CHAPTER 3

CHAOTIC DYNAMICS OF SECOND-ORDER NON-AUTONOMOUS SYSTEMS WITH THRESHOLD NONLINEARITY

3.1 INTRODUCTION

A detailed experimental investigations along with numerical studies of third-order autonomous systems and second-order non-autonomous systems with nonlinearities $x^2$, $|x|$, max($x$) and min($x$) have been made in chapter 2. Further in this chapter, chaotic dynamics of certain second-order non-autonomous systems with a new class of nonlinearity, known as threshold nonlinearity, is considered. The threshold mechanism is simple and easily implementable (Sinha 2000). Control will be triggered whenever the value of state variable exceeds a selected threshold value and the variable then be reset to the threshold value. That is, clipping in the signal with respect to the state variable takes place at the threshold value. The characteristic of threshold nonlinearity has been depicted in Figures 3.1a and 3.1b. This threshold type nonlinearity has been used to design two simple second-order non-autonomous systems to exhibit order and chaos phenomena. The implementation and results are discussed as follows:
Figure 3.1  (a) Timing waveforms of state variable $x$ and thresholded variable $x^*$ to the value of 0.5 and (b) Plot between $x$ and $x^*$
3.2 SIMPLE SECOND-ORDER NON-AUTONOMOUS SYSTEM

The dynamical equation of the second order non-autonomous system is given by

\[ \frac{d^2x}{dt^2} = -\alpha \frac{dx}{dt} - \beta x - \gamma G(x) + f \sin \omega t. \]  

(3.1)

This equation can be written as two first-order coupled nonlinear differential equations as:

\[ \frac{dx}{dt} = y, \]  

(3.2a)

\[ \frac{dy}{dt} = -\alpha y - \beta x - \gamma G(x) + f \sin \omega t. \]  

(3.2b)

Here \( \alpha, \beta, \gamma \) are constants, \( f \) is the amplitude and \( \omega \) is the angular frequency of the driving sinusoidal force. \( G(x) \) is the nonlinear characteristic function or piecewise linear function corresponding to threshold controller and is given by

\[ G(x) = \begin{cases} 
  x^*, & \text{when } x > x^* \\
  x, & \text{when } -x^* \leq x \leq x^* \\
  -x^*, & \text{when } x < -x^* 
\end{cases} \]  

(3.3)

where \( x^* \) is the threshold value and is equal to 0.7. If variable \( x \) is greater than threshold value \( x^* (= 0.7) \) then the variable is adjusted to \( x^* (= 0.7) \). When \( x < -x^* \), there is no controlling action. The other parameter values in equation (3.2) are fixed as \( \alpha = 0.5, \beta = 1.8, \gamma = -3 \) and \( \omega = 1.1 \). The amplitude of the driving force ‘f’ is considered as the single control parameter.
3.2.1 Stability analysis.

The fixed points of equation (3.2) in the absence of external force are determined by equating \( \frac{dx}{dt} = \frac{dy}{dt} = 0 \), which gives

\[
\frac{dx}{dt} = y = 0, \quad (3.4a)
\]

\[
\frac{dy}{dt} = -\beta x - \gamma G(x) = 0. \quad (3.4b)
\]

Roots of equations (3.4a) and (3.4b) are the fixed points being admitted by this system. The fixed points \((x_0, y_0)\) are corresponding to each of the three conditions of threshold function \(G(x)\), represented as in equation (3.3). For the fixed parameter values of \(\alpha = 0.5, \beta = 1.8, \gamma = -3\), the fixed points are given as \((1.2, 0), (0,0)\) and \((-1.2, 0)\). Fixed points are obviously further classified into distinct types, depending on the eigen values \(\lambda_1\) and \(\lambda_2\) of the Jacobian matrix, satisfying the characteristic equation

\[
\text{det}( J - \lambda I ) = 0, \quad (3.5)
\]

where \(I\) is the unit matrix (Lakshmanan and Murali 1996). The eigen values of the fixed points for these three different conditions are calculated using equation (3.5) as \((-0.25 \pm i1.31815), (0.8735, -1.3735)\) and \((0.25 \pm i1.31815)\).

(i) Corresponding to the eigenvalues \((-0.25 \pm i1.31815)\), the fixed point is a stable focus because the eigen values are complex conjugate to each other with negative real part.

(ii) Corresponding to the eigenvalues \((0.25 \pm i1.31815)\), the fixed point is an unstable focus because the eigen values are complex conjugate to each other with positive real part.

(iii) Corresponding to the real eigenvalues with one positive and one negative values \((0.8735, -1.3735)\), the fixed point is unstable saddle or hyperbolic equilibrium point.
Naturally these fixed points can be observed depending upon the initial condition $x(0)$ and $y(0)$ of equation (3.2) when $f = 0$. When the forcing signal is included ($f > 0$), these fixed points give rise to limit cycles through Hopf bifurcation and as ‘$f$’ is increased further the system exhibits period doubling bifurcations from the period-1 limit cycle to chaos.

3.2.2 Numerical study

Numerical analysis of equation (3.2) is carried out by using the standard 4th-order Runge-Kutta integration method for the parameter values of $\alpha = 0.5$, $\beta = 1.8$, $\gamma = -3$, $\omega = 1.1$ and $f$ as the control parameter. The results are shown in the Figures 3.2 and 3.3. In Figure 3.2, period-1, period-2, period-4, one-band chaos, double-band chaos and power spectrum, Poincare map corresponding to double-band chaos are shown. Figure 3.2(a) depicts the one-parameter bifurcation diagram in the ($f - x$) plane, which clearly indicates the familiar period-doubling bifurcation sequences, chaos, windows, etc. Also, in Figure 3.2(b), the maximal Lyapunov exponent spectrum in the ($f - \lambda_m$) plane is plotted.
Figure 3.2  Period doubling scenario of the second-order
non-autonomous system. Phase portraits in the (x–y) plane
for the parameter values $\alpha = 0.5$, $\beta = 1.8$, $\gamma = -3$ and
$\omega = 1.1$. (a) Period-1 limit cycle; $f = 0.48$, (b) Period-2 limit
cycle; $f=0.5$, (c) Period-4 limit cycle; $f = 0.51$, (d) One-band
chaos; $f = 0.536$, (e(i)) Double-band chaos; $f = 0.55$,
(f) Period-1 boundary; $f = 1.86$, (e(ii)) Power spectrum.
Frequency ($F = \omega / 2\pi$) is represented in Hz and (e(iii))
Poincare map corresponding to double-band chaos for
$f = 0.55$
Figure 3.3  (a) One parameter \( f \) bifurcation diagram in the \( (f - x) \) plane and (b) Maximal Lyapunov exponent in the \( (f - \lambda_m) \) plane for the parameter values \( \alpha = 0.5, \beta = 1.8, \gamma = -3 \) and \( \omega = 1.1 \)

3.2.3  Experimental study

An electronic analog simulator of equation (3.1) can be easily constructed using conventional operational amplifiers (OPAMP-\( \mu \)A741C), resistors (100 KΩ, 10 KΩ, 1 KΩ), capacitors (10nF) and diodes (IN 4007) and this experimental circuit confirms the results obtained numerically for this system. The basic simple circuit is depicted in Figure 3.4. It consists of two OPAMP-integrators \( (U_2, U_3) \), one adder \( (U_1) \), two inverters \( (U_3, U_4) \) and two diodes \( (D_1, D_2) \).
Figure 3.4 Analog simulation circuit model for the system 3.1
In the circuit of Figure 3.4, the outputs of two successive inverting integrators at the nodes are labeled as $V_2$ and $V_1$ as well as, output of a summing amplifier as $V_3$. When Kirchhoff’s rules at nodes a, b and c and golden rules for operational amplifiers (Horowitz and Hill 1989) are applied, the following relations among the voltages are obtained.

Node a: $V_2 = -R_3 C_2 \frac{dV_1}{dt}$, \hspace{1cm} (3.6)

Node b: $V_3 = -R_1 C_1 \frac{dV_2}{dt}$, \hspace{1cm} (3.7)

Node c: $V_3 = -(R_2/R_{12})V_1 - (R_2/R_{11})f(t) - (R_6/R_5)G(V_1) - (R_2/R_9)(-V_2)$, \hspace{1cm} (3.8)

where $f(t) = f \sin \omega t$. A precision clipping circuit with two diodes and corresponding reverse bias voltages as depicted in the box in Figure 3.4 is employed for threshold control mechanism to implement nonlinear function $G(x)$. Redefining $V_1 = -x$, rescaling $t \rightarrow R_1 C_1 \tau$ and substitution of equations (3.6) and (3.7) in equation (3.8) yields

$$\frac{d^2x}{dt^2} = -(R_2/R_{12})x - (R_2/R_{11})f(t) - (R_6/R_5)G(x) - (R_2/R_9)\frac{dx}{dt}. \hspace{1cm} (3.9)$$

Now choosing for the various resistors and capacitors the values $R_1=R_3=10K\Omega$, $R_2=R_5=R_{11}=100K\Omega$, $R_6=300K\Omega$, $R_9=200K\Omega$, $R_{12}=55K\Omega$, $C_1=C_2=10nF$, equation (3.9) may be compared to equation (3.1). It is straightforward to generalize equation (3.9) to the case where the resistors and capacitors differ slightly from their nominal values. In the actual experiment, the frequency of the external sinusoidal force has been fixed at 1752 Hz, so that the corresponding redefined frequency $\omega = R_1 C_1 \omega' = (10K\Omega) x (10nF) x (2\pi x 1752Hz) \approx 1.1$.

The amplitude ‘$f$’ of the forcing signal is used as the bifurcation parameter. By increasing the amplitude $f$ from zero upwards, for fixed frequency at1752 Hz, the circuit of Figure 3.4 is found to exhibit a sequence
of bifurcations. Starting from the equilibrium point the solution bifurcates through a Hopf bifurcation to a limit cycle and then by period-doubling sequences to one-band attractor, double-band attractor and boundary crisis. These are illustrated in Figure 3.5.

Figure 3.5 Experimental analog simulation result of second-order non-autonomous system. (a) Period-1 limit cycle; $f = 0.48$, (b) Period-2 limit cycle; $f = 0.5$, (c) Period-4 limit cycle; $f = 0.51$, (d) One-band chaos; $f = 0.536$, (e(i)) Double-band chaos; $f = 0.55$, (f) Period-1 boundary; $f = 1.86$ and (e(ii)) Power spectrum corresponding to double-band chaos; $f = 0.55$
3.3 REALIZATION OF MURALI – LAKSHMANAN - CHUA (MLC) CIRCUIT WITH THRESHOLD NONLINEARITY

Another second-order non-autonomous system with threshold nonlinerarity has been considered, in order to study the effectiveness of this type of nonlinearity to realize an analog simulation circuit equivalent of the familiar Murali - Lakshmanan - Chua (MLC) circuit. The dynamical state equation of this system is given by

\[
dx / dt = 1.65\ \text{G}(x) - 0.55\ x + y, \quad (3.10a)
\]

\[
dy / dt = -y - x + f\ \sin\ \omega\ t. \quad (3.10b)
\]

Here \(f\) is the amplitude and \(\omega\) is the angular frequency of the driving sinusoidal force. \(\text{G}(x)\) is the nonlinear characteristic function and is given by

\[
\text{G}(x) = \begin{cases} 
   x^*, \text{ when } x > x^* \\
   x, \text{ when } -x^* \leq x \leq x^* \\
   -x^*, \text{ when } x < -x^*
\end{cases} \quad (3.11)
\]

where \(x^*\) is the threshold value and is equal to 0.52. The angular frequency is \(\omega = 0.75\). The amplitude of the driving force ‘f’ is the control parameter.

3.3.1 Stability analysis

The fixed points of equation (3.10) in the absence of external force are determined by equating \(dx / dt = dy / dt = 0\), which gives

\[
1.65\ \text{G}(x) - 0.55x + y = 0, \quad (3.12a)
\]

\[
- y - x = 0. \quad (3.12b)
\]
Roots of equations (3.12a) and (3.12b) are the fixed points. The fixed points 
\((x_0, y_0)\) corresponding to each of the three conditions of threshold function 
\(G(x)\) are \((0.6, -0.6)\), \((0,0)\), \((-0.6, 0.6)\). The stability determining eigen values 
for the equilibrium points are calculated from the Jacobian matrix, whose 
characteristic equation is given in equation (3.5). The calculated eigen values 
associated with three conditions of threshold mechanism are 
\((-0.775 \pm i 0.975)\), \((0.37, -0.27)\) and \((-0.775 \pm i 0.975)\). Therefore fixed 
points are

(i) stable focus, because the eigen values \((-0.775 \pm i 0.975)\) are 
complex conjugate to each other and with negative real part 
and

(ii) saddle and unstable or hyperbolic equilibrium point, because 
the eigen values \((0.37, -0.27)\) are real.

These fixed points can be observed depending upon the initial 
condition \(x(0)\) and \(y(0)\) of equation (3.12) when \(f = 0\). When the forcing 
signal is included \((f > 0)\), these fixed points give rise to limit cycles through 
Hopf bifurcation and as ‘\(f\)’ is increased further the system exhibits period 
doubling bifurcations from the period-1 limit cycle to chaos.

3.3.2 Numerical study

Numerical analysis of equations (3.10a) and (3.10b) is carried out 
by using the standard 4th-order Runge-Kutta integration method for the 
angular frequency \(\omega = 0.75\) and considering ‘\(f\)’ as the control parameter. The 
results are shown in the Figures 3.6 and 3.7. In Figure 3.6, period-1 limit 
cycle, period-2 limit cycle, double-band chaos and power spectrum, Poincare 
map corresponding to double-band chaos are shown. Figure 3.7 (a) depicts the 
one-parameter bifurcation diagram in \((f-x)\) plane, which clearly indicates the
familiar period-doubling bifurcation sequences, chaos, windows, etc. Also, in Figure 3.7(b), the maximal Lyapunov exponent spectrum in \((f - \lambda_m)\) plane is plotted.

Figure 3.6 Period doubling scenario of the second-order non-autonomous system. Phase portraits in the \((x-y)\) plane for \(\omega = 0.75\). (a) Period-1 limit cycle; \(f = 0.09\), (b) Period-2 limit cycle: \(f=0.1\), (c(i)) Double-band chaos; \(f = 0.115\), (d) Period-3 window; \(f = 0.15\), (f) Period-1 boundary; \(f = 0.35\), (c(ii)) Power spectrum. Frequency \((F = \omega / 2\pi)\) is represented in Hz and (c(iii)) Poincare map corresponding to double-band chaos for \(f=0.115\)
3.3.3 Experimental study

An experimental analog simulation circuit confirms the numerical results obtained for this system. The circuit uses only operational amplifiers, resistors, capacitors and diodes. The basic simple circuit is given in the Figure 3.8. It consists of two OPAMP-integrators, two adders, two inverters and two diodes.
Figure 3.8 Analog simulation circuit model for the system (3.10)
In the circuit of Figure 3.8, the outputs of two inverting integrators at the nodes are labeled as $V_1$ and $V_3$ as well as, output of a summing amplifiers as $V_2$ and $V_4$. When Kirchhoff’s rules at nodes a, b, c and d and rules for operational amplifiers are applied, the following relations among the voltages are obtained,

**Node a:** \[ V_2 = -R_5 \, C_1 \, \frac{dV_1}{dt}, \]  \hspace{1cm} (3.13)

**Node b:** \[ V_2 = -(R_4/ R_1)G(V_1)-(R_4/ R_2)V_3 -(R_4/ R_3) (-V_1), \] \hspace{1cm} (3.14)

**Node c:** \[ V_4 = -R_{14} \, C_2 \, \frac{dV_3}{dt}, \] \hspace{1cm} (3.15)

**Node d:** \[ V_4 = (R_{13}/ R_9)V_1 - (R_{13}/ R_{10})f(t) + (R_{13}/ R_{15})V_3, \] \hspace{1cm} (3.16)

where $f(t) = fsin\omega t$. A precision clipping circuit with two diodes and corresponding reverse bias voltages as depicted in the box in Figure 3.8 is employed for threshold control mechanism to implement the nonlinear function $G(x)$. Redefining $V_1=x$, $V_3=y$, rescaling $t \rightarrow R_1C_1\tau$ and substitution of equations (3.13) and (3.15) in equations (3.14) and (3.16) yield equations (3.17) and (3.18) respectively,

\[
\frac{dx}{dt} = (R_4/ R_1) \, G(x) + (R_4/ R_2) \, y - (R_4/ R_3) \, x, \hspace{1cm} \text{(3.17)}
\]

\[
\frac{dy}{dt} = - (R_{13}/ R_9) \, x + (R_{13}/ R_{10}) \, f(t) - (R_{13}/ R_{15}) \, y. \hspace{1cm} \text{(3.18)}
\]

Now choosing for the various resistors and capacitors the values $R_1=60.6K\Omega$, $R_5=R_{14}=10K\Omega$, $R_2=R_{12}=R_9= R_{10}= R_{13}= R_{15}=100K\Omega$, $R_3=182K\Omega$, $C_1=C_2=10nF$, equations (3.17) and (3.18) are compared to equations (3.10a) and (3.10b) respectively. In the actual experiment, the frequency of the external sinusoidal force has been fixed at 1195Hz, so that the corresponding redefined frequency $\omega = R_1C_1\omega' = (10K\Omega) \times (10nF) \times (2\pi x 1195Hz) \approx 0.75$.

The amplitude ‘$f$’ of the forcing signal is used as the bifurcation parameter. By increasing the amplitude ‘$f$’ from zero upwards, for fixed
frequency at 1195 Hz, the circuit of Figure 3.8 is found to exhibit a sequence of bifurcations. Starting from the equilibrium point, the solution bifurcates through a Hopf bifurcation to a limit cycle and then by period-doubling sequences to one-band, double-band attractors and boundary crisis. These are illustrated in Figure 3.9.

Figure 3.9  Experimental analog simulation result of second-order non-autonomous system. (a) Period-1; f = 0.09, (b) Period-2; f=0.1, (c) One-band attractor; f=0.112, (d) & (e) double-band attractor; f=0.115, (f) period-2 window, (g) period-1 boundary; f=0.35
3.4 CONCLUSION

In this chapter, two new non-autonomous analog simulation circuits with threshold controller nonlinearity to exhibit chaos have been reported. Detailed numerical and experimental investigations have been made. For chosen circuit parameters, these systems found to reveal the familiar period doubling route to chaos, double scroll type chaotic attractors, periodic windows, boundary crisis, etc. The existence of such simple non-autonomous systems exhibiting rich variety of attractors can be profitably used in certain application areas such as stochastic resonance, secure communication and so on.