose T and b_i are random variables in equation. (3.4) i.e. $T \sim N(u_{ij}, s_{ij}^2)$ and $b_i \sim N(u_{bi}, s_{bi}^2)$, $i = 1, 2, ..., m,$ $j = 1, 2, ..., n,$ where u_{ij} and u_{bi} are means, and s_{ij}^2 and s_{bi}^2 are variances respectively. With the similar argument that led to the inequality in (3.6), one can obtain inequality (3.8), the i^{th} deterministic constraint for equation (3.4) as follows:

$$\sum_{j=1}^{n} u_{ij}x_j - Kp_i \sqrt{\sum_{j=1}^{n+1} s_{ij}^2 x_j^2} \leq u_{bi} \tag{3.8}$$

where $x_{n+1} = -1$.

3.3.4 Conversion of Objective Functions into Constraints

This section considers all the objective functions in the form of constraints (Charles and Dutta 2003, 2006b). The main feature of the model is that it takes into account the probability distribution of the objective functions by maximizing the lower allowable limit of the objective function under chance constraints where numerator and/or denominator coefficients are random.

The unknown parameter λ_y which is less than or equal to R_y(X) is defined by,

$$R_y(X) \geq \lambda_y \quad \text{i.e.,} \quad \frac{N_y(X)+\alpha_y}{D_y(X)+\beta_y} \geq \lambda_y \quad \Rightarrow \quad 0 \leq N_y(X) + \alpha_y - \lambda_y\,[D_y(X)+\beta_y]$$

There are two cases in this problem.

Case 1. $\alpha_y > 0$

Sub case 1a :

Assumption: $N_y(X) \sim N(\sum_{j=1}^{n} u_{cyj}x_j^2, \sum_{j=1}^{n} s_{cyj}^2 x_j^4)$ and $D_y(X) \sim N(\sum_{j=1}^{n} u_{dyj}x_j^2, \sum_{j=1}^{n} s_{dyj}^2 x_j^4)$,

where u_{cyj} and u_{dyj} are means and s_{cyj}^2 and s_{dyj}^2 are variances.

Let $f(X, \lambda_y; \alpha_y > 0) = \lambda_y\,[D_y(X)+\beta_y] - N_y(X) \leq \alpha_y$

$$E[f(X, \lambda_y; \alpha_y > 0)] = F^E(X, \lambda_y; \alpha_y > 0) = \lambda_y\,[D_y^E(X)+\beta_y] - N_y^E(X)$$

$$= \lambda_y\,[\sum_{j=1}^{n} u_{dyj}x_j^2 + \beta_y] - \sum_{j=1}^{n} u_{cyj}x_j^2 \qquad (3.9)$$

$$V[f(X, \lambda_y; \alpha_y > 0)] = F^V(X, \lambda_y; \alpha_y > 0) = \lambda_y^2 D_y^V(X) + N_y^V(X)$$

$$= \lambda_y^2 \sum_{j=1}^{n} s_{dyj}^2 x_j^4 + \sum_{j=1}^{n} s_{cyj}^2 x_j^4 = \sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^4 \quad (3.10)$$

$$\Pr[f(X, \lambda_y; \alpha_y > 0) \le \alpha_y] \ge 1 - p_y^{(2)} \quad (3.11)$$

$$\Rightarrow \lambda_y (D_y^E(X) + \beta_y) - N_y^E(X) + \phi^{-1}(q_y^{(2)}) \sqrt{\lambda_y^2 D_y^V(X) + N_y^V(X)} \le \alpha_y$$

$$\lambda_y [\sum_{j=1}^{n} u_{dyj} x_j^2 + \beta_y] - \sum_{j=1}^{n} u_{cyj} x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^4} \le \alpha_y \quad (3.12)$$

Sub case 1b:

Assumption: $N_y(X) \sim N(\sum_{j=1}^{n} u_{cyj} x_j^2, \sum_{j=1}^{n} s_{cyj}^2 x_j^4)$ and $D_y(X) \sim N(\sum_{j=1}^{n} u_{dyj} x_j, \sum_{j=1}^{n} s_{dyj}^2 x_j^2)$,

where u_{cyj} and u_{dyj} are means, and s_{cyj}^2 and s_{dyj}^2 are variances.

Similar to sub case 1a, one can obtain the constraint given below:

$$\Pr[f(X, \lambda_y; \alpha_y > 0) \le \alpha_y] \ge 1 - p_y^{(2)}$$

$$\lambda_y [\sum_{j=1}^{n} u_{dyj} x_j + \beta_y] - \sum_{j=1}^{n} u_{cyj} x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 x_j^2 + s_{cyj}^2 x_j^4)} \le \alpha_y \quad (3.13)$$

Sub case 1c:

Assumption: $N_y(X) \sim N(\sum_{j=1}^{n} u_{cyj} x_j, \sum_{j=1}^{n} s_{cyj}^2 x_j^2)$ and $D_y(X) \sim N(\sum_{j=1}^{n} u_{dyj} x_j^2, \sum_{j=1}^{n} s_{dyj}^2 x_j^4)$,

where u_{cyj} and u_{dyj} are means, and s_{cyj}^2 and s_{dyj}^2 are variances.

Similar to sub case 1a, one can obtain the constraint given below

$$\Pr[f(X, \lambda_y; \alpha_y > 0) \le \alpha_y] \ge 1 - p_y^{(2)}$$

$$\lambda_y [\sum_{j=1}^{n} u_{dyj} x_j^2 + \beta_y] - \sum_{j=1}^{n} u_{cyj} x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 x_j^4 + s_{cyj}^2 x_j^2)} \le \alpha_y \quad (3.14)$$

Case 2. $\alpha_y \leq 0$

Similar to sub cases 1a, 1b and 1c, one can obtain the constraints 2a, 2b and 2c given below:

Sub case 2a: $\sum_{j=1}^{n} u_{cyj} x_j^2 - \lambda_y [\sum_{j=1}^{n} u_{dyj} x_j^2 + \beta_y] + \phi^{-1}(p_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^4} \geq \alpha_y$ (3.15)

Sub case2b: $\sum_{j=1}^{n} u_{cyj} x_j^2 - \lambda_y [\sum_{j=1}^{n} u_{dyj} x_j + \beta_y] + \phi^{-1}(p_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2 x_j^2) x_j^2} \geq \alpha_y$ (3.16)

Sub case2c: $\sum_{j=1}^{n} u_{cyj} x_j - \lambda_y [\sum_{j=1}^{n} u_{dyj} x_j^2 + \beta_y] + \phi^{-1}(p_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 x_j^2 + s_{cyj}^2) x_j^2} \geq \alpha_y$ (3.17)

3.3.5 Definitions

The following definitions are defined in consent of subcase 1a of section 3.3.4. Similarly, one can define for Case 2(2a, 2b, 2c).

Let scalar $\lambda = \min\{\lambda_y \simeq R_y(X) \mid X$ be the unit vector and $y = 1, 2, ..., k\}$ (3.18)

Let the decision space be

$$S^O = \{ X \in \mathbb{R}^n \mid \lambda \sum_{j=1}^{n} u_{dyj} x_j^2 - \sum_{j=1}^{n} u_{cyj} x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^4}$$

$$\leq \alpha_y - \lambda \beta_y,\ y = 1, 2, ..., k,\ x_j \geq 0;\ j = 1, 2, \ldots, n \}$$

and

$$S_w = \{ X \in \mathbb{R}^n \mid \lambda \sum_{j=1}^{n} u_{dyj} x_j^2 - \sum_{j=1}^{n} u_{cyj} x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^4}$$

$$\leq \alpha_y - \lambda \beta_y,\ x_j \geq 0;\ j = 1, 2, ..., n,\ y \neq w \}$$

Definitions are:

(i) The constraint form of objective function

$$\lambda \sum_{j=1}^{n} u_{dwj} x_j^2 - \sum_{j=1}^{n} u_{cwj} x_j^2 + \phi^{-1}(q_w^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda^2 s_{dwj}^2 + s_{cwj}^2) x_j^4} \leq \alpha_w - \lambda \beta_w$$

is redundant in the system (3.3) if and only if $S^o = S_w$. The equivalent definition is $S_w = S^o$ if and only if

$$\lambda \sum_{j=1}^{n} u_{dwj} x_j^2 - \sum_{j=1}^{n} u_{cwj} x_j^2 + \phi^{-1}(q_w^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda^2 s_{dwj}^2 + s_{cwj}^2) x_j^4} \leq \alpha_w - \lambda \beta_w \tag{3.19}$$

for all $X \in S_w$ and hence,

$$s_w(X) = \alpha_w - \lambda \beta_w - \lambda \sum_{j=1}^{n} u_{dwj} x_j^2 + \sum_{j=1}^{n} u_{cwj} x_j^2 - \phi^{-1}(q_w^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda^2 s_{dwj}^2 + s_{cwj}^2) x_j^4} .$$

(ii) The constraint form of the w^{th} objective function (3.19) is redundant in the system (3.3) if and only if $\hat{s}_w$ = minimum $\{ s_w(X) \mid X \in S_w \} \geq 0$.

(iii) The constraint form of the w^{th} objective function (3.19) is strongly redundant in the system (3.3) if and only if $\hat{s}_w > 0$. However, the constraint can be redundant without strongly redundant.

(iv) The constraint form of the w^{th} objective function (3.19) is weakly in the system (3.3) if and only if $\hat{s}_w = 0$.

3.3.6 Identification of Redundant Constraints for NLSFP

In this section, an algorithm is provided that helps to identify redundant fractional objective function(s) in multi-objective linear stochastic fractional programming problems. Charles and Dutta (2003) using sequential linear programming provided a method for linearising the constraint version of the fractional objective function as defined in section 3.3.4.

Consider linearising the constraint form of fractional objective function.

$$\hat{R}_y(\mathrm{X}) = \left\{ \begin{array}{l} \lambda \sum_{j=1}^{n} u_{dyj} x_j^2 - \sum_{j=1}^{n} u_{cyj} x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^4} - \alpha_y + \lambda \beta_y \le 0, \\ \text{(based on assumption 1a)} \\ \lambda \sum_{j=1}^{n} u_{dyj} x_j - \sum_{j=1}^{n} u_{cyj} x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2 x_j^2) x_j^2} - \alpha_y + \lambda \beta_y \le 0, \\ \text{(based on assumption 1b)} \\ \lambda \sum_{j=1}^{n} u_{dyj} x_j^2 - \sum_{j=1}^{n} u_{cyj} x_j + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 x_j^2 + s_{cyj}^2) x_j^2} - \alpha_y + \lambda \beta_y \le 0, \\ \text{(based on assumption 1c)} \end{array} \right\}$$

then , $\hat{R}_y(\mathrm{X}_{\text{int}}) + \nabla \hat{R}_y(\mathrm{X})^T (\mathrm{X} - \mathrm{X}_{\text{int}}) \le 0,\ y = 1, 2, ..., k,\ \mathrm{X} \ge 0$. (3.20)

Inequality (3.20) can be viewed as $\hat{R}_y^{(1)}\mathrm{X} \le \alpha_y - \lambda \beta_y$, $\mathrm{X} \ge 0$, $y = 1, 2, ..., k$ for the following steps:

The matrix form of the above inequality can be viewed as

$\hat{R}_y^{(1)}\mathrm{X} \le \alpha - \lambda\beta$, $\mathrm{X} \ge 0$, where $\hat{R}^{(1)} \in \mathbb{R}^{kxn}$ and $(\alpha - \lambda\beta) \in \mathbb{R}^k$.

Adding slack variables to the k constraints form of objective functions, pre-multiplying by the inverse of an appropriate basis and redefining the variables (both slacks and structural variables) as x_j^{NB} (or) x_j^B according to their status (*NB* for non-basic, and *B* for basic), yields an equivalence system

$$[\hat{R}_{NB}^{(1)\,-1} \quad I] \begin{bmatrix} x^{NB} \\ x^{B} \end{bmatrix} = \eta \;, \qquad x^{B}, x^{NB} \geq 0.$$

The matrix $\hat{R}_{NB}^{(1)\,-1}$ is usually referred to as the Contracted Simplex Tableau Dantzig (1963). Let us refer to the elements of $\hat{R}_{NB}^{(1)\,-1}$ as γ_{ij}, η is the "updated right hand side" $\hat{R}^{(1)\,-1}(\alpha - \lambda\beta)$.

Theorem 3.1: *A constraint form of objective function is redundant if and only if its associated slack variable* s_w *has the property* $s_w = x_f^B$ *in a basic solution in which* $\gamma_{fj} \leq 0$, *j = 1, 2,…, n and* $\eta_f \geq 0$.

Proof IF: In a basic solution $x_f^B = \eta_f - \sum_{j=1}^{n} \gamma_{fj} x_j^B$, since in any feasible solution the value of the x_j^{NB} will be at least zero, the sum is at least zero and hence, $s_w = x_f^B \geq \eta_f \geq 0$. Therefore $\hat{s}_w \geq 0$.

Only IF: Let us consider the f^{th} row of tableau as the objective function for the sequential linear programming minimum $\{ s_w(X) \mid X \in S_w \}$; then if $\hat{s}_w \geq 0$, it follows that in the optimal solution $\gamma_{fj} \leq 0$ for all $j = 1, 2, \ldots, n$ with $\eta_f \geq 0$. Since this optimal solution is a feasible extreme point of S_w, it is a basic feasible solution for the original set of constraint form of objective functions. □

Since, in the theorem above $\hat{s}_w = \eta_f$, the constraint form of objective function is strongly redundant if $\eta_f > 0$ and weakly redundant if $\eta_f = 0$.

3.3.6.1 Redundancy Algorithm

1. A matrix of intercept is constructed with decision and slack variables as rows and columns respectively. This matrix is of order *m* x *n*.

if $\alpha_y \le 0$ then $\theta_{ji} = (\alpha_y - \lambda\ \beta_y)/\hat{R}^{(1)}_{yij}$; $\hat{R}^{(1)}_{yij} \ge 0$, $i = 1, 2, ..., k$, $j = 1, 2, ..., n$.

else $\theta_{ji} = (\alpha_y - \lambda\ \beta_y)/\hat{R}^{(1)}_{yij}$; $\hat{R}^{(1)}_{yij} < 0$, $i = 1, 2, ..., k$, $j = 1, 2, ..., n$.

2. Identify the pivot element in each row if $\alpha_y \le 0$ then $\Psi_j = \max_i\{\theta_{ji}\}$ else $\Psi_j = \min_i\{\theta_{ji}\}$, for all j while the objective is maximum or vice versa.

3. Score out the row and column corresponding to the entering and leaving variables. If a column has more than one maximum/minimum, score out those rows also.

4. The constraints corresponding to the slack variables in the unscored column, if any, *ab initio* are assumed and predicted to be redundant.

5. Remove these redundant constraint forms of fractional objective functions tentatively from the original model.

3.3.6.2 Proposed Algorithm

1. Convert the stochastic fractional objective functions into constraint form using section 3.3.4.

2. Adopting the technique of SLP Rao (2000) or Jeeva et al (2002) or Charles and Dutta (2003), linearise the constraint form of the objective functions.

3. Apply the algorithm under section 3.3.6.1 to identify the redundant objective function and ignore that objective function from the system.

4. Solve the reduced MOSFP to get the optimal solution as in Charles and Dutta (2001, 2003, 2006b) or using any stochastic programming solver.

3.3.7 Numerical Example

Example 3.7

$$\text{Max } R(X) = \left[\frac{c_{11}x_1^2+c_{12}x_2^2+\alpha_1}{d_{11}x_1^2+d_{12}x_2^2+\beta_1}, \frac{c_{21}x_1^2+c_{22}x_2^2+\alpha_2}{d_{21}x_1^2+d_{22}x_2^2+\beta_2}\right] \tag{3.21}$$

subject to $3x_1+5x_2 \leq 15$, $5x_1+2x_2 \leq 10$, $a_1x_1+a_2x_2 \leq 5$,

where $\alpha_1=\alpha_2=0$, $\beta_1=\beta_2=1$, $x_1, x_2 \geq 0$.

Let the third constraint satisfy at least 90%. The mean and variance of the random variables are given in Table 3.2.

Table 3.2 Mean and Variance of Random Variables for Example 3.7

Random variables	c_{11}	c_{12}	c_{21}	c_{22}	d_{11}	d_{12}	d_{21}	d_{22}	a_1	a_2
Mean	6	3	15	10	5	2	1	1	2	3
Variance	2	1	1	1	2	1	1	1	1	1

Take $p_1^{(2)}$ = 0.10 and $p_2^{(2)}$ = 0.90. The deterministic equivalent of constraint form of fractional objective functions is given below

$$6x_1^2+3x_2^2-\lambda_1(5x_1^2+2x_2^2+1)+1.28\sqrt{(2\lambda_1^2+2)x_1^4+(\lambda_1^2+1)x_2^4} \geq 0 \tag{3.22}$$

$$15x_1^2+10x_2^2-\lambda_2(x_1^2+x_2^2+1)-1.28\sqrt{(\lambda_2^2+1)x_1^4+(\lambda_2^2+1)x_2^4} \geq 0 \tag{3.23}$$

Let $\lambda = \min\{1.125, 8.333\} = 1.125$ at $(x_1, x_2) = (1, 1)$ from equation (3.18).

Therefore, inequalities (3.22)-(3.23) reduces to (3.24)-(3.25)

$$0.3750x_1^2 + 0.7500x_2^2 + 1.28\sqrt{4.5313x_1^4 + 2.2656x_2^4} \geq 1.1250 \quad (3.24)$$

$$13.8750x_1^2 + 8.8750x_2^2 - 1.28\sqrt{2.2656x_1^4 + 2.2656x_2^4} \geq 1.1250 \quad (3.25)$$

Using the SLP Charles and Dutta (2003) the following linear constraints are obtained:

$$5.2x_1 + 3.725x_2 \geq 5.5879 \quad (3.26)$$

$$25.0252x_1 + 15.026x_2 \geq 21.1212 \quad (3.27)$$

Adapt the redundancy algorithm given in section 3.3.6.1.

Table 3.3 Matrix-of-Intercept for Example 3.7

Slack/Decision Variables	S_1	S_2	Ψ
x_1	**1.074**	0.845	1.074
x_2	**1.5**	1.408	1.5

From the Table 3.3, it can be inferred that the constraint due to second objective function is strongly redundant. Therefore, the problem is solved ignoring the second objective function. The bi-objective stochastic fractional programming problem reduces to SFP problem as follows:

$$\text{Max } 0.90\lambda_1 \quad (3.28)$$

subject to $6x_1^2 + 3x_2^2 - \lambda_1(5x_1^2 + 2x_2^2 + 1) + 1.28\sqrt{(2\lambda_1^2 + 2)x_1^4 + (\lambda_1^2 + 1)x_2^4} \geq 0$

$3x_1 + 5x_2 \leq 15$, $5x_1 + 2x_2 \leq 10$, $2x_1 + 3x_2 + 1.28\sqrt{x_1^2 + x_2^2} \leq 5$

$x_1, x_2, \lambda_1 \geq 0.$

The solution is obtained as x_1 = 1.5244, x_2 = 0.0000, λ_1= 1.6890. The corresponding value of objective functions in equation (3.21) is [1.0608, 9.0580]. Here, it is to be noted that 90% of importance is given to the first objective function.

3.4 AN APPROACH TO FIND REDUNDANT OBJECTIVE FUNCTION(S) AND REDUNDANT CONSTRAINT(S) IN MONLSFP PROBLEMS

Medium to large scale multi-objective non-linear fractional programming problems, when modelled embed redundant objective function(s) and constraint(s) due to various reasons. Obviously, a problem with redundancy will be larger or contain more details than one without it. The consequence of the presence of redundancy that comes to mind is the effect it has on the problem solving stages of the process. All information contained in the problem has to be processed and if some information is redundant, they need not be processed at all. This is all the more important in iterative methods, such as Simplex method, where such redundant information is processed repeatedly and thereby wasting computational effort. Therefore in this section, an integrated heuristic algorithm has been developed to identify redundant objective function(s) and redundant constraint(s) *apriori* to improve the computational efficiency.

The methodology used here is as follows. Firstly, the non-linear stochastic objective functions are converted into constraint forms and then they are linearised. Secondly, deterministic equivalent of stochastic constraints are obtained and then linearised. Finally, the proposed redundancy algorithm is applied for (i) linearised constraint form of objective functions (ii) linearised constraint form of stochastic constraints and redundancies are identified separately. As a result, the original problem is downsized, then solved by using a solver.

3.4.1 Motivation

Among the various methods proposed by researchers of the past to detect redundancy, none came out with an algorithm for *apriori* identification of redundant objective functions and redundant constraints. By taking this as a clue, a heuristic algorithm has been developed which performs two operations in one stroke viz. identification of redundant objective functions and redundant constraints. This detection nips some of the redundant objective function(s) and redundant constraint(s) in their bud so that the problem can be solved with less computational effort i.e., the problem is solved quickly.

3.4.2 Problem Definition

Another version of MONLSFP in a criterion space is defined as follows: Max G(X) = [G_1(X), G_2(X), …, G_k (X)], where $G_y(X) = \frac{N_y(X)+\alpha_y}{D_y(X)+\beta_y}$,

$$y = 1, 2, \dots, k \qquad (3.29)$$

subject to $\Pr[\sum_{j=1}^{n} a_{ij}^{(1)} x_j \le b_i^{(1)}] \ge 1-p_i$ $i = 1, 2, ..., p, p+1, \dots, q, q+1, \dots, m$ (3.30)

$$\sum_{j=1}^{n} a_{ij}^{(2)} x_j \le b_i^{(2)} \qquad i = m+1, ..., h \qquad (3.31)$$

where $0 \le X_{nx1} = \| x_j \| \subset \mathbb{R}^n$ is a feasible set, and R: $\mathbb{R}^n \to \mathbb{R}^k$, $A_{mxn}^{(1)} = \| a_{ij}^{(1)} \|$, $b_{mx1} = \| b_i^{(1)} \|$, $i = 1, 2, ..., m,$ $j = 1, 2, ..., n,$ α_y, β_y are scalars, $A_{(h-m+1)xn}^{(2)} = \| a_{ij}^{(2)} \|$, $b_{(h-m+1)x1} = \| b_i^{(2)} \|$, $i = m+1, \dots, h,$ $j = 1, 2, \dots, n,$ $N_y(X) = \sum_{j=1}^{n} c_{yj} x_j^2$ and $D_y(X) = \sum_{j=1}^{n} d_{yj} x_j^2$.

In this model, out of $N_y(X)$, $D_y(X)$, $A^{(1)}$ and $b_i^{(1)}$, at least one may be a random variable and S = {X | Equations (3.30) - (3.31), $X \geq 0$, $X \subset \mathbb{R}^n$ } is non-empty, convex and compact set in $\mathbb{R}^n$.

The problem here is to develop a redundancy algorithm which identifies redundant objective function(s) and redundant constraint(s) in MONLSFP problems and to develop an algorithm for solving it.

3.4.3 Deterministic Equivalents of Probabilistic Constraints

Let $a_{ij}^{(1)}$ be a random variable in equation. (3.30) and it follows $N(u_{ij}, s_{ij}^2)$, i = 1, 2, ..., p, j = 1, 2,..., n, where u_{ij} is the mean and s_{ij}^2 is the variance. Let $l_i = \sum_{i=1}^{n} a_{ij}^{(1)} x_j$, i =1, 2, ..., p.

$$E(l_i) = \sum_{j=1}^{n} u_{ij} x_j \, ; \; V(l_i) = X' V_i X = \sum_{j=1}^{n} s_{ij}^2 x_j^2 \, ,$$

where V_i - i^{th} covariance matrix. When $a_{ij}^{(1)}$ is independent, the covariance terms become zero. The i^{th} deterministic constraint for equation (3.30) is obtained from Charles and Dutta (2001,2003) as follows:

$$\Pr(l_i \leq b_i) \geq 1 - p_i \text{ (or) } \Pr(Z_i \leq z_i) \geq 1 - p_i \, ,$$

where $Z_i = (l_i - E(l_i))/\sqrt{V(l_i)}$ follows standard normal distribution and $z_i = (b_i^{(1)} - E(l_i))/\sqrt{V(l_i)}$. Thus, $\phi(z_i) \geq \phi(Kq_i)$, where $1 - p_i = q_i = \phi(Kq_i)$ is the cumulative distribution function of standard normal distribution. Clearly, $\phi(.)$ is a non-decreasing continuous function, hence $z_i \geq Kq_i$. Substituting in this equation the values of $E(l_i)$ and $V(l_i)$,

$$\sum_{j=1}^{n} u_{ij} x_j + \mathrm{Kq}_i \sqrt{\sum_{j=1}^{n} s_{ij}^2 x_j^2} \leq b_i^{(1)} \tag{3.32}$$

Suppose $a_{ij}^{(1)}$ and $b_i^{(1)}$ are random variables in equation (3.30) i.e. $a_{ij}^{(1)} \sim \mathrm{N}\ (u_{ij}, s_{ij}^2)$ and $b_i^{(1)} \sim \mathrm{N}\ (u_{b_i^{(1)}}, s_{b_i^{(1)}}^2)$, $i = p+1, p+2, ..., q,\ j = 1, 2, ..., n,$ where u_{ij} and $u_{b_i^{(1)}}$ are means, and s_{ij}^2 and $s_{b_i^{(1)}}^2$ are variances respectively. With the similar argument that led to the inequality in (3.32), one can obtain inequality (3.33), the i^{th} deterministic constraint for equation (3.30) as follows:

$$\sum_{j=1}^{n} u_{ij} x_j - \mathrm{Kp}_i \sqrt{\sum_{j=1}^{n+1} s_{ij}^2 x_j^2} \leq u_{b_i^{(1)}} \tag{3.33}$$

where $x_{n+1} = -1$.

If $b_i^{(1)}$ is a random variable in equation (3.30), i.e., $b_i^{(1)} \sim \mathrm{N}(u_{b_i^{(1)}}, s_{b_i^{(1)}}^2)$, $i = q+1, q+2, ..., m,$ where $u_{b_i^{(1)}}$, $s_{b_i^{(1)}}^2$ are the mean and variance respectively. With the similar argument that led to the inequality in (3.32), one can obtain inequality (3.34), the i^{th} deterministic constraint for equation (3.30) as follows:

$$\sum_{j=1}^{n} a_{ij}^{(1)} x_j \leq u_{bi}^{(1)} + \mathrm{Kp}_i\, s_{b_i^{(1)}}^2 \tag{3.34}$$

3.4.4 Conversion of Objective Functions into Constraints

This section considers all the objective functions in the form of constraints (Charles and Dutta 2003, 2006b). The main feature of the model is that it takes into account the probability distribution of the objective functions by maximizing the lower allowable limit of the objective function under chance constraints where numerator and/or denominator coefficients are random.

The unknown parameter λ_y, which is less than or equal to $G_y(X)$ is defined by,

$$G_y(X) \geq \lambda_y \quad \text{i.e.,} \ \frac{N_y(X)+\alpha_y}{D_y(X)+\beta_y} \geq \lambda_y \quad \Rightarrow \quad 0 \leq N_y(X) + \alpha_y - \lambda_y \ [D_y(X) + \beta_y]$$

There are two cases in this problem.

Case 1. $\alpha_y > 0$

Under this case, there are three sub cases namely sub cases 1a, 1b and 1c. Assumptions and derivations of constraint forms of objective functions are the same as in sub cases 1a, 1b and 1c described in section 3.3.4. Therefore

Sub case 1a: $\lambda_y \left[\sum_{j=1}^{n} u_{dyj} x_j^2 + \beta_y \right] - \sum_{j=1}^{n} u_{cyj} x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^4} \leq \alpha_y$ (3.35)

Sub case 1b: $\lambda_y \left[\sum_{j=1}^{n} u_{dyj} x_j + \beta_y \right] - \sum_{j=1}^{n} u_{cyj} x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 x_j^2 + s_{cyj}^2 x_j^4)} \leq \alpha_y$ (3.36)

Sub case 1c: $\lambda_y \left[\sum_{j=1}^{n} u_{dyj} x_j^2 + \beta_y \right] - \sum_{j=1}^{n} u_{cyj} x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 x_j^4 + s_{cyj}^2 x_j^2)} \leq \alpha_y$ (3.37)

Case 2. $\alpha_y \leq 0$

Similar to sub cases 1a, 1b and 1c one can obtain the constraints 2a, 2b and 2c given below:

Sub case 2a: $\sum_{j=1}^{n} u_{cyj} x_j^2 - \lambda_y \left[\sum_{j=1}^{n} u_{dyj} x_j^2 + \beta_y \right] + \phi^{-1}(p_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^4} \geq \alpha_y$ (3.38)

Sub case 2b: $\sum_{j=1}^{n} u_{cyj}x_j^2 - \lambda_y [\sum_{j=1}^{n} u_{dyj}x_j + \beta_y] + \phi^{-1}(p_y^{(2)})\sqrt{\sum_{j=1}^{n}(\lambda_y^2 s_{dyj}^2 + s_{cyj}^2 x_j^2)x_j^2} \geq \alpha_y$ (3.39)

Sub case 2c: $\sum_{j=1}^{n} u_{cyj}x_j - \lambda_y [\sum_{j=1}^{n} u_{dyj}x_j^2 + \beta_y] + \phi^{-1}(p_y^{(2)})\sqrt{\sum_{j=1}^{n}(\lambda_y^2 s_{dyj}^2 x_j^2 + s_{cyj}^2)x_j^2} \geq \alpha_y$ (3.40)

3.4.5 Definitions

The following definitions are defined in consent of subcase 1a of section 3.4.4. Similarly, one can define for Case 2(2a, 2b, 2c) and for the constraints (3.31) to (3.34).

Let scalar $\lambda = \min\{\lambda_y \simeq G_y(X) \mid X$ be the unit vector and $y = 1, 2, ..., k\}$ (3.41)

Let the decision space be

$$S^O = \{ X \in \mathbb{R}^n \mid \lambda \sum_{j=1}^{n} u_{dyj}x_j^2 - \sum_{j=1}^{n} u_{cyj}x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n}(\lambda_y^2 s_{dyj}^2 + s_{cyj}^2)x_j^4} \leq \alpha_y - \lambda \beta_y,$$

Equations (3.31)–(3.34), $i = 1, 2, ..., h$, $y = 1, 2, ..., k$, $x_j \geq 0$; $j = 1, 2, \ldots, n$ }

and $S_w = \{X \in \mathbb{R}^n \mid \lambda \sum_{j=1}^{n} u_{dyj}x_j^2 - \sum_{j=1}^{n} u_{cyj}x_j^2 + \phi^{-1}(q_y^{(2)})\sqrt{\sum_{j=1}^{n}(\lambda_y^2 s_{dyj}^2 + s_{cyj}^2)x_j^4} \leq \alpha_y - \lambda \beta_y$,

Equations (3.31) – (3.34), $i = 1, 2, ..., h$, $x_j \geq 0$; $j = 1, 2, ..., n$, $y \neq w$ }

Definitions are:

(i) The constraint form of objective function

$$\lambda \sum_{j=1}^{n} u_{dwj}x_j^2 - \sum_{j=1}^{n} u_{cwj}x_j^2 + \phi^{-1}(q_w^{(2)})\sqrt{\sum_{j=1}^{n}(\lambda^2 s_{dwj}^2 + s_{cwj}^2)x_j^4} \leq \alpha_w - \lambda \beta_w$$

is redundant in the system (3.29) if and only if $S^o = S_w$. The equivalent definition is $S_w = S^o$ if and only if

$$\lambda \sum_{j=1}^{n} u_{dwj} x_j^2 - \sum_{j=1}^{n} u_{cwj} x_j^2 + \phi^{-1}(q_w^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda^2 s_{dwj}^2 + s_{cwj}^2) x_j^4} \leq \alpha_w - \lambda \beta_w \qquad (3.42)$$

for all $X \in S_w$ and hence

$$s_w(X) = \alpha_w - \lambda \beta_w - \lambda \sum_{j=1}^{n} u_{dwj} x_j^2 + \sum_{j=1}^{n} u_{cwj} x_j^2 - \phi^{-1}(q_w^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda^2 s_{dwj}^2 + s_{cwj}^2) x_j^4} .$$

(ii) The constraint form of the w^{th} objective function (3.42) is redundant in the system (3.29) if and only if $\hat{s}_w$ = minimum $\{ s_w(X) \mid X \in S_w \} \geq 0$.

(iii) The constraint form of the w^{th} objective function (3.42) is strongly redundant in the system (3.29) if and only if $\hat{s}_w > 0$. However, the constraint can be redundant without being strongly redundant.

(iv) The constraint form of the w^{th} objective function (3.42) is weakly redundant in the system (3.29) if and only if $\hat{s}_w = 0$.

3.4.6 Identification of Redundant Constraints for NLSFP

In this section, an algorithm is provided that helps to identify redundant fractional objective functions and redundant constraints in multi-objective linear stochastic fractional programming problems. Charles and Dutta (2003) using sequential linear programming provided a method for linearizing the constraint version of fractional objective function given in section 3.4.4 and the constraint version of stochastic constraint given in section 3.4.3.

Consider linearizing the constraint and constraint form of fractional objective function.

$$\hat{G}_y(X) = \begin{cases} \lambda \sum_{j=1}^{n} u_{dyj}x_j^2 - \sum_{j=1}^{n} u_{cyj}x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^4} - \alpha_y + \lambda \beta_y \le 0, \\ \qquad \text{(based on assumption in Case 1a)} \\ \lambda \sum_{j=1}^{n} u_{dyj}x_j - \sum_{j=1}^{n} u_{cyj}x_j^2 + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2 x_j^2) x_j^2} - \alpha_y + \lambda \beta_y \le 0, \\ \qquad \text{(based on assumption in Case 1b)} \\ \lambda \sum_{j=1}^{n} u_{dyj}x_j^2 - \sum_{j=1}^{n} u_{cyj}x_j + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 x_j^2 + s_{cyj}^2) x_j^2} - \alpha_y + \lambda \beta_y \le 0, \\ \qquad \text{(based on assumption in Case 1c)} \end{cases}$$

then , $\hat{G}_y(\mathrm{X}_{\mathrm{int}}) + \nabla \hat{G}_y(\mathrm{X})^T (\mathrm{X} - \mathrm{X}_{\mathrm{int}}) \le 0$, $y = 1, 2, ..., k$, $\mathrm{X} \ge 0$. (3.43)

Inequality (3.43) can be viewed as $\hat{G}_y^{(1)}\mathrm{X} \le \alpha_y - \lambda\beta_y$, $\mathrm{X} \ge 0$, $y = 1, 2, ..., k$ for the following steps:

The matrix form of the above inequality can be viewed as

$$\hat{G}^{(1)}\mathrm{X} \le \alpha - \lambda\beta, \ \mathrm{X} \ge 0, \text{where } \hat{G}^{(1)} \in \mathbb{R}^{kxn} \text{ and } (\alpha - \lambda\beta) \in \mathbb{R}^k. \quad (3.44)$$

Let $T_i^{(1)}(\mathrm{X}) = \begin{cases} \sum_{j=1}^{n} u_{ij}x_j + \mathrm{Kq}_i \sqrt{\sum_{j=1}^{n} s_{ij}^2 x_j^2} - b_i^{(1)} \le 0, & i = 1, 2, \ldots, p \\ \sum_{j=1}^{n} u_{ij}x_j - \mathrm{Kp}_i \sqrt{\sum_{j=1}^{n+1} s_{ij}^2 x_j^2} - u_{b_i^{(1)}} \le 0, & i = p+1, \ldots, q \end{cases}$

$$T_i^{(2)}(\mathrm{X}) = \begin{cases} \sum_{j=1}^{n} a_{ij}^{(1)} x_j - u_{bi}^{(1)} + \mathrm{Kp}_i s_{b_i^{(1)}}^2 \le 0, & i = q+1, \ldots, m \\ \sum_{j=1}^{n} a_{ij}^{(2)} x_j - b_i^{(2)} \le 0, & i = m+1, \ldots, h, \end{cases}$$

where q non-linear constraints and h-q+1 linear constraints are represented by $T_i^{(1)}(\mathrm{X})$ and $T_i^{(2)}(\mathrm{X})$ respectively.

Linearize the constraint function $T_i^{(1)}(\mathrm{X})$ about the point $\mathrm{X}_{\mathrm{int}}$ as

$$T_i^{(1)}(\mathrm{X}_{\mathrm{int}}) + \nabla T_i^{(1)}(\mathrm{X})^T(\mathrm{X}-\mathrm{X}_{\mathrm{int}}) \leq 0, \ \mathrm{X} \geq 0, \ i = 1, 2, \ldots, p, \ p+1, \ldots, q. \quad (3.45)$$

The matrix form of the above inequality can be viewed as

$T^{(1)}\mathrm{X} \leq b_{T^{(1)}}$ $\mathrm{X} \geq 0$, where $T^{(1)} \in \mathbb{R}^{q \mathrm{x} n}$ and $b_{T(1)} \in \mathbb{R}^q$.

The matrix form of h-q+1 linear inequalities can be viewed as

$T^{(2)}\mathrm{X} \leq b_{T^{(2)}}$ $\mathrm{X} \geq 0$, where $T^{(2)} \in \mathbb{R}^{(h-q+1) \mathrm{x} n}$ and $b_{T(2)} \in \mathbb{R}^{h-q+1}$.

Hence the matrix form of h inequalities can be viewed as

$$T\,\mathrm{X} = \begin{bmatrix} T^{(1)} \\ T^{(2)} \end{bmatrix} X \leq b = \begin{bmatrix} b_{T(1)} \\ b_{T(2)} \end{bmatrix}, \text{ where } T \in \mathbb{R}^{h \mathrm{x} n} \text{ and } b \in \mathbb{R}^{h \mathrm{x} 1} \quad (3.46)$$

The matrix form of k+h inequalities can be viewed as

$$A\,\mathrm{X} = \begin{bmatrix} \widehat{G}^{(1)} \\ --- \\ T \end{bmatrix} X \leq b = \begin{bmatrix} \alpha - \lambda\beta \\ ------ \\ b_{T(1)} \\ b_{T(2)} \end{bmatrix} \text{ where } A \in \mathbb{R}^{(k+h) \mathrm{x} n} \text{ and } b \in \mathbb{R}^{(k+h) \mathrm{x} 1} \quad (3.47)$$

Adding slack variables to the k constraints form of objective functions and to the h constraints, pre-multiplying by the inverse of an appropriate basis and redefining the variables (both slacks and structural variables) as x_j^{NB} (or) x_j^{B} according to their status (*NB* for non-basic, and *B* for basic) yields an equivalence system

$$[\,A_{NB}^{-1} \quad \mathrm{I}\,] \begin{bmatrix} x^{NB} \\ x^{B} \end{bmatrix} = \eta\,, \qquad x^B, x^{NB} \geq 0\,.$$

The matrix A_{NB}^{-1} is usually referred to as the Contracted Simplex Tableau Dantzig (1963). Let us refer to the elements of A_{NB}^{-1} as γ_{ij}, η is the "updated right hand side".

Theorem 3.2: *A regular constraint or constraint form of objective function is redundant if and only if its associated slack variable* s_w *has the property* $s_w = x_f^B$ *in a basic solution in which* $\gamma_{fj} \leq 0$, $j = 1, 2, \ldots, n$ *and* $\eta_f \geq 0$.

Proof IF: In a basic solution $x_f^{B}` = \eta_f - \sum_{j=1}^{n} \gamma_{fj} x_j^B$, since in any feasible solution the value of the x_j^{NB} will be at least zero, the sum is atleast zero and hence, $s_w = x_f^B \geq \eta_f \geq 0$. Therefore $\hat{s}_w \geq 0$.

Only IF: Let us consider the f^{th} row of tableau as the objective function for the sequential linear programming minimum $\{s_w(X) \mid X \in S_w\}$; then if $\hat{s}_w \geq 0$, it follows that in the optimal solution $\gamma_{fj} \leq 0$ for all $j = 1, 2, \ldots, n$ with $\eta_f \geq 0$. Since this optimal solution is a feasible extreme point of S_w, it is a basic feasible solution for the original set of constraints and constraint form of objective functions. Since, in the theorem above $\hat{s}_w = \eta_f$, the regular constraint or constraint form of objective function is strongly redundant if $\eta_f < 0$ and weakly redundant if $\eta_f = 0$. □

3.4.6.1 Redundancy algorithm

1. Construct a matrix of intercept with decision variables and slack variables as rows and columns respectively.

   1.1 For finding redundant objective function(s)

   if $\alpha_y \leq 0$ then $\theta_{jy} = (\alpha_y - \lambda\, \beta_y) / \hat{G}_{yj}^{(1)}$; $\hat{G}_{yj}^{(1)} \geq 0$, $y = 1, 2, \ldots, k$,

$j = 1, 2, ..., n.$

else $\theta_{jy} = (\alpha_y - \lambda\ \beta_y)/\hat{G}_{yj}^{(1)}$; $\quad \hat{G}_{yj}^{(1)} < 0$, $y = 1, 2, ..., k,$

$j = 1, 2, ..., n.$

1.2 For finding redundant constraint(s)

$\theta_{ji} = b_i / T_{ij}$; $\quad i = 1, 2, ..., h, \ j = 1, 2, ..., n.$

2. Identify the pivot element in each row

   2.1 For finding redundant objective function(s): if $\alpha_y \le 0$ then $\Psi_j^{obj} = \max_i \{\theta_{ji}\}$ else $\Psi_j^{obj} = \min_i \{\theta_{ji}\}$, for all j while the objective is maximum, vice versa.

   2.2 For finding redundant constraint(s):

   $\Psi_j^{const} = \max_i \{\theta_{ji} / \theta_{ji} > 0\}$ for all j, while the objective is minimization and constraints are lower bound;

   $\Psi_j^{const} = \min_i \{\theta_{ji} / \theta_{ji} > 0\}$ for all j, while the objective is maximization and constraints are upper bound.

3. Score out the row and column corresponding to the entering and leaving variables. If a column has more than one maximum/minimum, score out those rows also.

4. The constraints corresponding to the slack variables in the unscored column, if any, ab initio are assumed and predicted to be redundant.

5. Remove these redundant constraint forms of fractional objective functions and redundant constraints tentatively from the original model.

3.4.6.2 Proposed algorithm

1. Convert the stochastic fractional objective functions into constraint form using section 3.4.4.
2. Convert the stochastic constraints into deterministic constraints using section 3.4.3.
3. Adopting the technique of SLP Rao (2000) or Jeeva et al (2002) or Charles and Dutta (2003), linearise

   (i) the constraint form of the objective functions.

   (ii) the nonlinear constraints.
4. Apply the algorithm under section 3.4.6.1 to identify the redundant objective function(s) and redundant constraint(s) and ignore them from the system.
5. Solve the reduced MOSFP to get the optimal solution as in Charles and Dutta (2001, 2003, 2006b) or by using any other stochastic programming solvers such as SLP-IOR and AMPL.

3.4.7 Numerical Examples

Example 3.8

$$\text{Max } R(X) = \left[\frac{c_{11}x_1^2+c_{12}x_2^2+\alpha_1}{d_{11}x_1^2+d_{12}x_2^2+\beta_1}, \frac{c_{21}x_1^2+c_{22}x_2^2+\alpha_2}{d_{21}x_1^2+d_{22}x_2^2+\beta_2}\right] \quad (3.48)$$

subject to $3x_1+5x_2 \le 15$, $5x_1+2x_2 \le 10$, $a_1x_1+a_2x_2 \le 5$,

where $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 1$, $x_1, x_2 \ge 0$.

Let the third constraint satisfy at least 90%. The mean and variance of the random variables are given in Table 3.4.

Table 3.4 Mean and Variance of Random Variables for Example 3.8

Random variables	c_{11}	c_{12}	c_{21}	c_{22}	d_{11}	d_{12}	d_{21}	D_{22}	a_1	a_2
Mean	6	3	15	10	5	2	1	1	2	3
Variance	2	1	1	1	2	1	1	1	1	1

Take $p_1^{(2)} = 0.10$ and $p_2^{(2)} = 0.90$. The deterministic equivalent of constraint form of fractional objective functions is given below:

$$6x_1^2 + 3x_2^2 - \lambda_1(5x_1^2 + 2x_2^2 + 1) + 1.28\sqrt{(2\lambda_1^2 + 2)x_1^4 + (\lambda_1^2 + 1)x_2^4} \geq 0 \qquad (3.49)$$

$$15x_1^2 + 10x_2^2 - \lambda_2(x_1^2 + x_2^2 + 1) - 1.28\sqrt{(\lambda_2^2 + 1)x_1^4 + (\lambda_2^2 + 1)x_2^4} \geq 0 \qquad (3.50)$$

Let $\lambda = \min\{1.125, 8.333\} = 1.125$ at $(x_1, x_2) = (1, 1)$ from equation (3.41)

Therefore, inequalities (3.49)-(3.50) reduces to (3.51)-(3.52)

$$0.375x_1^2 + 0.750x_2^2 + 1.28\sqrt{4.531x_1^4 + 2.266x_2^4} \geq 1.125 \qquad (3.51)$$

$$13.875x_1^2 + 8.875x_2^2 - 1.28\sqrt{2.266x_1^4 + 2.266x_2^4} \geq 1.125 \qquad (3.52)$$

Using the SLP Charles and Dutta (2003), the following linear constraints are obtained:

$$5.200x_1 + 3.725x_2 \geq 5.587$$

$$25.025x_1 + 15.026x_2 \geq 21.121$$

The deterministic equivalent of stochastic constraint in system (3.48) is given below:

$$2x_1 + 3x_2 + 1.28\sqrt{x_1^2 + x_2^2} \leq 5$$

Using the SLP Charles and Dutta (2003), the following linear constraint is obtained:

$$2.910x_1 + 3.910x_2 \geq 5.010$$

Solution to the system (3.48) with deterministic constraints is $x_1 = 1.718$, $x_2 = 0.000$, $\lambda_1 = 1.653$, $\lambda_2 = 8.858$ and the corresponding value of objective functions in system (3.48) is [1.124,11.204].

Solution using Redundancy algorithm is given below.

Table 3.5 Matrix-of-Intercept for Example 3.8

	Objective functions			Constraints			
Slack/Decision Variables	S_1	S_2	Ψ_j^{obj}	S_3	S_4	S_5	Ψ_j^{const}
x_1	**1.074**	0.845	1.074	5	2	**1.722**	1.722
x_2	**1.500**	1.408	1.500	3	5	**1.281**	1.281

From the Table 3.5, it can be inferred that the constraint due to second objective function is strongly redundant and first, second constraints are strongly redundant. Therefore, ignoring the second objective function and the first two constraints from the original problem, the problem is then solved. The bi-objective stochastic fractional programming problem reduces to SFP problem as follows:

$$\text{Max } 0.90\ \lambda_1 \tag{3.53}$$

$$\text{subject to } 6x_1^2+3x_2^2-\lambda_1(5x_1^2+2x_2^2+1)+1.28\sqrt{(2\lambda_1^2+2)x_1^4+(\lambda_1^2+1)x_2^4}\geq 0$$

$$2x_1+3x_2+1.28\sqrt{x_1^2+x_2^2}\leq 5$$

$$x_1, x_2, \lambda_1 \geq 0.$$

The solution is obtained as x_1=1.718, x_2 =0.000, λ_1=1.653. The corresponding value of objective functions in equation (3.48) is [1.124,11.204]. Here, it is to be noted that 90% of importance is given to the first objective function.

Example 3.9

$$\text{Max } R(X) = [\frac{c_{11}x_1^2+c_{12}x_2^2+\alpha_1}{d_{11}x_1^2+d_{12}x_2^2+\beta_1}, \frac{c_{21}x_1^2+c_{22}x_2^2+\alpha_2}{d_{21}x_1^2+d_{22}x_2^2+\beta_2}] \tag{3.54}$$

subject to $a_{11}x_1+a_{12}x_2\leq 1$, $a_{21}x_1+a_{22}x_2\leq b_2$, $16x_1+x_2\leq 4$

where $\alpha_1= 7, \alpha_2= 2, \beta_1= 4, \beta_2= 4$, $x_1, x_2\geq 0$.

Let the first constraint be satisfied at least 90% and the second constraint be satisfied at least 80%. The mean and variance of the random variables are given in Table 3.6.

Table 3.6 Mean and Variance of Random Variables for Example 3.9

Random variables	c_{11}	c_{12}	c_{21}	c_{22}	d_{11}	d_{12}	d_{21}	d_{22}	a_{11}	a_{12}	a_{21}	a_{22}	b_2
Mean	1	2	1	1	2	2	2	3	2	1	3	4	3
Variance	1	0.5	0	0.5	1	1	1	1	1	1	2	3	2

Take $q_1^{(2)} = 0.10$ and $q_2^{(2)} = 0.90$.

The deterministic equivalent of constraint form of fractional objective functions is given below:

$$\lambda_1(2x_1^2+2x_2^2+4)-x_1^2-2x_2^2-1.28\sqrt{(\lambda_1^2+1)x_1^4+(\lambda_1^2+0.5)x_2^4}\le 7 \tag{3.55}$$

$$\lambda_2(2x_1^2+3x_2^2+4)-x_1^2-x_2^2+1.28\sqrt{(\lambda_2^2+0)x_1^4+(\lambda_2^2+0.5)x_2^4}\le 2 \tag{3.56}$$

Let λ =min{1.250, 0.444}=0.444 at (x_1, x_2) = (1,1) form (3.41)

Therefore, inequalities (3.55)-(3.56) reduces to (3.57)-(3.58)

$$-0.112x_1^2-1.112x_2^2-1.28\sqrt{1.197x_1^4+0.697x_2^4}\le 5.224 \tag{3.57}$$

$$-0.112x_1^2+0.332x_2^2+1.28\sqrt{0.198x_1^4+0.697x_2^4}\le 0.224 \tag{3.58}$$

Using the SLP Charles and Dutta (2003), the following linear constraints are obtained:

$$-2.451x_1-3.521x_2\le 2.238$$

$$0.309x_1+2.550x_2\le 1.653$$

The deterministic equivalent of stochastic constraints is given below:

$$2x_1+x_2+1.645\sqrt{x_1^2+x_2^2}\le 1$$

$$3x_1+4x_2+0.84\sqrt{2x_1^2+3x_2^2+2}\le 3$$

Using the SLP Charles and Dutta (2003), the following linear constraints are obtained:

$$3.163x_1 + 2.163x_2 \leq 1$$

$$3.635x_1 + 4.952x_2 \leq 2.365$$

$$16x_1 + x_2 \leq 4$$

Solution to the system (3.54) with deterministic constraints is $x_1 = 0.000$, $x_2 = 0.462$, $\lambda_1 = 1.545$, $\lambda_2 = 0.419$ and the corresponding value of objective functions in system (3.54) is $[1.678, 0.477]$.

Solution using Redundancy algorithm is given below.

Table 3.7 Matrix-of-Intercept for Example 3.9

	Objective functions			Constraints			
Slack/Decision Variables	S_1	S_2	Ψ_j^{obj}	S_3	S_4	S_5	Ψ_j^{const}
x_1	**-0.913**	5.350	-0.913	0.316	0.651	**0.250**	0.250
x_2	**-0.636**	0.648	-0.636	**0.462**	0.478	4	0.462

From the Table 3.7, it can be inferred that the constraint due to second objective function is strongly redundant and second constraints is strongly redundant. Therefore, ignoring the second objective function and second constraint from the original problem, the problem is then solved. The bi-objective stochastic fractional programming problem reduces to an SFP problem as follows:

$$\text{Max } 0.90\lambda_1 \tag{3.59}$$

subject to $\lambda_1(2x_1^2+2x_2^2+4)-x_1^2-2x_2^2-1.28\sqrt{(\lambda_1^2+1)x_1^4+(\lambda_1^2+0.5)x_2^4}\le 7$

$$2x_1+x_2+1.645\sqrt{x_1^2+x_2^2}\le 1$$

$$16x_1+x_2\le 4$$

$$x_1,x_2,\lambda_1\ge 0.$$

The solution is obtained as $x_1=0.000$, $x_2=0.462$, $\lambda_1=1.545$. The corresponding value of objective functions in equation (3.54) is $[1.678, 0.477]$. Here, it is to be noted that 90% of importance is given to the first objective function.

3.4.8 Summary

It may be observed from all the illustrative examples given in section 3.2.6 that the proposed Direct method has taken consistently lesser number of iterations/no iteration than the Simplex method. It is hoped that the efficiency will improve in large scale problems. The method presented in section 3.3 identifies redundant stochastic fractional objective function(s). The method has been developed with the intention of solving MONLSFP problems and in the process redundant stochastic objective functions are identified and removed from the original problem, provided redundancy exists. The method illustrated in section 3.4 has its uniqueness i.e., the novelty of this method is the simultaneous identification of redundant objective function(s) and constraint(s) in MONLSFP problems. In summary, the attraction of all the three proposed algorithms lies in its simplicity of operation. A downsized problem would involve lesser computational effort.

Proceedings of Ph.D. Viva-voce Examination of Mr. A. Udhayakumar held at 11.00 a.m. on 15.12.2010 in Ramanujan Computing Centre, Anna University, Chennai-600025.

The Ph.D Viva-Voce examination of **Mr. A. Udhayakumar** (Reg. No.: 2003719745) on his Ph.D thesis entitled, "APPROACHES FOR SOME SPECIAL CLASSES OF STOCHASTIC OPTIMIZATION PROBLEMS" was conducted on 15.12.2010 at 11.00 a.m in the Ramanujan Computing Centre, Anna University, Chennai-600025.

The following members of the Oral Examination Board were present:

1. Dr. C. Rajendran - Indian Examiner
2. Dr. R. Nadarajan - Expert Examiner
3. Dr. V. Rhymend Uthariaraj - Supervisor/Convenor

The candidate Mr. A. Udhayakumar presented the salient features of his Ph.D work. This was followed by questions from the board members. The queries and clarifications raised by the Foreign and Indian examiners were also put to the candidate. The candidate answered the questions to the fullest satisfaction of the board members.

The corrections and suggestions pointed out by the Indian/Foreign Examiners have been carried out and duly incorporated in the thesis before the Oral Examination.

Based on the candidate's research work, his presentation and also the clarifications and answers by the candidate to the questions, the board recommends that Mr. A. Udhayakumar be awarded the Ph.D. degree in the Faculty of Science and Humanities.

(Dr. C. Rajendran) **(Dr. R. Nadarajan)**

(Dr. V. Rhymend Uthariaraj)