CHAPTER 4

MODELLING AND ANALYTICAL SOLUTION TO HEAT TRANSFER AND BOUNDARY LAYER FLOW WITH SUCTION AND INJECTION*

4.1 INTRODUCTION

This chapter demonstrates the practical utility of the governing principle of dissipative processes to heat transfer in constant property boundary layer with suction and injection when free stream and wall temperature are non-uniform over a wedge whose apex angle is $\pi \beta$. The energy equation has been solved numerically by Evans (1967) for constant surface temperature. Levy (1952) has carried out extensive numerical computations to obtain heat transfer solutions for several values of $m$ and Prandtl number $Pr$. An approximate calculation of heat transfer has been developed by Lighthill (1950) with the assumption that the velocity profile in the boundary layer control could be approximated by its tangent at the wall.

Imai (1958) has developed a new approximate method and found the solution to the energy equation assuming \( m = 0 \) and 1. An exact solution of the energy equation for \( Pr = 1 \) has been obtained by Edgerton (1963) with the help of the vorticity transport theorem and Reynold’s analogy. Extensive tables for boundary layers on a plate \((m=0)\) with suction covering a wide range of values for the parameter \( H \) have been calculated by Emmons and Leigh (1954). For cases when there exist additional numerical solutions extending over a wide range of values of the parameters have been investigated by Nickel (1962).

The purpose of the present analysis is to obtain rapid analytical solutions for the flow and heat transfer in boundary layer with uniform suction/injection over a wedge surface. The results obtained by the present analysis are compared with the numerical solution of Evans (1967), exact solution of Edgerton (1963) and Nickel (1962) and it is found that the difference is less than two percent and the comparison shows good agreement.

### 4.2 GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

After applying the usual boundary layer approximations, the conservation equations of mass, momentum and energy for steady, two dimensional, laminar flow, with constant physical properties and neglecting viscous dissipation effects
are
\[ u_x + v_y = 0, \quad \text{(mass)} \quad (4.1) \]
\[ uu_x + vu_y = \nu u_{yy} + U_\infty(U_\infty)_x, \quad \text{(momentum)} \quad (4.2) \]
\[ uT_x + vT_y = \alpha T_{yy}, \quad \text{(energy)} \quad (4.3) \]
The compatibility conditions are
\[
\begin{align*}
y = 0, & \quad u = 0, \quad v = v_0, \quad T = T_0 \quad \text{(uniform)} \\
y = \infty, & \quad u = U_\infty, \quad T = T_\infty. \quad \text{(uniform)}
\end{align*}
\quad (4.4)
\]
The balance equations of the present problem are used in the formulation of Gyarmati’s variational principle and hence the governing Equations (4.1)-(4.3) are written in the balance form as
\[
\nabla \cdot \vec{V} = 0, \quad (\vec{V} = iu + jv) \quad (4.5)
\]
\[
\rho (\vec{V} \cdot \nabla)\vec{V} + \nabla \cdot \vec{P} = 0, \quad (4.6)
\]
\[
\rho C_p (\vec{V} \cdot \nabla)T + \nabla \cdot (\vec{J}_q) = 0. \quad (4.7)
\]
These equations represent the mass, momentum and energy balances, respectively. The energy dissipation \( T\sigma \) for the present system is given by (Gyarmati (1969))
\[
T\sigma = -J_q (\partial \ln T/\partial y) - P_{12} (\partial u/\partial y), \quad (4.8)
\]
the heat flux \( J_q \) and \( P_{12} \) the only component of momentum flux \( \vec{P}^{us} \), satisfy the conservative relations connecting the independent fluxes and forces as
\[
J_q = -L_\lambda (\partial \ln T/\partial y) \quad \text{and} \quad P_{12} = -L_s (\partial u/\partial y). \quad (4.9)
\]
The dissipation potentials \( \Psi \) and \( \Phi \) in the energy picture are given by
\[
T\Psi = \Psi^* = (1/2)[L_\lambda (\partial \ln T/\partial y)^2 + L_s (\partial u/\partial y)^2], \quad (4.10)
\]
\[
T\Phi = \Phi^* = (1/2)[R_\lambda J_q^2 + R_s P_{12}^2]. \quad (4.11)
\]
Using the Equations (4.8)-(4.11), Gyarmati’s variational principle in the energy picture (2.12) is formulated and it has the following form

\[
\delta \int_0^l \int_0^\infty \left[ -J_q (\partial \ln T / \partial y) - P_{12} (\partial u / \partial y) - (L_\lambda / 2)(\partial \ln T / \partial y)^2 
\right. \\
\left. - (L_s / 2)(\partial u / \partial y)^2 - (R_\lambda / 2) J_q^2 - (R_s / 2) P_{12}^2 \right] dy dx = 0. \quad (4.12)
\]

### 4.3 CALCULATION PROCEDURE

It is considered that the system of two dimensional laminar, inviscid potential flow past an unlimited wedge placed symmetrically in a stream with apex at the origin and the center line on the positive x-axis. The main stream velocity \( U_\infty \) and the wall temperature are assumed to vary as power functions of distance from the start of the boundary layer respectively as

\[
U_\infty = ax^m \quad \text{and} \quad T_0 - T_\infty = bx^n, \quad (4.13)
\]

where \( a \) and \( b \) are constants and the exponent “\( m \)” is connected with the apex angle \( \pi \beta \) by the relation, (Eckert (1979)).

\[
m = \beta/(2 - \beta) \quad \text{or} \quad \beta = 2m/(m + 1). \quad (4.14)
\]

In Equation (4.13), \( T_0 \) is the temperature of the surface and \( T_\infty \) is the free stream temperature respectively. Here the analysis is carried out for the entire range of realistic flow, that means when \( 0 \leq m < \infty \) or \( 0 \leq \beta < 2 \). The velocity and temperature fields in their respective boundary layer regions are suitably described by the following functions
\[ u/U_\infty = \begin{cases} 
3y/d_1 - 3y^2/d_1^2 + y^3/d_1^3, & (y < d_1) \\
U_\infty, & (y \geq d_1)
\end{cases} \]
\[ (T - T_\infty)/(T_0 - T_\infty) = \begin{cases} 
1 - 3y/2d_2 + y^3/2d_2^3, & (y < d_2) \\
T = T_\infty. & (y \geq d_2)
\end{cases} \] (4.15)

Where \( d_1 \) and \( d_2 \) are the momentum and thermal boundary layer thicknesses respectively. The velocity and thermal profiles (4.15) satisfy the following compatibility conditions
\[ \begin{align*}
y &= 0, & u &= 0, & v &= v_0, & T &= T_0, & T_y &= 0, \\
y &= d_1, & u &= U_\infty, & u_y &= 0, & u_{yy} &= 0, \\
y &= d_2, & T &= T_\infty, & T_y &= 0.
\end{align*} \] (4.16)

The smooth fit conditions \( u_y = 0 \) and \( T_y = 0 \) correspond to \( P_{12} = 0 \) and \( J_q = 0 \) at the respective edges of the boundary layers. Here \( d_1 \) and \( d_2 \) are unknown parameters and they are to be determined by the present thermodynamic analysis. The transverse velocity component \( v \) is obtained from the mass balance Equation (4.1) as
\[ v = [U_\infty d_1'][3y^2/2d_1^2 - 2y^3/d_1^3 + 3y^4/4d_1^4] + [U_\infty m/x][-3y^2/2d_1 + y^3/d_1^2 - y^4/4d_1^3] + v_0. \] (4.17)

The velocity and temperature functions (4.15) are substituted in the momentum and energy balance Equations (4.2) and (4.3), and on direct integration with respect to \( y \) with the help of smooth fit boundary conditions the fluxes \( P_{12} \) and \( J_q \) are obtained respectively. The expression for \( P_{12} \) remains the same for any Prandtl number \( Pr \). But the energy flux \( J_q \) assumes different expression for \( Pr \leq 1 \) and \( Pr \geq 1 \) respectively. When \( Pr \leq 1 \) the expression for \( J_q \) in the range \( d_1 \leq y \leq d_2 \) is obtained first and the expression for \( J_q \) in the range \( 0 \leq y \leq d_1 \) is determined subsequently by matching
the $J_q$ expressions of the two regions at the interface. The expressions for momentum and energy fluxes $P_{12}$ and $J_q$ are as follows respectively.

\[ -P_{12}/L_s = \frac{[U_\infty / d_1] + [mU_\infty^2 / \nu x]53d_1/160 - y + 3y^3/2d_1^2 - 3y^4/2d_1^3 + 3y^5/4d_1^4 - y^6/4d_1^5 + y^7/28d_1^6]}{1} + [U_\infty^2 d_1'/\nu][9/160 - 3y^3/2d_1^2 + 3y^4/4d_1^3 - 9y^5/2d_1^3 + 3y^6/4d_1^6 - 3y^7/28d_1^7] + [v_0U_\infty / \nu][-3/4 + 3y/d_1] \\
-3y^2/d_1^2 + y^3/d_1^3], \quad (0 \leq y \leq d_1) \quad (4.18) \]

\[ -J_q/L_\lambda = \frac{[PrU_\infty(T_0 - T_\infty)/\nu]3y^3/2d_1d_2 - 9y^5/10d_1d_2^2 - 9y^4/8d_1^2d_2^2 + 3y^5/10d_1^2d_2^2 - 3y^7/14d_1^3d_2^3 + 3d_1^4/40d_2^4 - 3d_1^3/280d_2^3}{1} + [mPrU_\infty(T_0 - T_\infty)/\nu x][3y^3/4d_1d_2 - 9y^5/20d_1d_2^2 - 3y^4/8d_1^2d_2^3 + 3y^6/4d_1^2d_2^3 + y^7/14d_1^3d_2^3 + 3d_1^4/40d_2^4 - 3d_1^3/280d_2^3] \\
\times[3y^2/2d_1 - 3y^3/2d_1d_2 + 3y^4/10d_1^2d_2 - y^5/d_2^2 + 9y^4/8d_1^2d_2^2 - y^6/4d_1^2d_2^3 - 3y^5/10d_1^2d_2^3 - 3y^7/14d_1^3d_2^3 + 3d_1^4/40d_2^4 - 3d_1^3/280d_2^3] + [PrU_\infty(T_0 - T_\infty)/\nu] \\
\times[-3y^3/4d_1^2d_2 + 9y^5/20d_1^2d_2^2 + 3y^4/4d_1^2d_2 - y^6/2d_1^2d_2^3 - 9y^5/40d_1^2d_2 - 9y^7/56d_1^3d_2^3 - 9y^6/20d_1^2d_2^3 - 9y^7/20d_1^2d_2^3 - 3y^3/2d_2 - 3y^2/2d_2^2 + 3d_1/2d_2 - d_1^3/2d_2^3], \quad (0 \leq y \leq d_1), \quad (Pr \leq 1) \quad (4.19) \]
\[-J_q/L_\chi = [PrU_\infty(T_0 - T_\infty)/\nu][3y^2/4d_2^2 - \frac{3}{8}d_2^4/8d_2^2 - 3/8]d_2^2 \]
\[+ [mPrU_\infty(T_0 - T_\infty)/\nu x][3y^2/4d_2 - 3y^4/8d_2^3 - 3d_2/8] \]
\[+ [nPrU_\infty(T_0 - T_\infty)/\nu x][y - 3y^2/4d_2 + y^4/8d_2^3 - 3d_2/8], \quad (d_1 \leq y \leq d_2), \quad (Pr \leq 1) \quad (4.20) \]

\[-J_q/L_\chi = [PrU_\infty(T_0 - T_\infty)/\nu][3y^3/2d_1d_2^2 - 9y^5/10d_1d_2^2 \]
\[-9y^4/8d_1^2d_2 - 9d_1^2d_2 + 3y^6/4d_1^2d_4^1 + 3y^5/10d_1^3d_2 - 3y^7/14d_1^3d_2^2 - 3d_2/5d_1 \]
\[+ 3d_2/8d_1^1 - 3d_2^3/35d_1^3]d_2^4 + [mPrU_\infty(T_0 - T_\infty)/\nu x] \]
\[\times [3y^3/4d_1d_2 - 3y^4/8d_1^1d_2 + 3y^5/40d_1^3d_2 - 9y^5/20d_1d_2^3 \]
\[+ y^6/4d_1^2d_2^3 - 5y^7/56d_1^4d_2^2 - 3d_2/10d_1 + d_2^3/8d_1^1 - 3d_2^3/140d_1^3] \]
\[+ [nPrU_\infty(T_0 - T_\infty)/\nu x][3y^2/2d_1 - 3y^3/2d_1d_2 + 3y^5/10d_1d_2^5 \]
\[-y^3/d_1^2 + 9y^4/8d_1^3d_2 - y^6/4d_1^2d_2^3 + y^4/4d_1^3 - 3y^5/10d_1^3d_2 \]
\[+ y^7/14d_1^3d_2^3 - 3d_2^2/10d_1 + d_2^3/8d_1^1 - 3d_2^3/140d_1^3] \]
\[+ [PrU_\infty(T_0 - T_\infty)/\nu][-3y^3/4d_1^2d_2 + 9y^5/20d_1d_2^3 \]
\[+ 3y^4/4d_1^2d_2 - y^6/2d_1^3d_2^3 - 9y^5/40d_1^3d_2 + 9y^7/56d_1^4d_2^3 \]
\[+ 3d_2^2/10d_1^2 + d_2^4/4d_1^1 + 9d_2^4/140d_1^4]d_1^4 + [v_0 Pr(T_0 - T_\infty)/\nu] \]
\[\times [-3y^2/2d_1 + y^3/2d_1^3 + 1]. \quad (0 \leq y \leq d_2) \quad (Pr \geq 1) \quad (4.21) \]

The prime indicates the differentiation with respect to \( x \). Using the expressions of \( P_{12} \) and \( J_q \) along with the velocity and temperature functions \( (4.15) \) the variational principle \( (4.12) \) is formulated independently for \( Pr \geq 1 \) and \( Pr \leq 1 \) respectively. After performing the integration with respect to \( y \) one can obtain the variational principle in the following forms respectively.

\[ \delta \int_0^1 L_{1,2}[d_1, d_2, d_1', d_2']dx = 0, \quad (Pr \leq 1), \quad (Pr \geq 1). \quad (4.22) \]
The explicit expressions for the Lagrangian densities are given below.

\[
\delta \int_0^1 \left( [U_\infty(T_0 - T_\infty)^2 Pr/\nu] \left\{ \begin{array}{c}
[0.257142857 - 0.0375d_1^2/d_2^2] \\
+ 0.0107142861d_1^2/d_2^2 - 0.000892857286d_1^4/d_2^2] [d_1^2] + [0.139285714d_2^2] \\
- 0.075d_1^2/d_2 + 0.0375d_1^2/d_2^2 + 0.00357142857d_1^4/d_2^2 \\
- 0.00537142d_1^5/d_2^2 + 0.000297618762d_1^7/d_2^6] [n/x] \\
+ [0.257142857d_2 - 0.2625d_1^3/d_2^2 + 0.184821429d_1^3/d_2^2] \\
- 0.0311011908d_1^5/d_2^2] [m/x] + [-0.030208333d_1^6/d_2^6] \\
+ 0.174107142d_1^4/d_2^2 - 0.225d_1^3/d_2^2] [d_1^4] + [-0.125d_1^6/d_2^6] \\
+ 0.75d_1^4/d_2^4 - 1.125d_1^5/d_2^5] [v_0/U_\infty] \right\} + [(T_0 - T_\infty)^2 (-0.6/d_2)] \\
+ [U_\infty^2(T_0 - T_\infty)^2 Pr^2/\nu^2] \left\{ [-0.285714286d_2 + 0.009375d_1^2/d_2^2] \\
- 0.00400162348d_1^5/d_2^4 + 0.00081761996d_1^6/d_2^6 - 0.00000538975 \\
x d_1^5/d_2^5] [d_2^2] + [-0.0109126984d_1^2 + 0.01875d_1^3/d_2^2 - 0.021180556d_1^3] \\
+ 0.0100595238d_1^4/d_2^2 - 0.0013230522d_1^5/d_2^2 - 0.000711579d_1^6/d_2^6 \\
+ 0.00027523803d_1^4/d_2^2 - 0.00000889485965d_1^6/d_2^6] [n^2/x^2] \\
+ [-0.0285714286d_2 + 0.065625d_1^3 - 0.0774320215d_1^4/d_2^2 \\
+ 0.0394141798d_1^4/d_2^2 - 0.00837031280d_1^6/d_2^6] [m^2/x^2] \\
+ [-0.0137297078d_1^8/d_2^2 + 0.0151486794d_1^5/d_2^4 - 0.00426353774 \\
x d_1^7/d_2^2] d_1^2 + [-0.375d_1^4/d_2^2 + 0.3375d_1^5/d_2^2 - 0.080357143d_1^7/d_2^6] \\
x [v_0^2/U_\infty^2] + [-0.032738571d_2^2 + 0.01875d_1^3 - 0.0100595238d_1^4/d_2^2 \\
+ 0.00264610357d_1^6/d_2^6 + 0.00213474042d_1^5/d_2^4 \\
- 0.00109015976d_1^7/d_2^2 + 0.0000533692115d_1^8/d_2^6] [d_2 n/x] \\
+ [-0.0571428571d_2^2 + 0.075d_1^3/d_2^2 - 0.067703937d_1^5/d_2^3] \\
\right\] \]
\[ +0.0250831203d_1^2/d_2^2 - 0.00418682889d_1^2/d_2^2][d'/x] \]
\[ +[0.05625d_1^2/d_2 - 0.0597006897d_1^2/d_2^2 + 0.0234478801d_1^2/d_2^2 \]
\[ -0.0040262101d_1^2/d_2^2][d'/d_2] + [0.28125d_1^2/d_2 \]
\[ -0.263839286d_1^2/d_2^2 + 0.0991815478d_1^2/d_2^2 - 0.0165900261d_1^2/d_2^2 \]
\[ \times [d_2v_0/U_\infty] + [-0.0327380952d_1^2 + 0.01875d_1^2/d_2 + 0.05625d_3^3 \]
\[ -0.07625d_1^2/d_2^2 - 0.00294338517d_1^2/d_2^2 + 0.0433394213d_1^2/d_2^2 \]
\[ -0.020729305d_1^2/d_2^2 + 0.00139560952d_1^2/d_2^2][mn/x^2] \]
\[ +[0.05625d_1^2 - 0.0661904762d_1^3/d_2 - 0.00558948837d_1^4/d_2^4 \]
\[ +0.0412046803d_1^2/d_2^2 - 0.0197028750d_1^2/d_2^2 + 0.00134224028d_1^2/d_2^2 \]
\[ \times [nd_1^3/x] + [0.28125d_1^2 - 0.3125d_1^2/d_2 - 0.0174107143d_1^2/d_2^2 \]
\[ +0.174553571d_1^2/d_2^3 - 0.0828125d_1^2/d_2^4 + 0.00553300844d_1^2/d_2^6 \]
\[ \times [nv_0/xU_\infty] + [0.05625d_1^2 - 0.0871601053d_1^4/d_2^2 \]
\[ +0.0537452389d_1^2/d_2^4 - 0.0125537966d_1^2/d_2^6][md_1/x] + [0.281250d_1^2 \]
\[ -0.405803571d_1^2/d_2^2 + 0.242201905d_1^2/d_2^2 - 0.053591721d_1^2/d_2^6 \]
\[ \times [mv_0/xU_\infty] + [-0.141964286d_1^2/d_2^2 + 0.143080356d_1^2/d_2^4 \]
\[ -0.0369926945d_1^2/d_2^2][d'v_0/U_\infty] \] \}
\[ dx = 0. \quad \{Pr \leq 1\} \quad (4.23) \]
and

\[ \delta \int_0^l \left[ [U_\infty(T_0 - T_\infty)^2 Pr/v] \left\{ [0.075d_2^2/d_1^2 - 0.314285714d_2^2/d_1^2 \right. \right. \]
\[ +0.46875d_2^2/d_1^2][d_2^2] + [0.0160714281d_1^2/d_2^3 - 0.0845238095d_2^3/d_1^2 \]
\[ +0.16875d_2^2/d_1^2][n/x] + [-0.05625d_1^2/d_1^4 + 0.20952381d_2^3/d_1^2 \]
\[ -0.234375d_2^2/d_1^2][d_1^4] + [0.01875d_2^2/d_1^4 - 0.104761905d_2^3/d_1^2 \]
+0.234375d_2^2/d_1[m/x] - [0.5v_0/U_\infty]\left\{ [0.6(T_0 - T_\infty)^2/d_2 +
+U_\infty^2(T_0 - T_\infty)^2 Pr^2/\nu^2]\right\} +\left[0.00236013982d_2^2/d_1^5
+0.0197257651d_2^3/d_1^4 - 0.070641858d_2^5/d_1^4
+0.123348214d_2^4/d_1^3 - 0.0925324675d_2^3/d_1^4[2d_2^2]
\right]
+[−0.00011208231d_2^3/d_1^5 + 0.00118255254d_2^5/d_1^5
−0.005561938d_2^5/d_1^4 + 0.0129910714d_2^6/d_1^3
−0.0135551948d_2^3/d_1^4[2^n^2/x^2] + [−0.00132757866d_2^8/d_1^8
+0.0098628826d_2^6/d_1^7 - 0.0293628246d_2^7/d_1^6
+0.0411160714d_2^4/d_1^5 - 0.0231331169d_2^5/d_1^6[2d_2^2]
\right]
+[−0.000147508734d_2^6/d_1^6 + 0.00164381374d_2^6/d_1^7
−0.0082558066d_2^5/d_1^4 + 0.0205580357d_2^6/d_1^3
−0.0231331169d_2^5/d_1^4[m^2/x^2] + [−0.117857143d_2(v_0^2/U_\infty^2)
+0.00100087409d_2^8/d_1^8 + 0.00946042038d_2^7/d_1^5
−0.0389335662d_2^5/d_1^4 + 0.0779464286d_2^5/d_1^3
−0.067775974d_2^4/d_1^3][2n/x] + [0.00354020974d_2^6/d_1^6
−0.027944834d_2^7/d_1^6 + 0.0917488759d_2^6/d_1^7
−0.143906250d_2^5/d_1^4 + 0.0925324675d_2^6/d_1^4]
×[d_1^5d_2^2] + [−0.00118006989d_2^8/d_1^6 + 0.0115066963d_2^5/d_1^6
−0.0495348399d_2^5/d_1^4 + 0.102790179d_2^6/d_1^3
−0.0925324675d_2^4/d_1^4][2m/x] + [0.0305735928d_2^4/d_1^3
−0.130580357d_2^3/d_1^2 + 0.2d_2^3/d_1^2][d_2^2v_0/U_\infty]
+[0.000750655571d_2^6/d_1^7 − 0.00674227921d_2^6/d_1^6]
\[ +0.0257495628 \frac{d_2^6}{d_1^6} - 0.0463318452 \frac{d_2^5}{d_1^5} + 0.033887987d_2^5/d_1^5\right] [d_1' u/x] + \left[-0.000250218519d_2^5/d_1^5 \right. \\
+ 0.00271814117 \frac{d_2^5}{d_1^5} - 0.0131840034d_2^5/d_1^5 \\
+ 0.0316145833d_2^5/d_1^5 - 0.033887987d_2^5/d_1^5\right][mn/x^2] \\
+ \left[0.00702110374d_2^5/d_1^5 - 0.038169649d_2^4/d_1^4 \right. \\
+ 0.0797619048d_2^4/d_1^4\right][nu_0/xU_\infty] + \left[0.000885052423d_2^4/d_1^4 \right. \\
- 0.00821906786d_2^4/d_1^4 + 0.0330232266d_2^4/d_1^4 \\
- 0.0616741071d_2^4/d_1^4 + 0.0462662338d_2^4/d_1^4\right][d_1' m/x] \\
+ \left[-0.0229301947d_2^4/d_1^4 + 0.0870535714d_2^4/d_1^4 - 0.1d_2^4/d_1^4 \right. \\
\times [d_1' u_0/xU_\infty] + \left[0.007643398116d_2^4/d_1^4 - 0.0435267857d_2^4/d_1^4 \right. \\
+ 0.1d_2^4/d_1^4\right][nu_0/xU_\infty] \bigg) \bigg) dx = 0, \quad \{Pr \geq 1\} \quad (4.24) \\
\]

The variational principles (4.22) are found identical when \(d_1 = d_2\). Accordingly, the Euler-Lagrange equations are

\[
\left(\frac{\partial L_{1,2}}{\partial d_2} - \frac{d}{dx}\left(\frac{\partial L_{1,2}}{\partial d'_2}\right)\right) = 0. \quad (Pr \leq 1, \quad Pr \geq 1) \quad (4.25)
\]

Equations (4.25) are second order ordinary differential equations in terms of \(d_1, d_1', d_2\) and \(d_2'\) respectively. These equations are simplified by introducing the non-dimensional boundary layer thicknesses \(d_1'\) and \(d_2'\) using the relation

\[
d_1 = d_1' \sqrt{\nu x/U_\infty} \quad \text{and} \quad d_2 = d_2' \sqrt{\nu x/U_\infty}. \quad (4.26)
\]

The resulting Euler-Lagrange equations are

\[
\left(\frac{\partial L_{1,2}}{\partial d_1'}\right) = 0, \quad (4.27)
\]

\[
\left(\frac{\partial L_{1,2}}{\partial d_2'}\right) = 0, \quad (Pr \leq 1, \quad Pr \geq 1) \quad (4.28)
\]
whose coefficients depend on the parameters $Pr$, $m$, $n$, and $H$, where $H$ is the non-dimensional suction/injection speed and is given by

$$H = v_0 \sqrt{Re/U_\infty}.$$ (4.29)

Here $Re$ denotes the Reynolds number $(U_\infty x / \nu)$. Suction and injection are represented by $H < 0$ and $H > 0$ respectively. Equation (4.27) is a simple polynomial equation in terms of momentum boundary layer thickness whose coefficients depend on the wedge angle parameter $m$, and the suction/injection parameter $H$. This equation is solved easily for any given combinations of $m$ and $H$ and the corresponding hydrodynamical boundary layer thickness $d_1^*$ is obtained as the only positive root. The polynomial Equations (4.28) are solved for given values of $Pr$, $m$, $n$ and $H$ and it is found that for any value of $Pr$ there corresponds only one real root $d_2^*$ which is the thermal boundary layer thickness. After getting $d_1^*$ and $d_2^*$ for given values of $Pr$, $m$, $n$ and $H$ the values of shear stress and local heat transfer (Nusselt number) are calculated with the help of the following relations respectively.

$$\tau_\omega = \sqrt{2 \nu x / U_\infty y^3 (m + 1)} (-P_{12}/L_{s})_{y=0}, \quad (4.30)$$

$$Nu_x = \sqrt{2 \nu x / U_\infty y^3 (m + 1)} (-J_{q}/L_{\lambda})_{y=0}. \quad (4.31)$$

### 4.4 DISCUSSION

In engineering applications the skin friction and heat transfer play a vital role. The thermal energy equation has been solved for two cases $d_1^* \leq d_2^* (Pr \leq 1)$ and $d_1^* \geq d_2^* (Pr \geq 1)$. These two independent analyses
yield solutions which match at $Pr=1$. It is found that both the analyses lead to satisfactory results in the respective ranges of Prandtl number. Although the present analysis is based on the assumption $Pr \geq 1$, it is found that the analysis remains valid for Prandtl numbers as low as 0.05.

To verify the accuracy of the present results, they are compared with the numerical solution of Evans (1967) for the case of $\beta = 0.5$, $H = 0$ when the wall temperature is uniform ($n = 0$). The error in the present results lies within 2% as it can be seen in Table 4.1. According to Edgerton (1963) an exact solution exists for the energy equation when $Pr = 1$ and the exact value of the local Nusselt number for this Prandtl number is given by the relation

$$Nu_{exact} = (m\beta)^{1/2} \sqrt{2/(m+1)/f''(0)} \text{ when } n = (3m-1)/2. \quad (4.32)$$

Where $f''(0)$ is the local shear stress value. The heat transfer values obtained from the present method are compared with this exact solution for various combination of $m$ and $H$ and the values are exhibited in Table 4.2. It is noted that the simple polynomial functions assumed for velocity and temperature fields lead to result which are very close to exact value of Edgerton (1963). The local surface friction (skin friction) values calculated with the help of Equation (4.30) are compared with the exact solution obtained by Nickel (1962). The skin friction values are plotted in Figure 4.1 as a function of wedge parameter $\beta$ and it can be observed that the obtained skin friction values by the present variational technique are very close to Nickel’s values. Further, it is demonstrated that the skin friction values are decreasing function of $H$. The variation of heat transfer values with $n$, for $H = 0$, $Pr=0.7$, 10, 100 and $\beta = 0.5$, 1 and 1.6 are plotted in Figure 4.2.
Table 4.1  Comparison of heat transfer values with Evans values for \( n = 0, \ H = 0 \) and \( \beta = 0.5 \)

<table>
<thead>
<tr>
<th>( Pr )</th>
<th>Evans (1967) numerical values</th>
<th>Present values</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td>0.00793</td>
<td>0.007990042</td>
</tr>
<tr>
<td>0.001</td>
<td>0.02473</td>
<td>0.025227770</td>
</tr>
<tr>
<td>0.01</td>
<td>-</td>
<td>0.078645570</td>
</tr>
<tr>
<td>0.1</td>
<td>0.2132</td>
<td>0.224632070</td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>0.538779069</td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>1.238009513</td>
</tr>
<tr>
<td>100</td>
<td>2.7367</td>
<td>2.739256052</td>
</tr>
<tr>
<td>1000</td>
<td>-</td>
<td>5.971178718</td>
</tr>
<tr>
<td>8000</td>
<td>11.9696</td>
<td>12.00175441</td>
</tr>
<tr>
<td>10000</td>
<td>-</td>
<td>12.93306190</td>
</tr>
<tr>
<td>20000</td>
<td>16.2637</td>
<td>16.30998000</td>
</tr>
</tbody>
</table>

Table 4.2  Comparison of heat transfer values with exact solution for \( Pr = 1 \) and \( H = 0 \)

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( n )</th>
<th>( Nu_{exact} )</th>
<th>Present values</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>4/17</td>
<td>0.3872</td>
<td>0.3823</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>0.5390</td>
<td>0.5390</td>
</tr>
<tr>
<td>0.8</td>
<td>1/2</td>
<td>0.7141</td>
<td>0.7132</td>
</tr>
<tr>
<td>1.0</td>
<td>1</td>
<td>0.8113</td>
<td>0.8058</td>
</tr>
<tr>
<td>1.2</td>
<td>7/4</td>
<td>0.8984</td>
<td>0.8920</td>
</tr>
<tr>
<td>1.6</td>
<td>11/2</td>
<td>1.0516</td>
<td>1.0392</td>
</tr>
</tbody>
</table>

From this figure, it is evident that the heat transfer is vanishing for \( \beta = 1 \) and 1.6 when \( n=-1, \ -2.5 \) and \( H = 0 \) respectively. The vanishing of the heat transfer is irrespective of the Prandtl number. It is also indicated by Figure 4.2 that the heat transfer increases with \( \beta \) and Prandtl number.
Figure 4.3 shows the effect of $n$ on the local Nusselt number values. The heat transfer is found increasing with $n$. One can easily observe from this figure that for a given Prandtl number and $H = 0$, the local Nusselt value tends to a limit value when $\beta$ tends to 2.0. The local heat transfer values which are obtained by the present variational technique are presented graphically in Figures 4.4-4.11, for various combinations of $Pr, \beta, n$ and $H$. From these figures, it is noted that during suction and injection the local heat transfer values increase with Prandtl numbers for all the values of $\beta$. And also for higher Prandtl numbers, the local heat transfer values are very high irrespective of $\beta, n$ and $H$. Since the analysis is carried out for the entire range of realistic flows ($0 \leq \beta < 2$) the heat transfer values can be calculated for any given values of $Pr, \beta, n$ and $H$.

![Graph showing skin friction as a function of $\beta$ for various values of $H$](image)

**Figure 4.1** Skin friction as a function of $\beta$ for various values of $H$
Figure 4.2  Variation of local Nusselt number as a function of $n$ for $\beta = 0.5, 1.0, 1.6$ and $Pr = 0.7, 10, 100$ when $H = 0$

Figure 4.3  Variation of local Nusselt number as a function of $\beta$ for $Pr = 0.7, 100$ and 1000 when $H = 0$
Figure 4.4  Variation of local Nusselt number with $\log(Pr)$ for various $\beta$ when $n = 1$ and $H = -3$

Figure 4.5  Variation of local Nusselt number with $\log(Pr)$ for various $\beta$ when $n = 1$ and $H = 3$
Figure 4.6  Variation of local Nusselt number with $\log(Pr)$ for various $\beta$ when $n = -1$ and $H = -3$

Figure 4.7  Variation of local Nusselt number with $\log(Pr)$ for various $\beta$ when $n = -1$ and $H = 3$
Figure 4.8  Variation of local Nusselt number with $\log(Pr)$ for various $\beta$ when $n = 1$ and $H = -3$

Figure 4.9  Variation of local Nusselt number with $\log(Pr)$ for various $\beta$ when $n = 1$ and $H = 3$
Figure 4.10  Variation of local Nusselt number with $Log(Pr)$ for various $\beta$ when $n = -1$ and $H = -3$

Figure 4.11  Variation of local Nusselt number with $Log(Pr)$ for various $\beta$ when $n = -1$ and $H = 3$
4.5 CONCLUSION

This chapter gives an analytical answer to the flow and heat transfer with suction and injection over a wedge. The governing partial differential equations are transformed to polynomial equations the coefficients of which are functions of independent parameter $Pr$ and the suction/injection parameter $H$. By using these simultaneous equations engineers and scientists can obtain skin friction and heat transfer values rapidly at the wall surface for any given value of $Pr$, $\beta$, $n$, $m$ and $H$. Compared with the formidable task of solving the governing equations numerically, the present variational technique solves the problem in a much easier way. While comparing the conventional exact methods with the present method, the results in the latter are obtained with remarkable accuracy and moreover, the amount of calculation is less.