1 SOLUTION

1.1 Single Period (Model 1) Solution

For the models developed in Chapter 4, an attempt is made to obtain a closed form solution analytically. The following assumptions are made as assumed in Burke et al. (2009).

A1: The demand distribution is assumed to be uniformly distributed with parameters \([a, b]\), where \(a\) and \(b\) denote the finite lower and upper bounds of the distribution.

A2: The yield distribution for each of the suppliers has finite mean and variance.

It is first shown that the expected profit function is concave in the decision variables and then the unconstrained and the constrained solutions are derived. For the calculations, \(k\) denotes the indices for the regular suppliers, i.e. \(k = 1,2,\cdots,n\) and zero denotes the index for the back-up supplier.

1.1.1 Concavity of the Expected Profit Function

To identify a global maximum of the expected profit function, the concavity of the objective function of Model 1 given in problem (P1) with respect to the decision variables has been examined.

**Result 1:** The expected total profit function for single period as given in equation (1) is concave with respect to the decision variables \(\{q_i, \forall i = 0,1,2,\cdots,n\}\) under assumptions A1 and A2.

**Proof:** To show the concavity of the expected total profit function with respect to the decision variables, the Hessian matrix is constructed. To construct the Hessian matrix the derivatives of
the expected total profit function with respect to the decision variables are calculated as shown below.

\[
\Pi = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -c_0 q_0 - \sum_{k=1}^{n} c_k r_k q_k + \int_{0}^{Q} \left( p z - h \left( Q - \frac{z}{2} \right) + S(Q - z) \right) f(z) \, dz + \int_{Q}^{Q+q_0} \left( p z - h \left( \frac{Q}{2} \right) - c_0 (z - Q) \right) f(z) \, dz + \int_{Q+q_0}^{\infty} \left( p(Q + q_0) - h \left( \frac{Q}{2} \right) - c_0 q_0 - u(z -(Q + q_0)) \right) f(z) \, dz \right] \, dr_i
\]

\[
= \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -c_0 q_0 - \sum_{k=1}^{n} c_k r_k q_k + l_1 + l_2 + l_3 \right] \, dr_i
\]

The third, fourth and the fifth term of the integral are denoted by \( l_1, l_2 \) and \( l_3 \). Therefore,

\[
\frac{\partial \Pi}{\partial q_k} = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -c_k r_k + \frac{\partial l_1}{\partial q_k} + \frac{\partial l_2}{\partial q_k} + \frac{\partial l_3}{\partial q_k} \right] \, dr_i
\]

Now, using Leibnitz rule,

\[
\frac{\partial l_1}{\partial q_k} = \frac{\partial l_1}{\partial Q} \frac{\partial Q}{\partial q_k} = \left[ (pQ - hQ) f(Q) + (-h + S)F(Q) \right] \frac{\partial Q}{\partial q_k}
\]

\[
\frac{\partial l_2}{\partial q_k} = \left[ (p(Q + q_0) - hQ - c_0 q_0) f(Q + q_0) - \left( pQ - \frac{hQ}{2} \right) f(Q) + \left( \frac{h}{2} + c_0 \right) \left( F(Q + q_0) - F(Q) \right) \right] \frac{\partial Q}{\partial q_k}
\]
\[
\frac{\partial I_3}{\partial q_k} = \left[-\left(p(Q + q_0) - \frac{hQ}{2} - c_0q_0\right)f(Q + q_0) + \left(p - \frac{h}{2} + u\right)(1 - F(Q + q_0))\right]\frac{\partial Q}{\partial q_k}
\]

Therefore,

\[
\frac{\partial \Pi}{\partial q_k} = \prod_{i=1}^{n} \int_0^1 g_i(r_i) r_k \left[-c_k + (-h + S)F(Q) + \left(-\frac{h}{2} + c_0\right)(F(Q + q_0) - F(Q)) + \left(p - \frac{h}{2} + u\right)(1 - F(Q + q_0))\right]dr_i
\]

\[
= \prod_{i=1}^{n} \int_0^1 g_i(r_i) r_k \left[-\frac{h}{2} - c_0 + S\right]F(Q) + (c_0 - p - u)F(Q + q_0) + \left(p - \frac{h}{2} + u - c_k\right)\right]dr_i
\]

Assuming that the demand distribution is uniform with parameters \([a, b]\),

\[
\frac{\partial \Pi}{\partial q_k} = \prod_{i=1}^{n} \int_0^1 g_i(r_i) r_k \left[-\frac{h}{2} - c_0 + S\right]F(Q) + (c_0 - p - u)F(Q + q_0) + \left(p - \frac{h}{2} + u - c_k\right)\right]dr_i
\]

\[
= \prod_{i=1}^{n} \int_0^1 g_i(r_i) r_k \left[-\frac{p + \frac{h}{2} + u - S}{b - a}\left(\sum_{i=1}^{n} r_i q_i + x - a\right) + \frac{(c_0 - p - u)\sum_{i=1}^{n} r_i q_i + q_0 + x - a}{b - a}\right]dr_i
\]

\[
\quad + \left(p - \frac{h}{2} + u - c_k\right)\right]dr_i
\]

Therefore,

\[
\frac{\partial^2 \Pi}{\partial q_0 \partial q_k} = \frac{-(p + u - c_0)}{b - a}r_k
\]
\[ \frac{\partial^2 \Pi}{\partial q_k^2} = \frac{-(p + \frac{h}{2} + u - S)}{b - a} (r_k^{-2} + \sigma_k^{-2}) \]

\[ \left. \frac{\partial^2 \Pi}{\partial q_m \partial q_k} \right|_{m \neq k} = \frac{-(p + \frac{h}{2} + u - S)}{b - a} \frac{1}{r_m r_k} \]

Again,

\[ \frac{\partial \Pi}{\partial q_0} = \prod_{i=1}^{n} \int_0^1 g_i(r_i) \left[ -c'_0 + \frac{\partial I_2}{\partial q_0} + \frac{\partial I_3}{\partial q_k} \right] dr_i \]

Using Leibnitz rule,

\[ \frac{\partial I_2}{\partial q_0} = \left( p(Q + q_0) - \frac{hq_0}{2} - c_0 q_0 \right) f(Q + q_0) \]

\[ \frac{\partial I_3}{\partial q_k} = - \left( p(Q + q_0) - \frac{hq_0}{2} - c_0 q_0 \right) f(Q + q_0) + (p + u - c_0)(1 - F(Q + q_0)) \]

Therefore,

\[ \frac{\partial \Pi}{\partial q_0} = \prod_{i=1}^{n} \int_0^1 g_i(r_i) \left[ -c'_0 + (p + u - c_0)(1 - F(Q + q_0)) \right] dr_i \]

Again, assuming that the demand is distributed uniformly with parameters \([a, b]\),

\[ \frac{\partial \Pi}{\partial q_0} = \prod_{i=1}^{n} \int_0^1 g_i(r_i) \left[ -c'_0 + (p + u - c_0) \left( 1 - \frac{\sum_{i=1}^{n} r_i q_i + q_0 + x - a}{b - a} \right) \right] dr_i \]
Therefore, the Hessian Matrix is

\[
= \prod_{t=1}^{n} \int_{0}^{1} g_{i}(r_{i}) \left[ -\frac{(p + u - c_{0})}{b - a} \left( \sum_{t=1}^{n} r_{i}q_{t} + q_{0} + x - a \right) 
+ (p - c_{0} + u - c_{0}') \right] dr_{i}
\]

Therefore,

\[
\frac{\partial^{2} \Pi}{\partial q_{0}^{2}} = -\frac{(p + u - c_{0})}{b - a}
\]

\[
\frac{\partial^{2} \Pi}{\partial q_{k}\partial q_{0}} = -\frac{(p + u - c_{0})}{b - a} r_{k}
\]

Hence, the Hessian Matrix is,

\[
H = \frac{(p + \frac{h}{2} + u - S)}{b - a} \begin{bmatrix}
-M_{0} & -M_{0}r_{1} & -M_{0}r_{2} & \cdots & -M_{0}r_{n} \\
-M_{0}r_{1} & -M_{1} & -r_{1}r_{2} & \cdots & -r_{1}r_{n} \\
-M_{0}r_{2} & -r_{2}r_{1} & -M_{2} & \cdots & -r_{2}r_{n} \\
\cdots & \cdots & \cdots & \ddots & \cdots \\
-M_{0}r_{n} & -r_{n}r_{1} & -r_{n}r_{2} & \cdots & -M_{n}
\end{bmatrix}
\]

where,

\[
M_{0} = \frac{p + u - c_{0}}{p + \frac{h}{2} + u - S}
\]

\[
M_{k} = r_{k}^{2} + \sigma_{k}^{2} \quad \text{for} \quad k = 1, 2, \cdots, n
\]

It may be noted that, \(M_{k}\) and \(M_{0}\) will always have positive values. Since in many practical situations, the salvage value is less than the payment made to the back-up supplier at the time of delivery, i.e. \(S < c_{0}\), therefore, the value of \(M_{0}\) is less than unity.

The determinant of the Hessian matrix is,
Since the value of \( \frac{p + \frac{h}{2} + u - S}{b - a} \) is less than unity, \( M_0 \) will always have a positive value. Therefore, the signs of the principal minors of the Hessian matrix will be \( \{1, 1\} \). Hence it is concluded that the Hessian matrix is negative semi-definite. Therefore, the expected profit function as given in equation (1) is concave with respect to the decision variables when the demand is uniformly distributed over \([a, b]\).

\[
\text{det}(H) = (-1)^{n+1} \frac{(p + \frac{h}{2} + u - S)}{b - a} \begin{vmatrix}
M_0 & M_0 r_1 & M_0 r_2 & \cdots & M_0 r_n \\
M_0 r_1 & M_1 & r_1 r_2 & \cdots & r_1 r_n \\
M_0 r_2 & r_2 r_1 & M_2 & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M_0 r_n & r_n r_1 & \cdots & \cdots & M_n
\end{vmatrix}
\]

\[
= (-1)^{n+1} \frac{(p + \frac{h}{2} + u - S)}{b - a} \left[ M_0 \left( \left( r_1^{-2} + \sigma_1^{2}\right) \left( r_2^{-2} + \sigma_2^{2}\right) \cdots \left( r_n^{-2} + \sigma_n^{2}\right) - r_1^{-2} r_2^{-2} \sigma_3^{2} \sigma_4^{2} \cdots \sigma_n^{2} \right.ight.
\]
\[
\left. - \cdots - r_1^{-2} r_2^{-2} r_3^{-2} \sigma_4^{2} \cdots \sigma_n^{2} - \cdots - r_1^{-2} r_2^{-2} r_3^{-2} \cdots \sigma_n^{2}\right)\right]
\]
\[
- M_0 \left( r_1^{-2} \sigma_2^{2} \sigma_3^{2} \sigma_4^{2} \cdots \sigma_n^{2} + r_1^{-2} r_2^{-2} \sigma_3^{2} \sigma_4^{2} \cdots \sigma_n^{2} + \right.
\]
\[
+ \sigma_1^{2} \sigma_2^{2} \sigma_3^{2} \sigma_4^{2} \cdots \sigma_{n-1}^{2} \right)\]
\]
\[
= (-1)^{n+1} \frac{(p + \frac{h}{2} + u - S)}{b - a} \left[ (M_0 - M_0^2) \sum_{i=1}^{n} \frac{r_i}{r_i} \prod_{j \neq i}^{n} \sigma_j^{2} + M_0 \prod_{i=1}^{n} \sigma_i^{2}\right]
\]

Since the value of \( M_0 \) is less than unity, \( (M_0 - M_0^2) \sum_{i=1}^{n} \frac{r_i}{r_i} \prod_{j \neq i}^{n} \sigma_j^{2} + M_0 \prod_{i=1}^{n} \sigma_i^{2} \) will always have a positive value. Therefore, the signs of the principal minors of the Hessian matrix will be \( (-1)^{l+1} \), \( l = 0,1,2,\cdots, n \). Hence it is concluded that the Hessian matrix is negative semi-definite. Therefore, the expected profit function as given in equation (1) is concave with respect to the decision variables when the demand is uniformly distributed over \([a, b]\).

### 1.1.2 Unconstrained Solution

Presuming that the assumptions A1 and A2 holds true, to find the solution of the non-linear optimization problem as given in constrained problem P4, first the unconstrained solution is obtained for the unconstrained problem P1. Now, as the concavity of the objective function about the decision variables has been established, the Karush-Kuhn-Tucker conditions for the unconstrained maximization problem reduce to:
\[
\frac{\partial \Pi}{\partial q_k} = 0
\]  

(4)

And,

\[
\frac{\partial \Pi}{\partial q_0} = 0
\]  

(3)

Using equations (4) and (5),

\[
\frac{\partial \Pi}{\partial q_0} = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ \frac{-(p + u - c_0)}{b - a} \left( \sum_{l=1}^{n} r_i q_l + q_0 + x - a \right) + (p - c_0 + u - c_0') \right] dr_i = 0
\]

\[
\Rightarrow q_0 + \bar{r}_1 q_1 + \bar{r}_2 q_2 + \cdots + \bar{r}_n q_n = \frac{p - c_0 + u - c_0'}{p + u - c_0} (b - a) - (x - a)
\]

\[
= K_0 (b - a) - (x - a)
\]

\[
\Rightarrow q_0 + \bar{r}_1 q_1 + \bar{r}_2 q_2 + \cdots + \bar{r}_n q_n = \beta_0
\]  

(5)

where, \( \beta_0 = K_0 (b - a) - (x - a) \) and \( K_0 = \frac{p - c_0 - c_0' + u}{p - c_0 + u} \). It can be noted that \( \beta_0 \) is specific to back-up supplier and does not depend on the number of suppliers considered.

Using equations (3) and (6),

\[
\frac{\partial \Pi}{\partial q_k} = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) r_k \left[ \frac{-(p + \frac{h}{2} + u - S)}{b - a} \left( \sum_{l=1}^{n} r_i q_l + x - a \right) + \frac{-(p + u - c_0)}{b - a} q_0 \right]

\[
\quad + \left( p - \frac{h}{2} + u - c_k \right) \right] dr_i = 0
\]
After rearranging the terms, we obtain,

\[
\prod_{i=1}^{n} \int_0^1 g_i(r_i) r_k \left[ \left( \sum_{l=1}^{n} \eta_l q_l + x - a \right) + \frac{p + u - c_0}{p + \frac{h}{2} + u - S} q_0 - \frac{p - h + u - c_k}{p + \frac{h}{2} + u - S} (b - a) \right] dr_i = 0
\]

\[
\prod_{i=1}^{n} \int_0^1 g_i(r_i) r_k \left[ \left( \sum_{l=1}^{n} \eta_l q_l + x - a \right) + \frac{p + u - c_0}{p + \frac{h}{2} + u - S} q_0 \right] dr_i = \bar{r}_k \frac{p - \frac{h}{2} + u - c_k}{p + \frac{h}{2} + u - S} (b - a)
\]

\[
\Rightarrow \bar{r}_1 \bar{r}_k q_1 + \bar{r}_2 \bar{r}_k q_2 + \cdots + (\bar{r}_k^2 + \sigma_k^2) q_k + \cdots + \bar{r}_n \bar{r}_k q_n + K' \bar{r}_k q_0 = \bar{r}_k [K_k (b - a) - (x - a)]
\]

\[
= \bar{r}_k \beta_k
\]

After rearranging the terms, we obtain,

\[
K' \bar{r}_k q_0 + \bar{r}_1 \bar{r}_k q_1 + \bar{r}_2 \bar{r}_k q_2 + \cdots + M_k q_k + \cdots + \bar{r}_n \bar{r}_k q_n = \bar{r}_k \beta_k
\]

(6)

where, \( \beta_k = K_k (b - a) - (x - a), (k = 1, 2, \cdots, n), \quad K_k = \frac{p - h + u - c_k}{p + \frac{h}{2} + u - S}, (k = 1, 2, \cdots, n) \) and \( K' = \frac{p + u - c_0}{p + \frac{h}{2} + u - S} \) and \( M_k = \bar{r}_k^2 + \sigma_k^2, (k = 1, 2, \cdots, n) \). It can be noted that \( \beta_k \) is specific to supplier \( k \) and does not depend on the number of suppliers considered.

Therefore the set of simultaneous linear equations presented in equations (7) and (8) can be represented in the matrix form as,

\[
\begin{bmatrix}
1 & \bar{r}_1 & \bar{r}_2 & \cdots & \bar{r}_n \\
K' \bar{r}_1 & M_1 & \bar{r}_2 & \cdots & \bar{r}_n \\
K' \bar{r}_2 & \bar{r}_2 \bar{r}_1 & M_2 & \cdots & \bar{r}_n \bar{r}_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
K' \bar{r}_n & \bar{r}_n \bar{r}_1 & \bar{r}_n \bar{r}_2 & \cdots & M_n
\end{bmatrix}
\begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
\vdots \\
q_n
\end{bmatrix}
= \begin{bmatrix}
\beta_0 \\
\bar{r}_1 \beta_1 \\
\bar{r}_2 \beta_2 \\
\vdots \\
\bar{r}_n \beta_n
\end{bmatrix}
\]

(7)

The solution of this set of simultaneous equations serves as an unconstrained optimum solution for the problem.
The determinant of the characteristic matrix of equation (9) can be simplified to:

\[ D = \begin{vmatrix}
1 & \bar{r}_1 & \bar{r}_2 & \cdots & \bar{r}_n \\
K'\bar{r}_1 & M_1 & \bar{r}_2 \bar{r}_1 & \cdots & \bar{r}_1 \bar{r}_n \\
K'\bar{r}_2 & \bar{r}_2 \bar{r}_1 & M_2 & \cdots & \bar{r}_2 \bar{r}_n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
K'\bar{r}_n & \bar{r}_n \bar{r}_1 & \bar{r}_n \bar{r}_2 & \cdots & M_n
\end{vmatrix} \]

\[ = \begin{vmatrix}
1 & \bar{r}_1 & \bar{r}_2 & \cdots & \bar{r}_n \\
K' & \bar{r}_1^2 + \sigma_1^2 & \bar{r}_1 \bar{r}_2 & \cdots & \bar{r}_1 \bar{r}_n \\
K' & \bar{r}_2 \bar{r}_1 & \bar{r}_2^2 + \sigma_2^2 & \cdots & \bar{r}_2 \bar{r}_n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
K' & \bar{r}_n \bar{r}_1 & \bar{r}_n \bar{r}_2 & \cdots & \bar{r}_n^2 + \sigma_n^2
\end{vmatrix} \]

\[ = \left( \bar{r}_1^2 + \sigma_1^2 \right) \left( \bar{r}_2^2 + \sigma_2^2 \right) \cdots \left( \bar{r}_n^2 + \sigma_n^2 \right) - \bar{r}_1 \bar{r}_2 \bar{r}_3^2 \sigma_3 \sigma_4 \cdots \sigma_n - \cdots - \bar{r}_1 \bar{r}_2 \bar{r}_n^2 \sigma_3 \sigma_4 \cdots \sigma_{n-1} - \cdots \]

\[ - \bar{r}_1 \bar{r}_2 \bar{r}_3^2 \cdots \bar{r}_n^2 - \bar{r}_1 K' \bar{r}_1 \sigma_1^2 \sigma_3 \cdots \sigma_n^2 - \bar{r}_2 K' \sigma_1^2 \sigma_2 \sigma_3 \cdots \sigma_n^2 - \cdots \]

\[ - \bar{r}_n K' \sigma_1^2 \sigma_2 \sigma_3^2 \cdots \sigma_{n-1} \bar{r}_n \]

\[ = \sigma_1^2 \sigma_2^2 \sigma_3^2 \cdots \sigma_n^2 \]

\[ + (1 - K') \left( \bar{r}_1^2 \sigma_2 \sigma_3 \cdots \sigma_n^2 + \sigma_1^2 \bar{r}_2^2 \sigma_3 \cdots \sigma_n^2 + \cdots + \sigma_1^2 \sigma_2^2 \sigma_3 \cdots \sigma_{n-1}^2 \bar{r}_n^2 \right) \]

\[ = (1 - K') \sum_{i=1}^{n} \left( \bar{r}_i^2 \prod_{j \neq i} \sigma_j^2 \right) + \prod_{i=1}^{n} \sigma_i^2 \]

It should be noted that the determinant of the characteristic matrix is non-zero. Therefore, the characteristic matrix is non-singular and there exists a solution to the unconstrained problem P1.

Therefore, using Kramer’s Rule, to get the value of \( q_0^* \), we replace the 1\(^{st} \) column of the characteristic matrix with RHS and calculate \( q_0^* \) as:
Now, the numerator can be expanded as:

\[
\begin{align*}
= \beta_0 \left( \bar{r}_1^2 + \sigma_1^2 \right) \left( \bar{r}_2^2 + \sigma_2^2 \right) \cdots \left( \bar{r}_n^2 + \sigma_n^2 \right) - \beta_0 \bar{r}_1^2 \bar{r}_2^2 \sigma_3^2 \sigma_4^2 \cdots \sigma_n^2 - \\
- \beta_0 \bar{r}_1^2 \bar{r}_2^2 \bar{r}_3^2 \sigma_4^2 \cdots \sigma_n^2 - \cdots - \beta_0 \bar{r}_1^2 \bar{r}_2^2 \bar{r}_3^2 \cdots \bar{r}_n^2 - \bar{r}_1 \beta_1 \bar{r}_1 \sigma_2^2 \sigma_3^2 \cdots \sigma_n^2 \\
- \bar{r}_2 \beta_2 \sigma_1^2 \bar{r}_2 \sigma_3^2 \cdots \sigma_n^2 - \cdots - \bar{r}_n \beta_n \sigma_1^2 \sigma_2^2 \sigma_3^2 \cdots \sigma_{n-1}^2 \bar{r}_n \\
= \beta_0 \sigma_1^2 \sigma_2^2 \sigma_3^2 \cdots \sigma_n^2 + \sum_{i=1}^{n} \left( \beta_0 - \beta_i \right) \bar{r}_i^2 \prod_{j \neq i} \sigma_j^2
\end{align*}
\]

Therefore,

\[
q_0^* = \frac{\left( \sum_{i=1}^{n} (\beta_0 - \beta_i) \bar{r}_i^2 \prod_{j \neq i} \sigma_j^2 \right) + \beta_0 \prod_{i=1}^{n} \sigma_i^2}{(1 - K') \left( \sum_{i=1}^{n} \bar{r}_i^2 \prod_{j \neq i} \sigma_j^2 \right) + \prod_{i=1}^{n} \sigma_i^2} = \frac{\left( \sum_{i=1}^{n} (\beta_0 - \beta_i) V_i^2 \right) + \beta_0}{(1 - K') \left( \sum_{i=1}^{n} V_i^2 \right) + 1} \quad (8)
\]

Where \( V_i = \frac{\bar{r}_i}{\sigma_i} \) is the inverse of the co-efficient of variation for the yield of the \( i^{th} \) supplier.

Also, to get the value of \( q_k^* \), we replace the \((k + 1)^{th}\) column of the characteristic matrix with RHS and calculate \( q_k^* \) as:

\[
q_k^* = \frac{\left( \sum_{i=1}^{n} (\beta_0 - \beta_i) \bar{r}_i^2 \prod_{j \neq i} \sigma_j^2 \right) + \beta_0 \prod_{i=1}^{n} \sigma_i^2}{(1 - K') \left( \sum_{i=1}^{n} \bar{r}_i^2 \prod_{j \neq i} \sigma_j^2 \right) + \prod_{i=1}^{n} \sigma_i^2} = \frac{\left( \sum_{i=1}^{n} (\beta_0 - \beta_i) V_i^2 \right) + \beta_0}{(1 - K') \left( \sum_{i=1}^{n} V_i^2 \right) + 1}
\]
Again, the numerator can be expanded as,

\[
\begin{align*}
&\left(\bar{r}_1^2 + \sigma_1^2\right) \left(\bar{r}_2^2 + \sigma_2^2\right) \cdots \left(\bar{r}_{k-1}^2 + \sigma_{k-1}^2\right) \bar{r}_k \beta_k \left(\bar{r}_{k+1}^2 + \sigma_{k+1}^2\right) \cdots \left(\bar{r}_n^2 + \sigma_n^2\right) \\
&\quad - \bar{r}_1^2 \bar{r}_2^2 \sigma_3^2 \sigma_4^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - \bar{r}_1^2 \bar{r}_2^2 \bar{r}_3^2 \sigma_4^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - \cdots \\
&\quad - \bar{r}_1^2 \bar{r}_2^2 \bar{r}_3^2 \cdots \bar{r}_{k-1} \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - \bar{r}_1 \bar{r}_2 \bar{r}_3 \sigma_2^2 \sigma_4^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - \cdots \\
&\quad - \bar{r}_1 \bar{r}_2 \bar{r}_3 \sigma_2^2 \sigma_3^2 \sigma_4^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - \cdots - \bar{r}_1 \bar{r}_2 \bar{r}_3 \bar{r}_4 \sigma_3^2 \sigma_4^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - \cdots \\
&\quad - K' \bar{r}_1 \bar{r}_2 \sigma_2^2 \sigma_3^2 \sigma_4^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - K' \bar{r}_2 \bar{r}_3 \sigma_2^2 \sigma_3^2 \sigma_4^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - \cdots \\
&\quad - K' \bar{r}_n \bar{r}_1 \sigma_2^2 \sigma_3^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - K' \bar{r}_n \bar{r}_2 \sigma_2^2 \sigma_3^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - \cdots \\
&\quad + K' \bar{r}_n \bar{r}_1 \sigma_2^2 \sigma_3^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - K' \bar{r}_n \bar{r}_2 \sigma_2^2 \sigma_3^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - \cdots \\
&\quad + K' \bar{r}_n \bar{r}_2 \sigma_2^2 \sigma_3^2 \cdots \sigma_{k-1}^2 \bar{r}_k \beta_k \sigma_{k+1}^2 \cdots \sigma_n^2 - \cdots - \beta_0 K' \bar{r}_k \sigma_2^2 \sigma_3^2 \cdots \sigma_{k-1}^2 \sigma_{k+1}^2 \cdots \sigma_{n-1}^2 \\
&\quad = (1 - K') \left( \sum_{i=1}^{n} \bar{r}_i (\beta_k - \beta_i) \bar{r}_i \prod_{j \neq i, j \neq k} \sigma_j^2 \right) + (\beta_k - \beta_0 K') \bar{r}_k \prod_{i \neq k} \sigma_i^2
\end{align*}
\]

After simplification,

\[
q^*_k = \frac{V_k}{\sigma_k} \left[ (1 - K') \left( \sum_{i=1}^{n} \beta_i \sigma_i^2 \right) \right] + \frac{(\beta_k - \beta_0 K')}{(1 - K') \left( \sum_{i=1}^{n} \sigma_i^2 \right) + 1}
\]

Equations (10) and (11) gives the optimal solution values for \( \{q_i^*, i = 0, 1, 2, \cdots, n\} \) to the unconstrained problem P1.
1.1.3 Constrained Solution

To construct the Lagrangian for the constrained problem (P4), the slack variables $s_i$ corresponding to the non-negativity constraint (C4.1) of the order quantities to each of the suppliers are added which must satisfy,

$$q_i - s_i^2 = 0 \forall i = 0, \ldots, n.$$

Also, slack variable $s$ corresponding to the maximum back-up constraint (C4.2) is added which must satisfy the following condition:

$$q_0 + s^2 = B$$

The Lagrangian function is:

$$L = \Pi - \sum_{i=0}^{n} \lambda_i (q_i - s_i^2) - \lambda (q_0 + s^2 - B)$$

where $\lambda_i$’s denote the Lagrangian constants corresponding to the non-negativity constraints of each of the order quantities and $\lambda$ denotes the Lagrangian constant corresponding to the maximum limit to the back-up supplier constraint. The optimality conditions are:

$$\frac{\partial L}{\partial q_k} = 0 \forall k = 0, 1, 2, \ldots, n$$

$$\Rightarrow \begin{cases} \frac{\partial \Pi}{\partial q_0} - \lambda_0 - \lambda = 0 \\ \frac{\partial \Pi}{\partial q_k} - \lambda_k = 0 \forall k = 1, 2, \ldots, n \end{cases}$$
If the unconstrained solution to the problem P1 obtained in equation (10) and (11) satisfies the constraints (C4.1) and (C4.2), then the unconstrained solution to the problem P1 is an optimal solution to the constrained problem P4.

1.1.4 Non-negativity Constraint

**Result 2:** Suppose an optimal solution to the unconstrained problem P1 satisfies the maximum back-up constraint of the problem P4 but the non-negativity constraint is violated. An optimal solution to the constrained problem P4 can be obtained from the optimal solution to the unconstrained problem.

**Proof:** Let an optimal solution to the unconstrained problem be denoted by \( \{q^*_j; j = 0,1,\ldots,n\} \) of which some of the \( q^*_j \)'s are negative.

Let \( V \) denotes the set of suppliers for which \( q^*_j \) is positive, and \( V' \) denotes the complementary set. Therefore,

\[
\frac{\partial L}{\partial \lambda_k} = 0 \forall k = 0,1,2,\ldots,n \tag{11}
\]

\[\Rightarrow q_k - s_k^2 = 0
\]

\[
\frac{\partial L}{\partial s_k} = 0 \forall k = 0,1,2,\ldots,n \tag{12}
\]

\[\Rightarrow 2\lambda_k s_k = 0
\]

\[
\frac{\partial L}{\partial \lambda} = 0 \tag{13}
\]

\[\Rightarrow q_0 + s^2 - B = 0
\]

\[
\frac{\partial L}{\partial s} = 0 \tag{14}
\]

\[\Rightarrow 2\lambda s = 0
\]
Let us define a restricted problem $P_1'$ which considers only those suppliers who belong to the set $V$, i.e. the problem statement of the restricted problem $P_1'$ is:

- **Restricted Problem $P_1'$:**

  Maximize $\Pi_{R1}$ for all $q_j$ such that $j \in V$ \hspace{1cm} (P1')

where, $\Pi_{R1}$ is the expected total profit of the firm for the restricted problem. Now, under the assumption of uniformly distributed demand, the expression of $\Pi$ can be simplified to:

\[
\Pi = -p \frac{b+a}{2} - \frac{(r + \frac{h}{2} + p - S)a^2}{2(b - a)} + \sum_i (r - \frac{h}{2} + p - c_i) \bar{r}_i q_i + \frac{(r + \frac{h}{2} + p - S)}{b - a} \sum_i \bar{r}_i q_i \\
- \frac{(r - c_0 + p)}{b - a} q_0 \sum_i \bar{r}_i q_i + (r - \frac{h}{2} + p)x + \frac{(r + \frac{h}{2} + p - S)a}{b - a} x \\
- \frac{(r - c_0 + p)}{b - a} q_0 x - \frac{(r - c_0 + p)q_0^2}{2(b - a)} + (r - c_0 - c_0' + p)q_0 \\
+ \frac{(r - c_0 + p)}{b - a} aq_0 \\
- \frac{(r + \frac{h}{2} + p - S)}{2(b - a)} \left( \sum_i M_i q_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \bar{r}_i \bar{r}_j q_i q_j + x^2 + 2x \sum_i \bar{r}_i q_i \right)
\]

Similarly, $\Pi_{R1}$ is the expected profit function with those suppliers who belong to set $V$. It can be observed from the expression of $\Pi$ that, $\Pi_{R1} = \Pi|_{q_i=0} \forall i \in V'$.

The unconstrained solution to the restricted problem $P1'$, $q_{j^*}^r$, is arrived at using the same method which was used to identify the unconstrained solution to the unconstrained problem $P1$. Therefore, when the order quantity back-up supplier belongs to set $V$:
\[ q^*_0 = \frac{\left( \sum_{i=1; i \in V}^n (\beta_0 - \beta_i) V_i^2 \right) + \beta_0}{(1 - K') \left( \sum_{i=1; i \in V}^n V_i^2 \right) + 1} \]

\[ q^*_j = \frac{V_j}{\sigma_j} \left[ \frac{(1 - K') \left( \sum_{i=1; i \in V; i \neq j}^n (\beta_j - \beta_i) V_i^2 \right) + (\beta_j - \beta_0 K')}{(1 - K') \left( \sum_{i=1; i \in V}^n V_i^2 \right) + 1} \right]; \ j \neq 0 \]

When the back-up supplier does not belong to set \( V \):

\[ q^*_j = \frac{V_j}{\sigma_j} \left[ \frac{(\sum_{i=1; i \in V; i \neq j}^n (\beta_j - \beta_i) V_i^2) + \beta_j}{(\sum_{i=1; i \in V}^n V_i^2) + 1} \right]; \ j \neq 0 \]

**Result 2.1:** The indices of the regular suppliers who belong to set \( V' \) are greater than the indices of the regular suppliers who belong to set \( V \).

**Proof:** It can be noted that the regular suppliers are indexed in a non-decreasing order of cost quoted by the suppliers, i.e. \( c_1 \leq c_2 \leq \cdots \leq c_n \). Now, the beta-values for each of the regular suppliers are expressed as \( \beta_k = K_k (b - a) - (x - a), (k = 1, 2, \ldots, n) \), where \( K_k = \frac{p-h+u-c_k}{p-h+u-S} \).

Therefore, it can be concluded that \( \beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \).

Let us assume that, a regular supplier indexed by \( M \), who has quoted a cost \( c_M \), belonged to the set \( V' \) after the restriction is done. This happens only if that supplier receives a zero or negative order in the solution to the unconstrained problem \( P_1 \), i.e.

\[ q^*_M = \frac{V_M}{\sigma_M} \left[ \frac{(1 - K') \left( \sum_{i=1; i \neq M}^n (\beta_M - \beta_i) V_i^2 \right) + (\beta_M - \beta_0 K')}{(1 - K') \left( \sum_{i=1}^n V_i^2 \right) + 1} \right] \leq 0 \]

\[ \Rightarrow (1 - K') \left( \sum_{i=1; i \neq M}^n (\beta_M - \beta_i) V_i^2 \right) + (\beta_M - \beta_0 K') \leq 0 \]
Again, let us assume that there exists a regular supplier indexed by \( L \) who has quoted a cost greater than or equal to that by Supplier \( M \), i.e. \( c_L \geq c_M \). This implies that \( \beta_L \leq \beta_M \). Now since 

\[(1 - K')(\sum_{i=1,i \neq M}^{n}(\beta_M - \beta_i)V_i^2) + (\beta_M - \beta_0K') \leq 0, \] 

therefore,

\[
(1 - K')\left(\sum_{i=1,i \neq L}^{n}(\beta_L - \beta_i)V_i^2\right) + (\beta_L - \beta_0K') \leq 0
\]

\[\Rightarrow q_L^* = \frac{V_L}{\sigma_L} \left[ (1 - K')(\sum_{i=1,i \neq L}^{n}(\beta_L - \beta_i)V_i^2) + (\beta_L - \beta_0K') \right] \leq 0 \]

Therefore, Supplier \( L \) will belong to set \( V' \) after the restriction. Hence any supplier who quoted a cost greater than or equal to Supplier \( M \) will also belong to set \( V' \) after the restriction. Since the suppliers are indexed in a non-decreasing order of cost, all the suppliers who are indexed higher Supplier \( M \) belong to set \( V' \). Also there exists a minimum value of index \( M = n^* + 1 \) such that all the suppliers indexed below \( M \) belong to set \( V \) and all the suppliers indexed above or equal to \( M \) belong to set \( V' \).

Hence set \( V \) becomes such that the unconstrained solution to the restricted problem becomes:

If the back-up supplier belongs to set \( V \):

\[
q_0^{**} = \frac{\left(\sum_{i=1}^{n^*}(\beta_0 - \beta_i)V_i^2\right) + \beta_0}{(1 - K')(\sum_{i=1}^{n^*}V_i^2) + 1}
\]

\[q_j^{**} = \frac{V_j}{\sigma_j} \left[ (1 - K')(\sum_{i=1;i \neq j}^{n^*}(\beta_j - \beta_i)V_i^2) + (\beta_j - \beta_0K') \right] \quad ; j \neq 0 \]

If the back-up supplier does not belong to set \( V \):
Result 2.2: If the solution to the unconstrained problem P1 satisfies the maximum back-up supplier constraint, then the solution to the restricted problem P1' will also satisfy the maximum back-up supplier constraint.

Proof: Since the solution to the unconstrained problem P1 satisfies the maximum back-up supplier constraint,

\[
q_0^* = \frac{\sum_{i=1}^{n}(\beta_0 - \beta_i)V_i^2 + \beta_0}{(1 - K')(\sum_{i=1}^{n}V_i^2) + 1} \leq B
\]

The expression of \(q_0^*\) can be simplified to:

\[
q_0^* = \frac{\beta_0 \left(1 - K'\right) (\sum_{i=1}^{n}V_i^2) + 1 + \beta_0 K' (\sum_{i=1}^{n}V_i^2) - (\sum_{i=1}^{n} \beta_i V_i^2)}{(1 - K')(\sum_{i=1}^{n}V_i^2) + 1}
\]

\[
= \beta_0 + \frac{\beta_0 K' (\sum_{i=1}^{n}V_i^2) - (\sum_{i=1}^{n} \beta_i V_i^2)}{(1 - K')(\sum_{i=1}^{n}V_i^2) + 1}
\]

\[
= \beta_0 + \frac{\beta_0 K'}{(1 - K') \left(\sum_{i=1}^{n}V_i^2\right) + \frac{1}{(1 - K')}}
\]

\[
= \beta_0 + \frac{\beta_0 K'}{(1 - K')} \frac{\alpha_1 V_1^2 + \alpha_2 V_2^2 + \cdots + \alpha_n V_n^2}{V_1^2 + V_2^2 + \cdots + V_n^2 + \frac{1}{(1 - K')}}
\]

where \(\alpha_i = \left(1 - \frac{\beta_i}{\beta_0 K'}\right)\) for \(i = 1, 2, \cdots, n\). Also since \(\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n\), hence \(\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n\).
Now, in the optimal solution to the restricted problem \( P1' \),

\[
q^{**}_0 = \frac{(\sum_{i=1}^{n^*}(\beta_0 - \beta_i)V^2_i) + \beta_0}{(1 - K')(\sum_{i=1}^{n^*}V^2_i) + 1}
\]

\[
= \beta_0 + \frac{\beta_0K'}{(1 - K')} \frac{\alpha_1V^2_1 + \alpha_2V^2_2 + \cdots + \alpha_{n^*}V^2_{n^*}}{V^2_1 + V^2_2 + \cdots + V^2_{n^*} + \frac{1}{(1 - K')}}
\]

Now,

\[
q_0 - q^{**}_0 = \frac{\beta_0K'}{(1 - K')} \left[ \frac{\alpha_1V^2_1 + \alpha_2V^2_2 + \cdots + \alpha_{n^*}V^2_{n^*}}{V^2_1 + V^2_2 + \cdots + V^2_{n^*} + \frac{1}{(1 - K')}} - \frac{\alpha_1V^2_1 + \alpha_2V^2_2 + \cdots + \alpha_{n^*}V^2_{n^*}}{V^2_1 + V^2_2 + \cdots + V^2_{n^*} + \frac{1}{(1 - K')}} \right]
\]

\[
= \beta_0K' \frac{\left[ (\alpha_{n^*+1} - \alpha_1)V^2_{n^*+1} + (\alpha_{n^*+1} - \alpha_2)V^2_{n^*+1} + \cdots + (\alpha_{n^*+1} - \alpha_{n^*})V^2_{n^*+1} \right]}{(1 - K')\left(V^2_1 + V^2_2 + \cdots + V^2_{n^*} + \frac{1}{(1 - K')}\right)\left(V^2_1 + V^2_2 + \cdots + V^2_{n^*} + \frac{1}{(1 - K')}\right)}
\]

\[
= \frac{\beta_0K' \left[ \sum_{j=n^*+1}^{n^*} \left[ \frac{\alpha_j}{(1 - K')} \frac{V^2_j}{V^2_1 + V^2_2 + \cdots + V^2_{n^*} + \frac{1}{(1 - K')}} \right] \left( \sum_{i=1}^{n^*} \left( \alpha_j - \alpha_i \right) V^2_i \right) \right]}{(1 - K')\left(\sum_{i=1}^{n^*} V^2_i + \frac{1}{(1 - K')}\right)\left(\sum_{i=1}^{n^*} V^2_i + \frac{1}{(1 - K')}\right)}
\]

(15)

Now, since the suppliers indexed by \( n^* + 1, n^* + 2, \cdots, n \) belong to set \( V' \), therefore the order quantities to those suppliers in the optimal solution to the unconstrained problem \( P1 \) must be either zero or negative. Therefore,

\[
q_{n^*+1}' = \frac{V^2_{n^*+1}}{\sigma_{n^*+1}^2} \left[ (1 - K')(\sum_{i=1,n^*+1}^{n^*}(\beta_{n^*+1} - \beta_i)V^2_i) + (\beta_{n^*+1} - \beta_0K') \right] \leq 0
\]
Since \( K' < 1 \), therefore,

\[
(1 - K') \left( \sum_{i=1,i \neq n^*+1}^{n} (\beta_{n^*+1} - \beta_i) V_i^2 \right) + (\beta_{n^*+1} - \beta_0 K') \leq 0
\]

\[
(1 - K') \left( \sum_{i=1,i \neq n^*+1}^{n} (\beta_i - \beta_{n^*+1}) V_i^2 \right) + (\beta_0 K' - \beta_{n^*+1}) \geq 0
\]

Since \( 0 < K' < 1 \), therefore,

\[
\left( \sum_{i=1,i \neq n^*+1}^{n} (\beta_i - \beta_{n^*+1}) V_i^2 \right) + \frac{(\beta_0 K' - \beta_{n^*+1})}{(1 - K')} \geq 0
\]

\[
\left( \sum_{i=1,i \neq n^*+1}^{n} \left( \frac{\beta_i}{\beta_0 K'} - \frac{\beta_{n^*+1}}{\beta_0 K'} \right) V_i^2 \right) + \frac{1 - \frac{\beta_{n^*+1}}{\beta_0 K'}}{(1 - K')} \geq 0
\]

Now since \( \beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n^*} \geq \beta_{n^*+1} \cdots \geq \beta_n \), therefore,

\[
\left( \sum_{i=1}^{n^*} \left( \frac{\beta_i}{\beta_0 K'} - \frac{\beta_{n^*+1}}{\beta_0 K'} \right) V_i^2 \right) + \frac{1 - \frac{\beta_{n^*+1}}{\beta_0 K'}}{(1 - K')} \geq 0
\]

\[
\Rightarrow \left( \sum_{i=1}^{n^*} (\alpha_{n^*+1} - \alpha_i) V_i^2 \right) + \frac{\alpha_{n^*+1}}{(1 - K')} \geq 0
\]

This relation will hold true for all the suppliers indexed above \( n^* \). Therefore, all the terms in the numerator in equation (17) is greater than or equal to zero and the denominator is greater than zero, therefore, \( q_0^* - q_0^{**} \geq 0 \). Now, since \( q_0^* \leq B \), therefore, \( q_0^{**} \leq B \). Hence, it can be concluded that if the solution to the unconstrained problem P1 satisfies the maximum back-up supplier constraint, then the solution to the restricted problem P1’ will also satisfy the maximum back-up supplier constraint.
Now, $q_j^{**}$ again can have positive, zero or negative values. We reconstruct the set $V$ with only those suppliers who obtained a positive order quantity in the unconstrained solution to the restricted problem. Similarly, $V'$ will now include those additional suppliers who obtained zero or negative quantities in the unconstrained solution to the restricted problem. This process is repeated till either all the suppliers in set $V$ obtain positive order quantities in the unconstrained solution to the restricted problem or set $V$ becomes a null set; and $q_j^{**}$ denotes the unconstrained solution to the restricted problem with only those suppliers who belong to the final set $V$. Also, we have,

$$\left. \frac{\partial \Pi_{R1}}{\partial q_j} \right|_{q_j=q_j^{**}} = 0$$

Let $\{\bar{q}_j; j = 0, 1, \cdots, n\}$ denotes a set of values of the order quantities.

Set,

$$\bar{q}_j = \begin{cases} 
q_j^{**} & \text{for } j \in V \\
0 & \text{for } j \in V' 
\end{cases} \quad (16)$$

Also set,

$$\lambda_j = \begin{cases} 
0 & \text{for } j \in V \\
\left. \frac{\partial \Pi}{\partial q_j} \right|_{q_j=q_j^{**}} & \text{for } j \in V' 
\end{cases} \quad (17)$$

and,

$$s_j^2 = \begin{cases} 
\bar{q}_j & \text{for } j \in V \\
0 & \text{for } j \in V' 
\end{cases} \quad (18)$$
Also, \[
\lambda = 0
\]
and, \[
s^2 = B - \bar{q}_0
\]

It can be noted that, from equation (19), since \(\bar{q}_j \geq 0\), \(s_j\)’s have real values for all \(j\). Now, for \(jeV\),

\[
\left. \left( \frac{\partial \Pi}{\partial q_j} - \lambda_j \right) \right|_{q_j = \bar{q}_j} = \left. \frac{\partial \Pi}{\partial q_j} \right|_{q_j = \bar{q}_j} = \left. \frac{\partial \Pi_{R1}}{\partial q_j} \right|_{q_j = \bar{q}_j} = 0
\]

and, for \(jeV’\),

\[
\left. \left( \frac{\partial \Pi}{\partial q_j} - \lambda_j \right) \right|_{q_j = a_j} = \left. \frac{\partial \Pi}{\partial q_j} \right|_{q_j = a_j} - \lambda_j = 0
\]

Also, for all \(j\),

\[
\left. \frac{\partial L}{\partial \lambda_j} \right|_{q_j = \bar{q}_j} = \bar{q}_j - s^2 = 0
\]

\[
\left. \frac{\partial L}{\partial s_k} \right|_{q_j = \bar{q}_j} = 2\lambda_k s_k = 0
\]

\[
\left. \frac{\partial L}{\partial \lambda} \right|_{q_j = \bar{q}_j} = \bar{q}_0 + s^2 - B = 0
\]

\[
\left. \frac{\partial L}{\partial \lambda} \right|_{q_j = \bar{q}_j} = 2\lambda s = 0
\]

Therefore, it can be observed that, \(\bar{q}_j, \lambda_j, s^2, \lambda\) and \(s^2\), given by equations (18), (19), (20) and (21) satisfy the Lagrangian optimality conditions as given by equations (12), (13), (14), (15) and
(16). Hence $\bar{q}_j$, as expressed in equation (1*), is an optimal solution to the constrained problem P4.

**Result 3:** Suppose in an optimal solution to the constrained problem P4, a supplier $M: (M < n)$ who quoted a cost $c_M$ does not get an order; then all regular suppliers who quoted a cost higher than $c_M$ do not get orders in that solution under the assumption of uniformly distributed demand.

**Proof:** Let an optimal solution to the problem P4 be denoted by $\bar{q}_j; j = 0,1,2,\ldots, n$. Suppose there exists a supplier $M$, who does not get an order in the optimal solution, i.e. $\bar{q}_M = 0$. Here we are referring to the solution which is derived to obtain $\bar{q}_M = 0$.

The suppliers are indexed in the non-decreasing order of cost. Suppose there exists a supplier $L$, who has quoted a higher cost compared to supplier $M$, i.e. $c_M < c_L$. The order quantity received by supplier $L$ in this solution is denoted by $\bar{q}_L$.

Since $c_M < c_L$, and since the beta values for each of the supplier remain the same irrespective of the number of suppliers used, $\beta_M > \beta_L$ (as shown in Result 2.1).

The solution to the constrained problem P4 is obtained from an unconstrained solution to the restricted problem P1’ with only those suppliers who belong to the set $V$. The set $V$ is constructed with only those suppliers who receive a positive order from the firm. Since $\bar{q}_M = 0$, therefore, supplier $M$ belongs to the set $V'$.

Now supplier set $V$ is first constructed with only those suppliers who receive a positive order from the firm in the unconstrained solution P1. Then unconstrained solution is obtained for the restricted problem with those suppliers who belong to set $V$. If some order quantities in the solution to the restricted problem is zero or negative, the corresponding suppliers are moved
from set $V$ to set $V'$, and again an unconstrained solution is obtained for the restricted problem with those suppliers who belong to the updated set $V$. Let us assume that during $l^{th}$ restriction supplier $M$ is moved from set $V$ to set $V'$, and let us call the restricted set before the $l^{th}$ restriction as $V_1$.

Let $q_{M}^{**}$ denote the order quantity received by Supplier $M$ in the $l^{th}$ restriction with the suppliers who belong to the set $V_1$.

For the scenario where the back-up supplier belongs to set $V_1$, since Supplier $M$ moves from set $V$ to set $V'$ in the $l^{th}$ restriction:

$$q_{M}^{**} = \frac{V_{M}}{\sigma_{M}} \left[ (1 - K') \left( \sum_{l=1; i \in V_1; i \neq M}^{n} (\beta_{M} - \beta_{i})V_{i}^2 \right) + (\beta_{M} - \beta_{0}K') \right] \leq 0$$

$$\Rightarrow (1 - K') \left( \sum_{l=1; i \in V_1; i \neq M}^{n} (\beta_{M} - \beta_{i})V_{i}^2 \right) + (\beta_{M} - \beta_{0}K') \leq 0$$

A necessary, but not sufficient, condition for $q_{L}$ to be positive is that the Supplier $L$ remains in set $V$ after the $l^{th}$ restriction. Now,

$$(1 - K') \left( \sum_{l=1; i \in V_1; i \neq M}^{n} (\beta_{M} - \beta_{i})V_{i}^2 \right) + (\beta_{M} - \beta_{0}K') \leq 0$$

$$\Rightarrow (1 - K') \left( \sum_{l=1; i \in V_1}^{n} (\beta_{M} - \beta_{i})V_{i}^2 \right) + (\beta_{M} - \beta_{0}K') \leq 0$$

Since $\beta_{M} > \beta_{L}$, therefore,
Therefore, Supplier cannot remain in set after the \( l \)th restriction and hence cannot receive any order in the optimal solution considered here to the constrained problem P4.

For the scenario where the back-up supplier does not belong to set \( V_l \), since Supplier \( M \) moves from set \( V \) to set \( V' \) in the \( l \)th restriction:

\[
q_{M^*}^* = \frac{V_M}{\sigma_M} \left[ \frac{\left( \sum_{i=1; i \neq M}^n (\beta_M - \beta_i)V_i^2 \right) + \beta_M}{\left( \sum_{i=1; i \neq V}^n V_i^2 \right) + 1} \right] \leq 0
\]

\[
\Rightarrow \left( \sum_{i=1; i \neq V_l; i \neq M}^n (\beta_M - \beta_i)V_i^2 \right) + \beta_M \leq 0
\]

Since \( \beta_M > \beta_L \), therefore,
Therefore, Supplier cannot remain in set after the $l^{th}$ restriction and hence cannot receive any order in the optimal solution considered here to the constrained problem P4.

Hence, it can be concluded that, if, in an optimal solution to the constrained problem P4, a supplier $M$: $(M < n)$ who quoted a cost $c_M$ does not get an order; then all suppliers who quoted a cost higher than $c_M$ do not get orders in the optimal solution.

**Result 4:** The number of suppliers $n^*$, who gets a positive order in an optimal solution to the constrained problem P4 is either $n$ or satisfies the following conditions:

\[
\beta_{n^*} > \frac{(1 - K') \sum_{j=1}^{n^*-1} \beta_j V_j^2 + \beta_0 K'}{(1 - K') \sum_{j=1}^{n^*-1} V_j^2 + 1}
\]

And,

\[
\beta_{n^*+1} \leq \frac{(1 - K') \sum_{j=1}^{n^*} \beta_j V_j^2 + \beta_0 K'}{(1 - K') \sum_{j=1}^{n^*} V_j^2 + 1}
\]

**Proof:** From Result 3, it can be concluded that the indices of the regular suppliers who receive positive order quantities in the solution to the constrained problem P4 are consecutive starting
from 1. Hence, if \( n^* \) denotes the number of suppliers who gets a positive order in an optimal solution to the constrained problem P4, then the indices for those suppliers will be \( 1, 2, \cdots, n^* \). As the solution to the constrained problem P4 is obtained by identifying the unconstrained solution to the restricted problem P1 with suppliers belonging to set \( V \), therefore, each of the first \( n^* \) suppliers will receive a positive quantity only if the unconstrained solution to the restricted problem with set \( V \) consisting of the first \( n^* \) suppliers be the solution to the constrained problem P4. Therefore,

\[
(1 - K') \sum_{i=1, i \neq n^*}^{n^*} (\beta_{n^*} - \beta_i) V_i^2 + (\beta_{n^*} - \beta_0 K') > 0
\]

\[
\Rightarrow (1 - K') \sum_{i=1}^{n^*-1} (\beta_{n^*} - \beta_i) V_i^2 + (\beta_{n^*} - \beta_0 K') > 0
\]

\[
\Rightarrow \beta_{n^*} \left[ 1 + (1 - K') \sum_{i=1}^{n^*-1} V_i^2 \right] > \beta_0 K' + (1 - K') \sum_{i=1}^{n^*-1} \beta_i V_i^2
\]

\[
\Rightarrow \beta_{n^*} > \frac{(1 - K') \sum_{j=1}^{n^*-1} \beta_j V_j^2 + \beta_0 K'}{(1 - K') \sum_{j=1}^{n^*-1} V_j^2 + 1}
\]

The above relation must hold true for first \( n^* \) suppliers to ensure that all of them get positive orders. Now, \( n^* \) would be number of suppliers to be receiving positive order quantity only if all the suppliers quoting higher cost compared to supplier \( n^* \) does not get any order in the optimal solution to the constrained problem P4. Using Result 3, the previous statement holds true if the Supplier indexed by \( n^* + 1 \) does not receive any order in the optimal solution to the constrained problem P4. Since the solution to the constrained problem P4 is calculated by solving the
restricted unconstrained problem $P1'$ with the suppliers belonging to set $V$, hence, Supplier $n^* + 1$ must belong to set $V'$. Therefore,

\[
(1 - K') \sum_{i=1, i \neq n^* + 1}^{n^* + 1} (\beta_{n^* + 1} - \beta_i) V_i^2 + (\beta_{n^* + 1} - \beta_0 K') \leq 0
\]

Therefore by simplification,

\[
\beta_{n^* + 1} \leq \frac{(1 - K') \sum_{j=1}^{n^*} \beta_j V_j^2 + \beta_0 K'}{(1 - K') \sum_{j=1}^{n^*} V_j^2 + 1}
\]

### 1.1.5 Back-up Supplier Limit Constraint

If an optimal solution to the unconstrained problem $P1$ satisfies the non-negativity constraint of the problem $P4$ but the constraint regarding the maximum limit of the back-up supplier (C4.2) is violated, then an optimal solution to the constrained problem $P4$ can be obtained from the optimum solution to the unconstrained problem.

Let us define a restricted problem $P1''$ where the optimum order quantities to the regular suppliers are determined given that the order quantity booked from the back-up supplier is $B$. Therefore, the expected profit function for this restricted problem becomes,
The problem statement for the restricted problem is:

- **Restricted Problem P1''**: 

  \[
  \Pi_{R2} = \int_0^1 g_1(r_1) \left[ \int_0^1 g_2(r_2) \cdots \int_0^1 g_n(r_n) \left[ \int_0^Q \left( pz - h \left( \frac{Q}{2} \right) - c_0 B - \sum_{i=1}^n c_i r_i q_i \right) f(z) dz \right] \right] dz
  \]
  \[
  \left. + S(Q - z) f(z) dz \right) \left. + \int_Q^{Q+B} \left( pz - h \left( \frac{Q}{2} \right) - c_0 B - \sum_{i=1}^n c_i r_i q_i - c_0 (z - Q) \right) f(z) dz \right) \left. + \int_{Q+B}^{\infty} \left( p(Q + q_0) - h \left( \frac{Q}{2} \right) - c_0 B - \sum_{i=1}^n c_i r_i q_i - c_0 B \right) \right) \left. - u(z - (Q + B)) \right) f(z) dz \right] dr_n \cdots dr_2 dr_1
  \]

The problem statement for the restricted problem is:

- **Restricted Problem P1''**: 

  Maximize \( \Pi_{R2} \) \hspace{1cm} (P1'')

Let \( \{q_j^{**}; j = 1, \ldots, n\} \) denotes the unconstrained solution to the restricted problem P1''. Using the same method which was used to identify the unconstrained solution to the unconstrained problem P1, \( q_j^{**} \) is identified as,

\[
q_j^{**} = \frac{V_j}{\sigma_j} \left[ \sum_{i=1, i \neq j}^n (\beta_j' - \beta_i') V_i^2 + \beta_j' \right] \left/ \sum_{i=1}^n V_i^2 + 1 \right.; j = 1, 2, \ldots, n
\]

where \( \beta_k' = \beta_k - K'B \).
Result 5: If the unconstrained solution to problem P1 satisfies the non-negativity constraint C4.1, but does not satisfy the maximum limit of the back-up supplier constraint C4.2, i.e. \( q_0^* > B \), then the solution to the restricted problem P1'' will also satisfy the non-negativity constraint.

Proof: By simplification, the following expressions are obtained:

\[
\sum_{i=1}^{n} \beta_i V_i^2 = \frac{1}{p + \frac{h}{2} + u - S} \sum_{i=1}^{n} \left[ (pb + ub - px - ux) - \left( c_i + \frac{h}{2} \right) (b - a) + \left( S - \frac{h}{2} \right) (a - x) \right] V_i^2
\]

And,

\[
\sum_{i=1}^{n} K' \beta_0 V_i^2 = \frac{1}{p + \frac{h}{2} + u - S} \sum_{i=1}^{n} \left[ (pb + ub - px - ux) - \left( (c_0 + c'_0) (b - a) + c_0 (a - x) \right) \right] V_i^2
\]

Since for all practical purposes, \( c_0 + c'_0 > c_i + \frac{h}{2} \) and \( c_0 > S \), therefore, \( \sum_{i=1}^{n} \beta_i V_i^2 > \sum_{i=1}^{n} K' \beta_0 V_i^2 \). Now, since the unconstrained solution to problem P1 does not satisfy the maximum limit of the back-up supplier constraint C4.2, \( q_0^* > B \)

\[
\Rightarrow \frac{\sum_{i=1}^{n} (\beta_0 - \beta_i)V_i^2 + \beta_0}{(1 - K') \sum_{i=1}^{n} V_i^2 + 1} > B
\]

\[
\Rightarrow \beta_0 \left[ (1 - K') \sum_{i=1}^{n} V_i^2 + 1 \right] - B \left[ (1 - K') \sum_{i=1}^{n} V_i^2 + 1 \right] > \sum_{i=1}^{n} \beta_i V_i^2 - \sum_{i=1}^{n} K' \beta_0 V_i^2 > 0
\]
Therefore, $\beta_0 > B$.

Now, the unconstrained solution to problem P1 satisfies the non-negativity constraint C4.1, i.e.

$$ q_k^* = \frac{V_k}{\sigma_k} \frac{[ (1 - K') \sum_{i=1,i\neq k}^n (\beta_k - \beta_i) V_i^2 + (\beta_k - \beta_0 K') ]}{(1 - K') \sum_{i=1}^n V_i^2 + 1} \geq 0 $$

$$ \Rightarrow (1 - K') \sum_{i=1,i\neq k}^n (\beta_k - \beta_i) V_i^2 + (\beta_k - \beta_0 K') \geq 0 $$

$$ \Rightarrow \sum_{i=1,i\neq k}^n (\beta_k - \beta_i) V_i^2 + (\beta_k - \beta_0 K') \geq 0 $$

$$ \Rightarrow \sum_{i=1,i\neq k}^n (\beta_k - K'B - \beta_i + K'B) V_i^2 + (\beta_k - K'B) \geq 0 $$

$$ \Rightarrow \sum_{i=1,i\neq k}^n (\beta_k' - \beta_i') V_i^2 + \beta_k' \geq 0 $$

Therefore, the unconstrained solution to the restricted problem P1’’:

$$ q_{k''}^* = \frac{V_k}{\sigma_k} \frac{[ \sum_{i=1,i\neq k}^n (\beta_k' - \beta_i') V_i^2 + \beta_k' ]}{\sum_{i=1}^n V_i^2 + 1} \geq 0 $$

Hence, the unconstrained solution to the restricted problem P1’’ also satisfies the non-negativity constraint. Also,

$$ \frac{\partial \Pi_{R2}}{\partial q_k} \bigg|_{q_k=q_{k''}} = 0 $$
Let \( \{\bar{q}_j; j = 0,1,\cdots, n\} \) denotes a set of values for the order quantities to each of the suppliers, where,

\[
\bar{q}_0 = B
\]

\[
\bar{q}_j = q_{j}^{**}; j = 1,\cdots, n
\]

\[
s^2 = 0
\]

\[
\lambda = \frac{\partial \Pi}{\partial q_0}
\]

\[
s_j^2 = \bar{q}_j; j = 0,1,\cdots, n
\]

\[
\lambda_j = 0; j = 0,1,\cdots, n
\]

It can be noted that, since \( \bar{q}_j \geq 0 \), \( s_j \)'s have real values for \( j = 0,1,\cdots, n \). Now, for \( j = 1,\cdots, n \),

\[
\left( \frac{\partial \Pi}{\partial q_j} - \lambda_j \right) \bigg|_{q_j = \bar{q}_j} = \frac{\partial \Pi}{\partial q_j} \bigg|_{q_j = \bar{q}_j} = \frac{\partial \Pi_{R2}}{\partial q_j} \bigg|_{q_j = q_{j}^{**}} = 0
\]

and,

\[
\left( \frac{\partial \Pi}{\partial q_0} - \lambda_0 - \lambda \right) \bigg|_{q_j = \bar{q}_j} = \frac{\partial \Pi}{\partial q_0} - \lambda = 0
\]

Also, for \( j = 0,1,\cdots, n \),

\[
\frac{\partial L}{\partial \lambda_j} \bigg|_{q_j = \bar{q}_j} = \bar{q}_j - s_j^2 = 0
\]
Therefore, the given set of values of $\bar{q}_j, \lambda, s^2, \lambda$ and $s^2$, satisfy the Lagrangian optimality conditions as given by equations (12), (13), (14), (15) and (16). Hence $\bar{q}_j$ is as an optimal solution to the constrained problem P4.

1.2 Multi-Period Solution

In this section, the multi-period model is solved for both infinite time horizon and finite time horizon scenarios. It is assumed that the probability density function of the demand distribution for each period remains the same. For finite time horizon, using the dynamic programming approach for forward recursion, let $\Pi_j(x_j)$ denotes the maximum expected profit for periods $j, j+1, \ldots, N$, $j < N$ given that $x_j$ is the inventory level at the beginning of the $j^{th}$ period before the order is received, then, with the notations described in Section 4.3.2,
\[ \Pi_j(x_j) = \max_{q_i^j \geq 0, q_0^j \leq B} \prod_{i=1}^{n} \int_{0}^{r_i^j} g_i(r_i^j) \left[ -c_0^j q_0^j - \sum_{k=1}^{n} c_k r_k^j q_k^j \right. \\
+ \int_{0}^{y_j} \left( p z - h \left( y_j - \frac{z}{2} \right) \right) f(z) dz \\
+ \int_{y_j}^{y_j + q_0^j} \left( p z - h \left( \frac{y_j}{2} \right) - c_0 (z - y_j) \right) f(z) dz \\
+ \int_{y_j + q_0^j}^{\infty} \left( p (y_j + q_0^j) + \alpha (z - (y_j + q_0^j)) - h \left( \frac{y_j}{2} \right) - c_0 q_0^j \right) \\
- u \left( z - (y_j + q_0^j) \right) f(z) dz + \int_{0}^{\infty} \Pi_{j+1}(x_{j+1}) f(z) dz \left. \right] dr_i^j \]

However, as discussed during the model formulation, any inventory leftover at the end of last \(N^{th}\) period is sold at scrap value and any unmet demand in the last period is assumed to be lost.

Therefore, using the same notations as in Section 4.3.2, the maximum expected total profit for the \(N^{th}\) period is given by,

\[ \Pi_N(x_N) = \max_{q_i^N \geq 0, q_0^N \leq B} \prod_{i=1}^{n} \int_{0}^{r_i^N} g_i(r_i^N) \left[ -c_0^N q_0^N - \sum_{k=1}^{n} c_k r_k^N q_k^N \right. \\
+ \int_{0}^{y_N} \left( p z - h \left( y_N - \frac{z}{2} \right) + S(y_N - z) \right) f(z) dz \\
+ \int_{y_N}^{y_N + q_0^N} \left( p z - h \left( \frac{y_N}{2} \right) - c_0 (z - y_N) \right) f(z) dz \\
+ \int_{y_N + q_0^N}^{\infty} \left( p (y_N + q_0^N) - h \left( \frac{y_N}{2} \right) - c_0 q_0^N - u (z - (y_N + q_0^N)) \right) f(z) dz \left. \right] dr_i^N \]

Now, the inventory on hand at the end of \(j^{th}\) period,

\[ x_{j+1} = \begin{cases} 
    y_j - z, & \text{for } z \leq y_j \\
    0, & \text{for } y_j < z \leq y_j + q_0^j \\
    y_j + q_0^j - z, & \text{for } z > y_j + q_0^j
\end{cases} \]
Therefore, for all \( j < N \),

\[
\Pi_j(x_j) = \max_{q_i^j \geq 0, q_0^j \leq \delta} \prod_{i=1}^{n} \int_0^1 g_i(r_i^j) \left[ -c_0'q_0^j - \sum_{k=1}^{n} c_k r_k^j q_k^j \right. \\
+ \int_0^{y_j} \left( p z - h \left( y_j - \frac{z}{2} \right) + \alpha \Pi_{j+1}(y_j - z) \right) f(z) \, dz \\
+ \int_{y_j}^{y_j + q_0^j} \left( p z - h \left( \frac{y_j}{2} \right) - c_0 (z - y_j) \right) f(z) \, dz \\
+ \int_{y_j + q_0^j}^{\infty} \left( p (y_j + q_0^j) + \alpha p \left( z - (y_j + q_0^j) \right) - h \left( \frac{y_j}{2} \right) - c_0 q_0^j \right. \\
- u \left( z - (y_j + q_0^j) \right) + \alpha \Pi_{j+1}(y_j + q_0^j - z) \right) f(z) \, dz \left] \, dr_i^j \right)
\]  

Equation (23) serves as the recursive equation during the formulation of the finite \( N \)-period forward recursion dynamic programming model. The formulation of the dynamic programming model is discussed in Section 5.2.2.

### 1.2.1 Infinite Time Horizon Solution

If the number of periods to be considered is infinite, then the index \( j \), which indicates the period in the recursive relation, can be dropped. Also, for infinite periods, \( \Pi_j = \Pi_{j+1} = \Pi \). Therefore, the recursive equation given in equation (21) reduces to,
1.2.1.1 Concavity of the Expected Profit Function

Again, in the context of infinite time horizon multi-period problem, to check the existence of a global maximum of the expected profit function, the concavity of the expected profit function with respect to the decision variables is examined.

**Result 5.5:** The objective function for infinite time horizon problem as given in equation (24) is concave with respect to the decision variables \(q_t, \forall t = 0, 1, 2, \ldots, n\) under assumptions A1 and A2.

**Proof:** First, the Hessian matrix for the objective function for the infinite time horizon model is constructed. The objective function for this problem can be represented as,

\[
\Pi(x) = \max_{q_1 \geq 0, q_0 \leq B} \sum_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[-c_0q_0 - \sum_{k=1}^{n} c_k r_k q_k \right. \\
+ \int_{0}^{y} \left(pz - h \left(y - \frac{z}{2}\right) + \alpha \Pi(y - z)\right)f(z)dz \\
+ \int_{y}^{y+q_0} \left(pz - h \left(\frac{y}{2}\right) - c_0 (z - y)\right)f(z)dz \\
+ \int_{y+q_0}^{\infty} \left(p(y + q_0) + \alpha p(z - (y + q_0)) - h \left(\frac{y}{2}\right) - c_0 q_0 \right) \\
- u(z - (y + q_0)) + \alpha \Pi(y + q_0 - z)\right]f(z)dz dr_i
\]  

(22)
\[ O = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -c_0' q_0 - \sum_{k=1}^{n} c_k r_k q_k + \int_{0}^{y} \left( p z - h \left( y - \frac{z}{2} \right) + \alpha \Pi(y - z) \right) f(z) \, dz \right. \\
\left. + \int_{y}^{y+q_0} \left( p z - h \left( \frac{y'}{2} \right) - c_0(z - y) \right) f(z) \, dz \right. \\
\left. + \int_{y+q_0}^{\infty} \left( p(y + q_0) + \alpha p(z - (y + q_0)) - h \left( \frac{y}{2} \right) - c_0 q_0 - u(z - (y + q_0)) \right) + \alpha \Pi(y + q_0 - z) f(z) \, dz \right] \, dr_i \\
= \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -c_0' q_0 - \sum_{k=1}^{n} c_k r_k q_k + I_1 + I_2 + I_3 \right] \, dr_i \] 

Where the third, fourth and the fifth term of the integral are denoted by \( I_1, I_2 \) and \( I_3 \). Therefore,

\[ \frac{\partial O}{\partial q_k} = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -c_0' r_k + \frac{\partial I_1}{\partial q_k} + \frac{\partial I_2}{\partial q_k} + \frac{\partial I_3}{\partial q_k} \right] \, dr_i \]

And,

\[ \frac{\partial O}{\partial q_0} = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -c_0' + \frac{\partial I_1}{\partial q_0} + \frac{\partial I_2}{\partial q_0} + \frac{\partial I_3}{\partial q_0} \right] \, dr_i \]

Now, the differentiation component will consist terms in the form of \( \frac{\partial \Pi(y-z)}{\partial q_k}, \frac{\partial \Pi(y+q_0-z)}{\partial q_0}, \frac{\partial \Pi(y-z)}{\partial q_k} \) and \( \frac{\partial \Pi(y+q_0-z)}{\partial q_k} \). The value of these terms are determined as follows. If the order quantity from the \( k^{th} \) supplier is increased by \( d \) units, that means at the end of the period the firm will have \( r_k d \) units more at hand at the end of the period. This ensures that in the next period the firm has to
procure $r_k d$ units less, which means that the firm saves an amount of $c_w r_k d$, where $c_w$ is the weighted average of the costs quoted by the suppliers. Therefore,

$$\frac{\partial \Pi}{\partial q_k} = \alpha c_w r_k$$

However, since $q_0$ is only the maximum order booked to the back-up supplier, the effect of the back-up supplier will be visible only if $z > y + q_0$. Therefore,

$$\frac{\partial \Pi}{\partial q_0} = \begin{cases} 
0 & \text{for } z \leq y + q_0 \\
\alpha c_w & \text{for } z > y + q_0 
\end{cases}$$

Therefore, using Leibnitz rule,

$$\frac{\partial I_1}{\partial q_k} = \left[\left(p y \frac{h y}{2} f(y) + (-h + \alpha c_w) F(y)\right) r_k \right.$$

$$\frac{\partial I_2}{\partial q_k} = \left[\left(p (y + q_0) \frac{h y}{2} - c_0 q_0 \right) f(y + q_0) - \left(p y \frac{h y}{2} \right) f(y) \right.$$

$$+ \left(-\frac{h}{2} + c_0 \right) \left(F(y + q_0) - F(y)\right) \right] r_k$$

$$\frac{\partial I_3}{\partial q_k} = \left[-\left(p (y + q_0) \frac{h y}{2} - c_0 q_0 \right) f(y + q_0) + \left(p - \alpha p - \frac{h}{2} + u + \alpha c_w \right) \left(1 - F(y + q_0)\right) \right] r_k$$

Therefore,

$$\frac{\partial O}{\partial q_k} = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) r_k \left[-c_k + (-h + \alpha c_w) F(y) + \left(-\frac{h}{2} + c_0 \right) \left(F(y + q_0) - F(y)\right) \right.$$

$$+ \left(p - \alpha p - \frac{h}{2} + u + \alpha c_w \right) \left(1 - F(y + q_0)\right) \right] dr_i$$
\[= \prod_{l=1}^{n} \int_{0}^{1} g_l(r_l) r_k \left[ \left( \frac{-h}{2} - c_0 + \alpha c_w \right) F(y) + (c_0 - p(1 - \alpha) - u - \alpha c_w)F(y + q_0) \right. \\
+ \left. \left( p(1 - \alpha) - \frac{h}{2} + u + \alpha c_w - c_k \right) \right] dr_l \]

Foe uniformly distributed demand with parameters \([a, b],\)

\[
\frac{\partial O}{\partial q_k} = \prod_{l=1}^{n} \int_{0}^{1} g_l(r_l) r_k \left[ \left( \frac{-h}{2} - c_0 + \alpha c_w \right) \frac{\sum_{l=1}^{n} r_l q_l + x - a}{b - a} \right. \\
+ \left. (c_0 - p(1 - \alpha) - u - \alpha c_w) \frac{\sum_{l=1}^{n} r_l q_l + q_0 + x - a}{b - a} \right. \\
+ \left. \left( p(1 - \alpha) - \frac{h}{2} + u + \alpha c_w - c_k \right) \right] dr_l \\
\]

\[
= \prod_{l=1}^{n} \int_{0}^{1} g_l(r_l) r_k \left[ \frac{-p(1 - \alpha) + \frac{h}{2} + u}{b - a} \right. \left( \sum_{l=1}^{n} r_l q_l + x - a \right) \\
+ \left. \frac{-p(1 - \alpha) + u - c_0 + \alpha c_w}{b - a} \right. q_0 \\
+ \left. \left( p(1 - \alpha) - \frac{h}{2} + u - c_k + \alpha c_w \right) \right] dr_l \\
\]

(23)

Therefore,

\[
\frac{\partial^2 O}{\partial q_0 \partial q_k} = \frac{-p(1 - \alpha) + u - c_0 + \alpha c_w}{b - a} \frac{1}{r_k} \\
\frac{\partial^2 O}{\partial q_k^2} = \frac{-p(1 - \alpha) + \frac{h}{2} + u}{b - a} \left( \frac{1}{r_k^2} + \sigma_k^2 \right) \\
\frac{\partial^2 O}{\partial q_m \partial q_k} \mid_{m \neq k} = \frac{-p(1 - \alpha) + \frac{h}{2} + u}{b - a} \frac{1}{r_m r_k} \\
\]

Again,
Using Leibnitz rule,

\[
\frac{\partial l_2}{\partial q_0} = \left( p(y + q_0) - \frac{hy}{2} - c_0q_0 \right) f(y + q_0)
\]

\[
\frac{\partial l_3}{\partial q_k} = -\left( p(y + q_0) - \frac{hy}{2} - c_0q_0 \right) f(y + q_0) + (p(1 - \alpha) + u - c_0 + \alpha c_w)(1 - F(y + q_0))
\]

Therefore,

\[
\frac{\partial O}{\partial q_0} = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -c_0' + (p(1 - \alpha) + u - c_0 + \alpha c_w)(1 - F(y + q_0)) \right] dr_i
\]

So, for uniformly distributed demand with parameters \([a, b]\),

\[
\frac{\partial O}{\partial q_0} = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -c_0' + (p(1 - \alpha) + u - c_0 + \alpha c_w) \left( 1 - \frac{\sum_{l=1}^{n} r_lq_l + q_0 + x - a}{b - a} \right) \right] dr_i
\]

\[
= \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -(p(1 - \alpha) + u + \alpha c_w - c_0) \left( \frac{\sum_{l=1}^{n} r_lq_l + q_0 + x - a}{b - a} \right) \right. \\
\left. + (p(1 - \alpha) - c_0 + \alpha c_w + u - c_0') \right] dr_i
\]

(24)

Therefore,

\[
\frac{\partial^2 O}{\partial q_0^2} = \frac{-(p(1 - \alpha) + u + \alpha c_w - c_0)}{b - a}
\]

\[
\frac{\partial^2 O}{\partial q_k \partial q_0} = \frac{-(p(1 - \alpha) + u + \alpha c_w - c_0)}{b - a} \bar{r}_k
\]
So, the Hessian Matrix is,

\[ H = \frac{(p(1 - \alpha) + \frac{h}{2} + u)}{b - a} \begin{bmatrix}
-M_0 & -M_0\bar{r}_1 & -M_0\bar{r}_2 & \cdots & -M_0\bar{r}_n \\
-M_0\bar{r}_1 & -M_1 & -\bar{r}_1\bar{r}_2 & \cdots & -\bar{r}_1\bar{r}_n \\
-M_0\bar{r}_2 & -\bar{r}_2\bar{r}_1 & -M_2 & \cdots & -\bar{r}_2\bar{r}_n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-M_0\bar{r}_n & -\bar{r}_n\bar{r}_1 & -\bar{r}_n\bar{r}_2 & \cdots & -M_n
\end{bmatrix} \]

where,

\[ M_0 = \frac{p(1 - \alpha) + u + \alpha c_w - c_0}{p(1 - \alpha) + \frac{h}{2} + u} \]

\[ M_k = \bar{r}_k^2 + \sigma_k^2 \text{ for } k = 1, 2, \ldots, n \]

Again, it may be noted that, \( M_k \) and \( M_0 \) will always have positive values. Now, as it is an infinite time horizon model, therefore it is realistic to assume that there is long term relationship between the back-up supplier and the firm. Also, if some units of the back-up supplier remain unsold in a period due to the “booking” at the beginning of the period, those units can easily be sold in the next period. Therefore, it is practical to assume that the booking cost per unit at the beginning of the period \( c_0' \) is very small, making \( c_0 \) sufficiently large, so that, \( \alpha c_w - c_0 \leq \frac{h}{2} \). Hence, in all practical situations, \( M_0 \) will have a value of less than unity.

The determinant of the Hessian matrix is,

\[ \det(H) = (-1)^{n+1} \frac{(p(1 - \alpha) + \frac{h}{2} + u)}{b - a} \begin{vmatrix}
M_0 & M_0\bar{r}_1 & M_0\bar{r}_2 & \cdots & M_0\bar{r}_n \\
M_0\bar{r}_1 & M_1 & \bar{r}_1\bar{r}_2 & \cdots & \bar{r}_1\bar{r}_n \\
M_0\bar{r}_2 & \bar{r}_2\bar{r}_1 & M_2 & \cdots & \bar{r}_2\bar{r}_n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
M_0\bar{r}_n & \bar{r}_n\bar{r}_1 & \bar{r}_n\bar{r}_2 & \cdots & M_n
\end{vmatrix} \]
Since the value of $\beta$ is less than unity, $\delta$ will always have a positive value. Therefore, the signs of the principal minors of the Hessian matrix will be $(-1)^{n+1}$. Hence it is concluded that the Hessian matrix is negative semi-definite. Therefore, the objective function as given in equation (23) is concave with respect to the decision variables when the demand is uniformly distributed over $[a, b]$.

1.2.1.2 Unconstrained and Constrained Solutions

To solve the non-linear constrained optimization problem P8, first the unconstrained solution for problem P5 is obtained. Now, as the concavity of the objective function about the decision variables has been established, the Karush-Kuhn-Tucker conditions for the unconstrained maximization problem reduce to:

$$\frac{\partial O}{\partial q_0} = 0$$  \hspace{1cm} (25)

And,
\[ \frac{\partial O}{\partial q_k} = 0 \]  (26)

Using equations (26) and (27),

\[
\frac{\partial O}{\partial q_0} = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) \left[ -\frac{(p(1 - \alpha) + u + \alpha c_w - c_0)}{b - a} \left( \sum_{l=1}^{n} r_l q_l + q_0 + x - a \right) 
\right. \\
\left. + \frac{(p(1 - \alpha) - c_0 + u + \alpha c_w - c_0')}{p(1 - \alpha) + u + \alpha c_w - c_0} (b - a) - (x - a) \right] \, dr_i = 0
\]

\[ \Rightarrow q_0 + \bar{r}_1 q_1 + \bar{r}_2 q_2 + \cdots + \bar{r}_n q_n = \frac{p(1 - \alpha) - c_0 + u + \alpha c_w - c_0'}{p(1 - \alpha) + u + \alpha c_w - c_0} (b - a) - (x - a) = K_0 (b - a) - (x - a) = \beta_0
\]

\[ \Rightarrow q_0 + \bar{r}_1 q_1 + \bar{r}_2 q_2 + \cdots + \bar{r}_n q_n = \beta_0 \]  (27)

Using equations (25) and (28),

\[
\frac{\partial O}{\partial q_k} = \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) r_k \left[ -\frac{(p(1 - \alpha) + u + \alpha c_w - c_0)}{b - a} \left( \sum_{l=1}^{n} r_l q_l + x - a \right) 
\right. \\
\left. + \frac{(p(1 - \alpha) + u + \alpha c_w - c_0)}{b - a} q_0 + \left( \frac{p(1 - \alpha) - \frac{h}{2} + u + \alpha c_w - c_k}{p(1 - \alpha) + \frac{h}{2} + u} \right) q_0 \right] \, dr_i = 0
\]

\[ \Rightarrow \prod_{i=1}^{n} \int_{0}^{1} g_i(r_i) r_k \left[ \left( \sum_{l=1}^{n} r_l q_l + x - a \right) + \frac{p(1 - \alpha) + u + \alpha c_w - c_0}{p(1 - \alpha) + \frac{h}{2} + u} q_0 \right] \, dr_i = \bar{r}_k \frac{p(1 - \alpha) - \frac{h}{2} + u + \alpha c_w - c_k}{p(1 - \alpha) + \frac{h}{2} + u} (b - a)
\]
\[ \Rightarrow \bar{r}_1 \bar{r}_k q_1 + \bar{r}_2 \bar{r}_k q_2 + \cdots + (\bar{r}_k^2 + \sigma_k^2) q_k + \cdots + \bar{r}_n \bar{r}_k q_n + K' \bar{r}_k q_0 = \bar{r}_k [K_k(b - a) - (x - a)] = \bar{r}_k \beta_k \]

\[ \Rightarrow K' \bar{r}_k q_0 + \bar{r}_2 \bar{r}_k q_1 + \bar{r}_2 \bar{r}_k q_2 + \cdots + M_k q_k + \cdots + \bar{r}_n \bar{r}_k q_n = \bar{r}_k \beta_k \]

(28)

where, \( \beta_k = K_k(b - a) - (x - a), (k = 1, 2, \cdots, n) \), \( K_k = \frac{p^{(1-\alpha)\frac{b}{2}+u+\alpha c_w-c_k}}{p^{(1-\alpha)\frac{b}{2}+u}}, (k = 1, 2, \cdots, n) \)

and \( K' = \frac{p^{(1-\alpha)+u+\alpha c_w-c_0}}{p^{(1-\alpha)+\frac{b}{2}+u}} \) and \( M_k = \bar{r}_k^2 + \sigma_k^2, (k = 1, 2, \cdots, n) \).

Therefore the set simultaneous equations presented in equations (29) and (30) can be represented in the matrix form as,

\[
\begin{bmatrix}
1 & \bar{r}_1 & \bar{r}_2 & \cdots & \bar{r}_N \\
K' \bar{r}_1 & M_1 & \bar{r}_2 & \cdots & \bar{r}_N \\
K' \bar{r}_2 & \bar{r}_2 \bar{r}_1 & M_2 & \cdots & \bar{r}_N \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
K' \bar{r}_N & \bar{r}_N \bar{r}_1 & \bar{r}_N \bar{r}_2 & \cdots & M_N
\end{bmatrix}
\begin{bmatrix}
q_0 \\
q_1 \\
q_2 \\
\vdots \\
q_N
\end{bmatrix} =
\begin{bmatrix}
\beta_0 \\
\bar{r}_1 \beta_1 \\
\bar{r}_2 \beta_2 \\
\vdots \\
\bar{r}_N \beta_N
\end{bmatrix}
\]

(29)

Since equation (31) is identical to equation (9) differing only in the expressions of \( K' \) and \( \beta_k \); form of the unconstrained and constrained solutions to problems P5, P6, P7 and P8 for the infinite time horizon scenario will be the same as the unconstrained and constrained solutions obtained for problems P1, P2, P3 and P4 for the single period scenario in Section 5.1.

1.2.2 Finite Time Horizon Solution

In case of finite time horizon multi-period problem, usage of calculus to directly obtain a solution for the problem becomes extremely cumbersome because of the reason that the non-linearity in the decision variables will increase with increase in number of periods considered. Therefore the objective function may not remain strictly concave.
For these reasons, forward recursion dynamic programming approach is used to find an optimal solution for finite $(N)$ period model. By definition, dynamic programming is a method for solving complex problems by sequentially solving simpler sub-problems. In dynamic programming approach, recursive equations are developed which sequentially solves the sub-problems, and ultimately identifies an optimal solution to the main problem.

In an optimization problem, dynamic programming refers to reducing the problem into a sequence of sub-problems over time. To do this, maximum value of the expected profit function for periods $j, j + 1, \ldots, N$ is defined as $\Pi_j(x_j)$. The expected profit function is associated with an argument $x_j$ which represents the level of inventory at the beginning of the $j^{th}$ period. Therefore, $\Pi_N(x_N)$ is defined as the maximum value of the expected profit for the $N^{th}$ period as a function of the state of the stage $x_N$ as expressed in equation (21). Subsequently, the values $\Pi_{N-1}, \Pi_{N-2}, \ldots, \Pi_1$ are calculated by working backwards, using a recursive relationship known as the Bellman equation as given by equation (22). The optimal values of the decision variables can be recovered individually by tracking back the previous calculations.

Following steps were followed to obtain a solution for finite time horizon multi-period problem:

Step 1: The value of the expected profit function for the $N^{th}$ period defined in equation (21) is calculated and optimized numerically to obtain a solution for $N^{th}$ period for a given level of inventory at the beginning of the $N^{th}$ period, $x_N$.

Step 2: Step 1 is repeated for different values of $x_N$'s to make a table of solutions for different initial conditions.

Step 3: The expected profit function for the $(N - 1)^{th}$ period is calculated for a given value of $x_{N-1}$ and using equation (22) and the expected amount of inventory on hand at the end of the
period (or, the expected amount of inventory on hand at the beginning of the next period, $x_N$) is calculated. For the calculated expected value of $x_N$, the maximum expected profit for the last period is obtained from the table made in step 2. The sum of the expected profit function for the $(N - 1)^{th}$ period and the maximum expected profit for the last period is maximized to obtain the solution for $(N - 1)^{th}$ period for a given value of $x_{N-1}$.

Step 4: Step 3 is repeated for different values of $x_{N-1}$’s to make a table of solutions for different initial conditions.

Step 5: Steps 3 and 4 are repeated for $N = N - 1$, till $N$ becomes 1. For $N = 1$, obtain the solution for the observed value of initial inventory level $x_1$. Finally, the entire calculation is backtracked to obtain the solution for each period for that observed value of $x_1$.

For illustration, dynamic programming approach is used to obtain an optimal solution for $N = 3$ (see Appendix 2). The package used for the non-linear optimization in this illustration is Microsoft Excel 2007. For non-linear optimization problems Excel 2007 uses generalized reduced gradient (GRG) algorithm developed by Lasdon et al (1978). By testing on 70 different non-linear optimization problems Kao (1998) showed that performance of GRG is better compared to other non-linear programming algorithms like sequential quadratic programming (SQP), reduced gradient (RG) etc.